Lefschetz trace formulas for flows on foliated manifolds

work in progress
joint with Jesús Álvarez López and Eric Leichtnam

Yuri A. Kordyukov

Institute of Mathematics, Ufa Science Center RAS, Ufa, Russia

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The setting of the problems

- M a closed manifold, $\dim M = n$.
- $\mathcal{F}$ a codimension one foliation on M.
- $\phi^t : M \to M, t \in \mathbb{R}$ a foliated flow
  (it takes each leaf to a leaf).

Problems:

- To define a Lefschetz number (distribution) of the flow ϕ:

$$L(\phi) = \sum_{j=0}^{n-1} (-1)^j \mathrm{Tr}\,(\phi^* : H^j \to H^j)$$

  H^j is some cohomology theory associated to $\mathcal{F}$, Tr is some trace.
- To prove the corresponding Lefschetz trace formula, an expression for $L(\phi)$ in terms of closed orbits and fixed points of the flow.

Simple flows

Assumption 1:

All fixed points and closed orbits of the flow are simple:

- A closed orbit c of period l (not necessarily minimal) of the flow ϕ is called simple, if

$$\det(\mathrm{id} - \phi_*^l : T_x\mathcal{F} \to T_x\mathcal{F}) \neq 0, \quad x \in c.$$

- A fixed point x of the flow ϕ is called simple if

$$\det(\mathrm{id} - \phi_*^t : T_xM \to T_xM) \neq 0, \quad t \neq 0.$$

Simple flows

- Fix(ϕ) the fixed point set of ϕ (closed in M).
- M^0 the $\mathcal{F}$-saturation of Fix(ϕ) (the union of leaves with fixed points).
  Observe that M^0 is ϕ-invariant, and, under Assumption 1, it is a finite union of compact leaves.
- $M^1 = M \setminus M^0$ the transitive point set.

Assumption 2:

The orbits of the flow in M^1 are transverse to the leaves:

$$T_x M = \mathbb{R}\, Z(x) \oplus T_x\mathcal{F}, \quad x \in M^1,$$

where Z is the infinitesimal generator of ϕ (a vector field on M).

Definition

If the foliated flow ϕ satisfies Assumptions 1 and 2, it is called simple.

Guiilemin-Sternberg formula

A canonical expression for the right-hand side of the Lefschetz formula, which follows from the Guiilemin-Sternberg formula:

$L(\phi)$ is a distribution on $\mathbb{R}^+$ given by:

$$L(\phi) = \sum_c l(c) \sum_{k=1}^{\infty} \varepsilon_{kl(c)}(c)\delta_{kl(c)} + \sum_p \varepsilon_p |1 - e^{\varkappa_p t}|^{-1},$$

c runs over all closed orbits and p over all fixed points of ϕ:

- $l(c)$ the minimal period of c,
- $\varepsilon_l(c) := \operatorname{sign} \det \left(\mathrm{id} - \phi_*^l : T_x\mathcal{F} \to T_x\mathcal{F}\right), x \in c$.
- $\varepsilon_p := \operatorname{sign} \det \left(\mathrm{id} - \phi_*^t : T_p\mathcal{F} \to T_p\mathcal{F}\right), t > 0$.
- $\varkappa_p \neq 0$ is a real number such that

$$\bar{\phi}_*^t : T_pM/T_p\mathcal{F} \to T_pM/T_p\mathcal{F}, \quad x \mapsto e^{\varkappa_p t} x.$$

The refined setting of the problems:

To define a Lefschetz distribution $L(\phi)$ of a simple foliated flow ϕ as a distribution on $\mathbb{R}$ in the form:

$$L(\phi) = \sum_{j=0}^{n-1} (-1)^j \mathrm{Tr}\,(\phi^* : H^j \to H^j)$$

- H^j is some cohomology theory associated with $\mathcal{F}$,
- Tr is a trace,

such that the above Guillemin-Sternberg formula holds.

Motivation:

Deninger's program to study zeta- and L-functions for algebraic schemes over the integers, in particular, the Riemann zeta-function (Berlin, ICM, 1998).

Nonsingular flows

ASSUMPTIONS:

- M a closed manifold, $\dim M = n$.
- $\mathcal{F}$ a codimension one foliation on M.
- $\phi^t : M \to M, t \in \mathbb{R}$ a simple foliated flow.
- ϕ has no fixed points:
  - all the closed orbits are simple,
  - all the orbits in M are transverse to the leaves.

Jesús A. Álvarez López, Y. K., Distributional Betti numbers of transitive foliations of codimension one. Foliations: geometry and dynamics (Warsaw, 2000), 159–183, World Sci. Publ., River Edge, NJ, 2002.

Leafwise de Rham complex

$(\Omega(\mathcal{F}), d_{\mathcal{F}})$ the leafwise de Rham complex of $\mathcal{F}$:

- $\Omega^{\cdot}(\mathcal{F}) = C^{\infty}(M, \Lambda^{\cdot} T^*\mathcal{F})$ smooth leafwise differential forms;
- $d_{\mathcal{F}} : \Omega^{\cdot}(\mathcal{F}) \to \Omega^{\cdot+1}(\mathcal{F})$ the leafwise de Rham differential.

In a foliated chart with coordinates $(x_1, \ldots, x_{n-1}, y) \in \mathbb{R}^{n-1} \times \mathbb{R}$ such that leaves are given by $y = c$, a p-form $\omega \in \Omega^p(\mathcal{F})$ is written as

$$\omega = \sum_{\alpha_1 < \alpha_2 < \ldots < \alpha_p} a_{\alpha}(x, y) dx_{\alpha_1} \wedge \ldots \wedge dx_{\alpha_p}$$

and $d_{\mathcal{F}}\omega \in \Omega^{p+1}(\mathcal{F})$ is given by

$$d_{\mathcal{F}}\omega = \sum_{j=1}^{n-1} \sum_{\alpha_1 < \alpha_2 < \ldots < \alpha_p} \frac{\partial a_{\alpha}}{\partial x_j}(x, y) dx_j \wedge dx_{\alpha_1} \wedge \ldots \wedge dx_{\alpha_p}.$$

Leafwise de Rham cohomology

- The reduced leafwise de Rham cohomology of $\mathcal{F}$:

$$\overline{H}(\mathcal{F}) = \ker d_{\mathcal{F}} / \overline{\operatorname{im} d_{\mathcal{F}}},$$

  the closure is in C^∞-topology.
- ϕ is a foliated flow $\Longrightarrow d_{\mathcal{F}} \circ \phi^t = \phi^t \circ d_{\mathcal{F}}$.
  The induced action:

$$\phi^{t*} : \overline{H}(\mathcal{F}) \to \overline{H}(\mathcal{F}).$$

Question

The trace of $\phi^{t*} : \overline{H}(\mathcal{F}) \to \overline{H}(\mathcal{F})$?

The leafwise Hodge decomposition

- $\mathcal{F}$ is a Riemannian foliation.
- g the Riemannian metric on M such that the infinitesimal generator Z of the flow ϕ is of length one and is orthogonal to the leaves — a bundle-like metric.
- $\Delta_{\mathcal{F}} = d_{\mathcal{F}}\delta_{\mathcal{F}} + \delta_{\mathcal{F}}d_{\mathcal{F}}$ the leafwise Laplacian on $\Omega(\mathcal{F})$ (a second order tangentially elliptic differential operator on M).
- $\mathcal{H}(\mathcal{F})$ the space of leafwise harmonic forms on M:

$$\mathcal{H}(\mathcal{F}) = \{\omega \in \Omega(\mathcal{F}) : \Delta_{\mathcal{F}}\omega = 0\}.$$

Theorem (Alvarez Lopez - Yu. K)

The Hodge isomorphism

$$\overline{H}(\mathcal{F}) \cong \mathcal{H}(\mathcal{F}).$$

Transverse ellipticity:

The leafwise de Rham complex $(\Omega(\mathcal{F}), d_{\mathcal{F}})$ of $\mathcal{F}$ as well as the leafwise Laplacian $\Delta_{\mathcal{F}}$ are transversally elliptic relative to the action of the group $\mathbb{R}$, given by the flow ϕ

$L(\phi)$ is a distribution on $\mathbb{R}$:

$$L(\phi) = \mathrm{ind}_{\mathbb{R}}(\Omega(\mathcal{F}), d_{\mathcal{F}}) \in \mathcal{D}'(\mathbb{R}).$$

We will use the leafwise Hodge theory.

The Lefschetz distribution

For any $f \in C_c^\infty(\mathbb{R})$, define

$$A_f = \int_{\mathbb{R}} \phi^{t*} \cdot f(t)\, dt \circ \Pi : L^2\Omega(\mathcal{F}) \to L^2\Omega(\mathcal{F}),$$

where $\Pi : L^2\Omega(\mathcal{F}) \to L^2\mathcal{H}(\mathcal{F})$ is the orthogonal projection.

Theorem

A_f is a smoothing operator. In particular, A_f is of trace class.

The Lefschetz distribution $L(\phi) \in \mathcal{D}'(\mathbb{R})$:

$$< L(\phi), f >= \mathrm{Tr}^s A_f := \sum_{j=1}^{n-1} (-1)^j \operatorname{Tr} A_f^{(i)}, \quad f \in C_c^\infty(\mathbb{R}),$$

where $A_f^{(i)}$ is the restriction of A_f to $\Omega^i(\mathcal{F})$.

The Lefschetz formula

Theorem (Alvarez Lopez - Y.K.)

Assume that ϕ is simple and has no fixed points.

- *On $\mathbb{R} \setminus \{0\}$*

$$L(\phi) = \sum_{c} l(c) \sum_{k \neq 0} \varepsilon_{kl(c)}(c) \delta_{kl(c)},$$

  when c runs over all closed orbits of ϕ and $l(c)$ denotes the minimal period of c.

- *In some neighborhood of 0 in $\mathbb{R}$:*

$$L(\phi) = \chi_{\Lambda}(\mathcal{F}) \cdot \delta_0.$$

$\chi_{\Lambda}(\mathcal{F})$ the Λ-Euler characteristic of $\mathcal{F}$ given by the holonomy invariant transverse measure Λ (Connes, 1979).

The setting

ASSUMPTION:

- M a closed manifold, $\dim M = n$.
- $\mathcal{F}$ a codimension one foliation on M.
- $\phi^t : M \to M, t \in \mathbb{R}$ a simple foliated flow.

- $\mathrm{Fix}(\phi)$ the fixed point set of ϕ (closed in M).
- M^0 the $\mathcal{F}$-saturation of $\mathrm{Fix}(\phi)$ (the union of leaves with fixed points).
- $M^1 = M \setminus M^0$ the transitive point set.

Definition

The foliated flow ϕ is simple, if:

- all of its fixed points and closed orbits are simple,
- its orbits in M^1 are transverse to the leaves.

Remarks

- The leafwise de Rham complex $(\Omega(\mathcal{F}), d_{\mathcal{F}})$ of $\mathcal{F}$ as well as the leafwise Laplacian $\Delta_{\mathcal{F}}$ are transversally elliptic only on the transitive point set M^1, not on M^0.
- As a consequence, the operator

$$A_f = \int_{\mathbb{R}} \phi^{t*} \cdot f(t)\, dt \circ \Pi : L^2\Omega(\mathcal{F}) \to L^2\Omega(\mathcal{F})$$

  is not a smoothing operator. Its Schwartz kernel is smooth on $M^1 \times M^1$ and singular near $M^0 \times M^0$.
  So its trace is not well-defined.
- $\mathcal{F}$ is not a Riemannian foliation.
  Indeed, $\mathcal{F}$ is a foliation almost without holonomy:
  - M^0 is a finite union of compact leaves,
  - only the leaves in M^0 may have non-trivial holonomy groups.

A singular Riemannian metric

There is a Riemannian metric g^1 on M^1:

- M^1_l equipped with $g_l := g^1|_{M^1_l}$ is a manifold of bounded geometry;
- g^1 is bundle-like for $\mathcal{F}^1$;
- $\mathcal{F}^1_l$ a Riemannian foliation of bounded geometry;
- ϕ^t_l a flow of bounded geometry.

Remarks:

- Observe that g^1 is singular at M^0.
- Each (M^1_l, g^1_l) is a Riemannian manifold with cylindrical ends.

Local stability for foliations

We use a very concrete choice of such a metric g^1. We need to describe a local structure of the foliation near M^0.
Fix a compact leaf L in M^0. Using the local stability theorem for foliations, one can show that $\mathcal{F}$ can be described around L by using the suspension construction.

The initial data for the suspension construction:

- L a connected closed manifold;
- a homomorphism (the holonomy homomorphism)

$$\bar{h} : \Gamma := \pi_1 L / \ker h \to \mathrm{Diffeo}_+(\mathbb{R}, 0), \gamma \mapsto \bar{h}_\gamma, \quad \bar{h}_\gamma(x) = a_\gamma x,$$

where $\gamma \in \Gamma \mapsto a_\gamma \in \mathbb{R}^+$ is a homomorphism.

Suspension manifold

The holonomy covering

$\pi : \widetilde{L} \to L$ the regular covering map with

$$\pi_1 \widetilde{L} \equiv \ker h \Leftrightarrow \mathrm{Aut}(\pi) \equiv \Gamma.$$

The canonical left action of each $\gamma \in \Gamma$ on $\widetilde{L}$ is denoted by $\tilde{y} \mapsto \gamma \cdot \tilde{y}$.

The suspension manifold:

$M_L = \widetilde{L} \times_\Gamma \mathbb{R}$ the orbit space for the diagonal Γ-action on $\widetilde{M}_L = \widetilde{L} \times \mathbb{R}$:

$$\gamma \cdot (\tilde{y}, x) = (\gamma \cdot \tilde{y}, a_\gamma\, x). \quad (\tilde{y}, x) \in \widetilde{L} \times \mathbb{R}.$$

Let $[\tilde{y}, x]$ denote the element in M_L represented by each $(\tilde{y}, x) \in \widetilde{M}_L$.

Foliated fiber bundle

The fiber bundle map

$\widetilde{\varpi} : \widetilde{M}_L = \widetilde{L} \times \mathbb{R} \to \widetilde{L}$ the Γ-equivariant map given by the first factor projection induces the map:

$$\varpi : M_L = \widetilde{L} \times_\Gamma \mathbb{R} \to L, \quad \varpi([\tilde{y}, x]) = \pi(\tilde{y}).$$

Note that the typical fiber of ϖ is $\mathbb{R}$.

The suspension foliation

$\mathcal{F}_L$ is the foliation on M_L transverse to the fibers of $\varpi : M_L \to L$, which is induced by the Γ-invariant foliation on $\widetilde{M}_L$ with leaves $\widetilde{L} \times \{x\}$ $(x \in \mathbb{R})$.

Since 0 is fixed by the Γ-action on $\mathbb{R}$, the leaf $\widetilde{L} \equiv \widetilde{L} \times \{0\}$ of $\widetilde{\mathcal{F}}_L$ projects to a leaf of $\mathcal{F}_L$ that can be canonically identified with L.

Local description near the compact leaf

According to the local stability theorem, there are tubular neighborhoods $\varpi : V_L \to L$ of L in M_L and $\varpi : V \to L$ of L in M and a diffeomorphism from V to V_L, which takes $\mathcal{F}|_V$ to $\mathcal{F}_L|_{V_L}$:

$$V \equiv V_L, \quad \mathcal{F}|_V \equiv \mathcal{F}_L|_{V_L}.$$

and the flow ϕ^t on $V \equiv V_L$ is given by

$$\phi^t([\tilde{y}, x]) = [\phi_x^t(\tilde{y}), e^{\varkappa_L t} x], \quad [\tilde{y}, x] \in V_L \subset M_L = \widetilde{L} \times_\Gamma \mathbb{R}.$$

Recall that $\varkappa_p \neq 0$ is a real number (depending only on L) such that

$$\bar{\phi}_*^t : T_pM/T_p\mathcal{F} \to T_pM/T_p\mathcal{F}, \quad x \mapsto e^{\varkappa_p t} x.$$

Construction of the singular Riemannian metric

- g^0 a Riemannian metric on L.
- $g_{\mathcal{F}_L}$ a leafwise Riemanian metric on $(M_L, \mathcal{F}_L)$, defined by requiring that the restrictions of the map

$$\varpi : M_L = \widetilde{L} \times_\Gamma \mathbb{R} \to L, \quad \varpi([\tilde{y}, x]) = \pi(\tilde{y}),$$

  to the leaves of $\mathcal{F}_L$ are local isometries.
- g_{M_L} a Riemannian metric on $M_L \setminus L = \widetilde{L} \times_\Gamma (\mathbb{R} \setminus \{0\})$:

$$g_{M_L} = g_{\mathcal{F}_L} + \frac{dx^2}{x^2}, \quad [\tilde{y}, x] \in \widetilde{L} \times_\Gamma (\mathbb{R} \setminus \{0\}),$$

  is bundle-like for $\mathcal{F}_L$.

Construction of the singular Riemannian metric

We fix an identification

$$V \equiv V_L, \quad \mathcal{F}|_V \equiv \mathcal{F}_L|_{V_L},$$

and easily get a bundle-like metric g^1 on $(M^1, \mathcal{F}^1)$ with the above properties:

- g^1 is bundle-like for $\mathcal{F}^1$;
- M_l equipped with $g_l := g^1|_{M_l^1}$ is a manifold of bounded geometry;
- $\mathcal{F}_l^1$ a Riemannian foliation of bounded geometry;
- ϕ_l^t a flow of bounded geometry.

The blow-up of M

- M^1_l, $l = 1, \ldots, r$, the connected components of the transitive point set $M^1 (= M \setminus M^0)$:

$$(M^1, \mathcal{F}^1) = \bigsqcup_l (M^1_l, \mathcal{F}^1_l).$$

- $M^l = \overline{M^1_l}$ is the closure of M^1_l.
  Thus, M_l is a connected compact manifold with boundary, endowed with a smooth foliation $\mathcal{F}_l$ tangent to the boundary.
- Put

$$M^{\mathrm{c}} := \bigsqcup_l M_l, \quad \mathcal{F}^{\mathrm{c}} := \bigsqcup_l \mathcal{F}_l.$$

- The flow lifts to a simple foliated flow $\phi^{\mathrm{c},t}$ of $\mathcal{F}^{\mathrm{c}}$ tangent to ∂M^{c}.

Differential operators on the blow-up

- The blow up of the transitive point set M^1:

$$M^{\mathrm{c}} = \bigsqcup_l M_l, \quad \mathcal{F}^{\mathrm{c}} = \bigsqcup_l \mathcal{F}_l,$$

  M_l a connected compact manifold with boundary,
  $\mathcal{F}_l$ a smooth foliation tangent to the boundary:

$$\mathring{M}_l \equiv M_l^1, \quad \mathring{\mathcal{F}}_l \equiv \mathcal{F}_l^1.$$

- We transfer the Riemannian metric g^1 to $\mathring{M}_l$.
  We get a b-metric (generally, non-exact).
- We also have $\mathring{M}_l$ to be a manifold of bounded geometry and $\mathring{\mathcal{F}}_l$ a Riemannian foliation of bounded geometry.
- $d_{\mathring{\mathcal{F}}_l}$ the leafwise de Rham differential on $\Omega(\mathring{\mathcal{F}}_l)$.
- $\delta_{\mathring{\mathcal{F}}_l}$ the leafwise de Rham codifferential on $\Omega(\mathring{\mathcal{F}}_l)$.
- $D_{\mathring{\mathcal{F}}_l} = d_{\mathring{\mathcal{F}}_l} + \delta_{\mathring{\mathcal{F}}_l}$.

Smoothing operators

For any $\psi \in \mathcal{A}$, $f \in C_c^\infty(\mathbb{R})$ and l, the operator

$$\mathring{P}_l = \int_{-\infty}^{\infty} \phi^{t*} \cdot f(t)\, dt \circ \psi(D_{\mathring{\mathcal{F}}_l})$$

is a smoothing operator on $\mathring{M}_l$, but its kernel is singular near $\partial \mathring{M}_l$.

The algebra $\mathcal{A}$:

$\mathcal{A}$ the Fréchet algebra of functions $\psi : \mathbb{R} \to \mathbb{C}$ such that the Fourier transform $\hat{\psi}$ satisfies: for every $k \in \mathbb{N}$, there is $A_k > 0$

$$|\hat{\psi}(\xi)| \leq A_k e^{-k|\xi|}, \quad \xi \in \mathbb{R}\,.$$

$\mathcal{A}$ contains all functions with compactly supported Fourier transform, as well as the Gaussians $x \mapsto e^{-tx^2}$ with $t > 0$.

Theorem (Alvarez Lopez, K., Leichtnam)

$$\mathring{P}_l = \int_{-\infty}^{\infty} \phi^{t*} \cdot f(t)\, dt \circ \psi(D_{\mathring{\mathcal{F}}_l})$$

gives rise to an element P_l of the algebra $\Psi_{\mathrm{b}}^{-\infty}(M_l; \bigwedge T\mathcal{F}_l^*)$ in b-calculus:

- *The Schwartz kernel K_{P_l} is smooth in the interior $\mathring{M}_l \times \mathring{M}_l$.*
- *K_{P_l} has a C^∞ extension to $M_l \times M_l \setminus \partial M_l \times \partial M_l$ that vanishes to all orders at $(\partial M_l \times M_l) \cup (M_l \times \partial M_l)$.*
- *In a tubular neighborhood of $L \subset \pi_0(\partial M_l)$ with coordinates $(\rho, y), \rho \in (0, \epsilon_0), y \in L$, the kernel K_{P_l} has the form*

$$K_{P_l}(\rho, y, \rho', y') = \kappa_{P_l}\left(\rho, y, \frac{\rho'}{\rho}, y'\right) \left|\frac{d\rho'}{\rho'}\right| |dy'|,$$

where $\kappa_{P_l}(\rho, y, s, y')$ is smooth up to L (that is, up to $\rho = 0$).

In a tubular neighborhood of L with coordinates $\rho \in (0, \epsilon_0), y \in L$,

$$P_l u(\rho, y) = \int \kappa_{P_l}\left(\rho, y, \frac{\rho'}{\rho}, y'\right) u(\rho', y') \left|\frac{d\rho'}{\rho'}\right| |dy'|,$$

and $\kappa_{P_l}(\rho, y, s, y')$ is smooth up to L (that is, up to $\rho = 0$).

The b-trace of P_l:

$$ {}^{\mathrm{b}}\mathrm{Tr}\,(P_l) = \lim_{\epsilon \to 0} \Big(\int_{\rho > \epsilon} K_{P_l}(\rho, y, \rho, y) |d\rho||dy| + \ln \epsilon \int \kappa_{P_l}(0, y, 1, y)|dy| \Big).$$

${}^{\mathrm{b}}\mathrm{Tr}$ doesn't have trace property, but ${}^{\mathrm{b}}\mathrm{Tr}\,[P, P']$ is expressed in terms of traces of some explicit integral operators on ∂M_l.

Since $M^{\mathrm{c}} = \bigsqcup_l M_l, \mathcal{F}^{\mathrm{c}} = \bigsqcup_l \mathcal{F}_l$, we get the operator

$$P \equiv \bigoplus_l P_l = \int_{-\infty}^{\infty} \phi^{t*} \cdot f(t)\, dt \circ \psi(D_{\mathcal{F}^{\mathrm{c}}})$$

$$\in \Psi_{\mathrm{b}}^{-\infty}(M^{\mathrm{c}}; \bigwedge T\mathcal{F}^{\mathrm{c}*}) \equiv \bigoplus_l \Psi_{\mathrm{b}}^{-\infty}(M_l; \bigwedge T\mathcal{F}_l^*)\ .$$

In particular, its b-trace ${}^{\mathrm{b}}\mathrm{Tr}\,(P)$ is well-defined.
The b-supertrace of P:

$${}^{\mathrm{b}}\mathrm{Tr}\,^{\mathrm{s}}(P) = \sum_{j=1}^{n-1} (-1)^j\, {}^{\mathrm{b}}\mathrm{Tr}\,(P^{(j)}),$$

where $P^{(j)}$ is the restriction to j-forms.

We follow the heat kernel approach to index theory:

- Fix an even $\psi \in \mathcal{A}$ and $f \in C_c^\infty(\mathbb{R})$.
- For $u > 0$, let

$$P_{\psi_u,f} = \int_{-\infty}^{\infty} \phi^{t*} \cdot f(t)\, dt \circ \psi(uD_{\mathcal{F}^c})$$

- Since the b-trace is not a trace, $\frac{d}{du} {}^{\mathrm{b}}\mathrm{Tr}^{\mathrm{s}}(P_{\psi_u,f}) \neq 0$.

Derivative of the b-supertrace

$$\frac{d}{du} {}^{\mathrm{b}}\mathrm{Tr}^{\mathrm{s}}(P_{\psi_u,f}) = \sum_{L\in\pi_0(M^0)} \frac{2}{|\varkappa_L|} \sum_{\gamma\in\Gamma_L} \mathrm{Tr}^{\mathrm{s}}_{\Gamma_L}\left(T_\gamma^* \widetilde{R}_{\widetilde{L},u,t_{L,\gamma}}\right) f(t_{L,\gamma}) \,.$$

Notation

Derivative of the b-supertrace

$$\frac{d}{du}{}^{\mathrm{b}}\mathrm{Tr}^{\mathrm{s}}(P_{\psi_u,f}) = \sum_{L\in\pi_0(M^0)} \frac{2}{|\varkappa_L|} \sum_{\gamma\in\Gamma_L} \mathrm{Tr}^{\mathrm{s}}_{\Gamma_L}\left(T^*_\gamma \widetilde{R}_{\widetilde{L},u,t_{L,\gamma}}\right) f(t_{L,\gamma}) ,$$

- $\widetilde{L}$ the universal covering of L, $\Gamma_L := \pi_1 L$.
- $\varkappa_L \neq 0$ a real number such that, for $p \in L$,

$$\bar{\phi}^t_* : N_p\mathcal{F} \to N_p\mathcal{F}, \quad x \to e^{\varkappa_L t}x.$$

- $t_{L,\gamma} = -\varkappa_L^{-1} \log a_{L,\gamma}$ relative periods, where a homomorphism $\gamma \in \Gamma_L \mapsto a_{L,\gamma} \in \mathbb{R}^+$ is given by the holonomy homomorphism

$$\gamma \in \Gamma_L \mapsto \bar{h}_{L,\gamma} \in \mathrm{Diffeo}_+(\mathbb{R}, 0), \quad \bar{h}_{L,\gamma}(x) = a_{L,\gamma}x.$$

More notation

Derivative of the b-supertrace

$$\frac{d}{du}{}^{\mathrm{b}}\mathrm{Tr}^{\,\mathrm{s}}(P_{\psi_u,f}) = \sum_{L\in\pi_0(M^0)} \frac{2}{|\varkappa_L|} \sum_{\gamma\in\Gamma_L} \mathrm{Tr}^{\,\mathrm{s}}_{\Gamma_L}\left(T^*_\gamma \widetilde{R}_{\widetilde{L},u,t_{L,\gamma}}\right) f(t_{L,\gamma})\ ,$$

- $\widetilde{R}_{\widetilde{L},u,t} = u\tilde{\eta}\wedge\tilde{\phi}^{t*}_L\psi'(uD_{\widetilde{L}})$ a Γ_L-invariant smoothing operator on $\widetilde{L}$.
  - $\tilde{\eta}$ a closed one-form on $\widetilde{L}$, the lift of a closed one-form η on L.
  - $\phi^t_L : L \to L$ the restriction of the flow to L.
  - $\widetilde{\phi}^t_L : \widetilde{L} \to \widetilde{L}$ its lift to $\widetilde{L}$.
- T^*_γ the induced action of $\gamma \in \Gamma_L$ on Γ_L-invariant operators on $\widetilde{L}$.
- Tr_{Γ_L} the Γ_L-trace on Γ_L-invariant operators on $\widetilde{L}$.

Definition of $\widetilde{\eta}$

- Fix generators $\gamma_1, \ldots, \gamma_k$ of Γ_L ($k = \mathrm{rank}\,\Gamma_L$).
- c_i a piecewise smooth loop in L based at p representing γ_i^{-1}.
- $\beta_1, \ldots, \beta_k$ closed 1-forms on L such that

$$\langle [\beta_i], \gamma_j \rangle = -\int_{c_j} \beta_i = \delta_{ij}, \quad \langle [\beta_i], \ker h \rangle = 0.$$

- η the closed 1-form on L:

$$\eta = \ln(a_{L,\gamma_1})\, \beta_1 + \cdots + \ln(a_{L,\gamma_k})\, \beta_k \ ,$$

  (the homomorphism $\gamma \in \Gamma_L \mapsto a_{L,\gamma} \in \mathbb{R}^+$ given by the holonomy)
- $\tilde{\eta}$ the lift of η to $\widetilde{L}$.
- If we consider η as a closed leafwise 1-form on the suspension manifold M_L, then there exists a 1-form ω on M_L satisfying $T\mathcal{F}_L = \ker \omega$ such that

$$d\omega = \eta \wedge \omega \ .$$

Variation of the b-supertrace:

For $u, v > 0$,

$$
\begin{aligned}
{}^{\mathrm{b}}\mathrm{Tr}^{\,\mathrm{s}}(P_{\psi_v,f}) - {}^{\mathrm{b}}\mathrm{Tr}^{\,\mathrm{s}}(P_{\psi_u,f}) \\
= \sum_{L\in\pi_0(M^0)} \frac{2}{|\varkappa_L|} \sum_{\gamma\in\Gamma_L} \mathrm{Tr}^{\,\mathrm{s}}_{\Gamma_L}\left(T_\gamma^* \widetilde{S}_{\widetilde{L},u,v,t_{L,\gamma}}\right) f(t_{L,\gamma})\ ,
\end{aligned}
$$

where $\widetilde{S}_{\widetilde{L},u,v,t} = \int_u^v \widetilde{R}_{\widetilde{L},w,t}\, dw = \widetilde{\eta}\wedge \tilde{\phi}_0^{t*}\, \frac{\psi(vD_{\widetilde{L}})-\psi(uD_{\widetilde{L}})}{D_{\widetilde{L}}}$.

Lefschetz distribution:

$$
\begin{aligned}
\langle L(\phi), f\rangle = {}^{\mathrm{b}}\mathrm{Tr}^{\,\mathrm{s}}(P_{\psi_v,f}) \\
- \lim_{u\to 0} \sum_{L\in\pi_0(M^0)} \frac{2}{|\varkappa_L|} \sum_{\gamma\in\Gamma_L} \mathrm{Tr}^{\,\mathrm{s}}_{\Gamma_L}\left(T_\gamma^* \widetilde{S}_{\widetilde{L},u,v,t_{L,\gamma}}\right) f(t_{L,\gamma}).
\end{aligned}
$$

The limit $u \to 0$

Theorem

There exists the limit of ${}^b\mathrm{Tr}^s(P_{\psi_u,f})$ *as* $u \to 0$*, which is given on* $\mathbb{R}_+$ *by*

$$\lim_{u\to 0} {}^b\mathrm{Tr}^s(P_{\psi_u,f}) = \sum_c l(c) \sum_{k=1}^{\infty} \varepsilon_{kl(c)}(c) \cdot f(kl(c))$$

where c *runs over all closed orbits of* ϕ^t*,* $l(c)$ *denotes the minimal period of* c*, and* x *is an arbitrary point of* c*.*

Corollary

$L(\phi)$ is a well-defined distribution on $\mathbb{R}_+$ and

$$\langle L(\phi), f\rangle = \lim_{u\to 0} {}^b\mathrm{Tr}^s(P_{\psi_u,f}).$$

Trace formula

On $\mathbb{R}_+$, we have

$$L(\phi) = \sum_c l(c) \sum_{k=1}^{\infty} \varepsilon_{kl(c)}(c) \cdot \delta_{kl(c)}$$

where c runs over all closed orbits of ϕ^t, $l(c)$ denotes the minimal period of c, and x is an arbitrary point of c.

Perspectives

- To give a cohomological interpretation of the limit as $v \to +\infty$ of

$$ {}^{b}\mathrm{Tr}^{\,s}(P_{\psi_v,f}) - \lim_{u\to 0} \sum_{L\in\pi_0(M^0)} \frac{2}{|\varkappa_L|} \sum_{\gamma\in\Gamma_L} \mathrm{Tr}^{\,s}_{\Gamma_L}\left(T^*_\gamma \widetilde{S}_{\widetilde{L},u,v,t_{L,\gamma}}\right) f(t_{L,\gamma}). $$

- To get the contribution of fixed points as in the Guillemin-Sternberg formula

$$ L(\phi) = \sum_c l(c) \sum_{k=1}^{\infty} \varepsilon_{kl(c)}(c)\delta_{kl(c)} + \sum_p \varepsilon_p |1 - e^{\varkappa_p t}|^{-1}, $$

- To describe $L(\phi)$ in a neghborhood of 0.