Blowups, deformations to normal cones and Lie groupoids (joint work with Claire Debord)

Claire Debord & GS: Blowup constructions for Lie groupoids and a Boutet de Monvel type calculus (In preparation)

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A geometric idea

A Lie groupoid $G \rightrightarrows M$ gives a family of longitudinal differential operators (its algebroid). Evolution along the groupoid.

$V \subset M$ a submanifold, seen as an obstacle.
It forces operators to "slow down" near V in the normal direction.
Propagation should preserve V and move along a subgroupoid $\Gamma \rightrightarrows V$.

We propose a construction of a Lie groupoid taking into account this kind of propagation.

The plan of this talk:

- Present two general constructions of groupoids:
  1. Deformation to the normal cone (DNC)
  2. Blowup ($Blup$).

- Compute, connecting maps and index elements arising in these constructions.

Two classical constructions
1. The Deformation to the Normal Cone

Let V be an immersed submanifold of a smooth manifold M with normal bundle N_V^M. The deformation to the normal cone is

$$DNC(M,V) = M \times \mathbb{R}^* \sqcup N_V^M \times \{0\}.$$

Smooth structure, generated by the following maps to be smooth ($x \in M$, $\lambda \in \mathbb{R}^*$, $y \in V$, $\xi \in T_yM/T_yV$):

- $p : DNC(M,V) \to M \times \mathbb{R} : \quad p(x,\lambda) = (x,\lambda)\,, \ \ p(y,\xi,0) = (y,0);$
- given $f : M \to \mathbb{R}$, smooth with $f_{|V} = 0$,
  $\tilde{f} : DNC(M,V) \to \mathbb{R}, \quad \tilde{f}(x,\lambda) = \dfrac{f(x)}{\lambda}, \ \tilde{f}(y,\xi,0) = (df)_y(\xi)$

Remark. Restricts to every (locally closed) subset of $\mathbb{R}$.

Define $DNC_+(M,V) = M \times \mathbb{R}_+^* \sqcup N_V^M \times \{0\}$.

Functoriality of DNC

Consider a commutative diagram of smooth maps

$$\begin{array}{ccc} V & \hookrightarrow & M \\ {\scriptstyle f_V}\downarrow & & \downarrow{\scriptstyle f_M} \\ V' & \hookrightarrow & M' \end{array}$$

Horizontal arrows are immersions of submanifolds.

We get a smooth map $DNC(f) : DNC(M, V) \to DNC(M', V')$ defined by

$$\begin{cases} DNC(f)(x, \lambda) = (f_M(x), \lambda) & \text{for } x \in M,\ \lambda \in \mathbb{R}_* \\ DNC(f)(x, \xi, 0) = (f_V(x), (\overline{df})_x(\xi), 0) & \text{for } x \in V,\ \bar{\xi} \in T_xM/T_xV \end{cases}$$

where $(\overline{df})_x : T_xM/T_xV \to T_{f_V(x)}M'/T_{f_V(x)}V'$ is the map induced by $(df_M)_x$.

Deformation groupoid

Let Γ be a subgroupoid and an immersed submanifold of a Lie groupoid $G \overset{r,s}{\rightrightarrows} G^{(0)}$.

By functoriality

$$DNC(G,\Gamma) \rightrightarrows DNC(G^{(0)},\Gamma^{(0)})$$

is naturally a Lie groupoid:

- source and range maps are $DNC(s)$ and $DNC(r)$;
- space of composable arrows (identifies with) $DNC(G^{(2)},\Gamma^{(2)})$ and its product with $DNC(m)$ ($m : G_i^{(2)} \to G_i$ is the product).

Remark

N_Γ^G is a groupoid over $N_{\Gamma^{(0)}}^{G^{(0)}}$ denoted $\mathcal{N}_\Gamma^G \rightrightarrows N_{\Gamma^{(0)}}^{G^{(0)}}$.

$$DNC(G,\Gamma) = (G \times \mathbb{R}^*) \sqcup \mathcal{N}_\Gamma^G \times \{0\} \rightrightarrows (G^{(0)} \times \mathbb{R}^*) \sqcup N_{\Gamma^{(0)}}^{G^{(0)}} \times \{0\}$$

Examples of DNC groupoids

1. Tangent groupoid of Alain Connes
$DNC(M \times M, \Delta_M) = (M \times M) \times \mathbb{R}^* \sqcup TM \times \{0\}$. Adiabatic groupoid ([Monthubert-Pierrot 99, Nistor-Weinstein-Xu 99]): restriction of $DNC(G, G^{(0)})$ over $G^{(0)} \times [0,1]$.
This groupoid encodes the index of M, of G.

2. V submanifold of $G^{(0)}$, saturated for G, $DNC(G, G_V^V)$ normal groupoid of immersion $G_V^V \hookrightarrow G$ which gives the shriek map [Hilsum-S 87].

3. K maximal compact subgroup of a Lie group G, $DNC(G, K)$ used by Higson to recover "Dirac induction".

4. Double deformation: $G_1 \subset G_2 \subset G_3$.
$DNC^2(G_3, G_2, G_1) = DNC\big(DNC(G_3, G_1), DNC(G_2, G_1)\big)$.

   Example: $\pi : E \to M$ a submersion; consider $\Delta E \subset E \underset{M}{\times} E \subset E \times E$:

   Used by [Debord-Lescure-Nistor] for a diagram chasing proof of the Atiyah-Singer index theorem ($E \to M$ is the normal bundle).

... Many more...

2. The Blowup construction

$V \subset M$ closed submanifold.
Scaling action of $\mathbb{R}^*$ on $M \times \mathbb{R}^*$ extends to the gauge action on $DNC(M,V) = M \times \mathbb{R}^* \sqcup N_V^M$:

$$\begin{array}{ccc} DNC(M,V) \times \mathbb{R}^* & \longrightarrow & DNC(M,V) \\ (z,t,\lambda) & \mapsto & (z,\lambda t) \ for\ t \neq 0 \\ (x,X,0,\lambda) & \mapsto & (x,\frac{1}{\lambda}X,0) \ for\ t = 0 \end{array}$$

The manifold $V \times \mathbb{R}$ embeds in $DNC(M,V)$.
The gauge action is free and proper on the open subset $DNC(M,V) \setminus V \times \mathbb{R}$ of $DNC(M,V)$. We let:

$$Blup(M,V) = \big(DNC(M,V) \setminus V \times \mathbb{R}\big)/\mathbb{R}^* = M \setminus V \sqcup \mathbb{P}(N_V^M).$$

Put also

$$SBlup(M,V) = \big(DNC_+(M,V) \setminus V \times \mathbb{R}_+\big)/\mathbb{R}_+^* = M \setminus V \sqcup \mathbb{S}(N_V^M).$$

Fonctoriality of *Blup*

$$\begin{array}{ccc} V & \hookrightarrow & M \\ \downarrow f_V & & \downarrow f_M \\ V' & \hookrightarrow & M' \end{array}$$ gives $DNC(f) : DNC(M,V) \to DNC(M',V')$

Equivariant under the gauge action: it passes to the quotient $Blup$... where it is defined.

Let $U_f(M,V) = DNC(M,V) \setminus DNC(f)^{-1}(V' \times \mathbb{R})$; define

$$Blup_f(M,V) = U_f/\mathbb{R}^* \subset Blup(M,V)$$

Then, by passing $DNC(f)$ to the quotient:

$$Blup(f) : Blup_f(M,V) \to Blup(M',V')$$

Analogous construction $SBlup(f) : SBlup_f(M,V) \to SBlup(M',V')$.

Blowup groupoid

Let Γ be a closed Lie subgroupoid of a Lie groupoid $G \overset{r,s}{\rightrightarrows} G^{(0)}$. Define

$$\widetilde{DNC(G,\Gamma)} = U_r(G,\Gamma) \cap U_s(G,\Gamma)$$

elements whose image by $DNC(s)$ and $DNC(r)$ are not in $\Gamma^{(0)} \times \mathbb{R}$.

Subgroupoid of $DNC(G,\Gamma)$. Gauge action: groupoid automorphisms, whence (or by functoriality)

$$Blup_{r,s}(G,\Gamma) = \widetilde{DNC(G,\Gamma)}/\mathbb{R}^* \rightrightarrows Blup(G^{(0)},\Gamma^{(0)})$$

is naturally a Lie groupoid;
source $= Blup(s)$, range $= Blup(r)$ and product $= Blup(m)$.

Analogous constructions hold for $SBlup$.

Examples of blowup groupoids

Let $V \subset M$ be a hypersurface.

$$\underbrace{G_b = SBlup_{r,s}(M \times M, V \times V)}_{\text{The b-calculus groupoid}} \subset \underbrace{SBlup(M \times M, V \times V)}_{\text{Melrose's b-space}}$$

$$\underbrace{G_0 = SBlup_{r,s}(M \times M, \Delta(V))}_{\text{The 0-calculus groupoid}} \subset \underbrace{SBlup(M \times M, \Delta(V))}_{\text{Mazzeo-Melrose's 0-space}}$$

Can take a groupoid $G \rightrightarrows M$ transverse to V.

$$G_b = SBlup_{r,s}(G, G_V^V)) \text{ and } G_0 = SBlup_{r,s}(G, V).$$

$M = V \times \mathbb{R}$, corresponding G_0: Gauge adiabatic groupoid [Debord, S].

Remarks

- Iterate these constructions: manifolds with corners [Monthubert].
- Also by iteration stratified manifolds [Debord, Lescure, Rochon].

$\mathcal{VB}$-groupoids [Pradines]

Lie groupoids E, Γ. Vector bundle structure $p : E \to \Gamma$.
p is a groupoid morphism; all the groupoid maps for E are linear bundle maps.

- $E^{(0)} \subset E_{|\Gamma^{(0)}}$ subbundle;
- $r_E : E_x \to E^{(0)}_{r_\Gamma(x)}$ and $s_E : E_x \to E^{(0)}_{s_\Gamma(x)}$,
- inverse:$E_x \to E_{x^{-1}}$ linear (for all $x \in \Gamma$) .
- product:$\{(u, v) \in E_x \times E_y;\ s_E(u) = r_E(v)\} \to E_{x\cdot y}$ linear for $(x, y) \in \Gamma^{(2)}$.

With a $\mathcal{VB}$-groupoid $p : (E, r_E, s_E) \to (\Gamma, r_\Gamma, s_\Gamma)$ are associated:
The projective $\mathcal{VB}$-groupoid. $\mathcal{P}(E) = (E \setminus (\ker r \cup \ker s))/\mathbb{R}^*$;
The spherical $\mathcal{VB}$-groupoid. $\mathcal{P}(E) = (E \setminus (\ker r \cup \ker s))/\mathbb{R}^*_+$.

The case Γ =point: linear groupoids

Suppose E is a (real) vector space and $F \subset E$ a vector subspace.
Let $r, s : E \to F$ be two linear retractions.

(Classical) facts

1. Unique linear groupoid structure on E: $\mathcal{E} \rightrightarrows F$ with source s, range r and units given by the inclusion $F \subset E$:
   Product $u \cdot v = u + v - s(u)$. Inverse of u is $(r + s - id)(u)$.
2. $\mathcal{E}$ is the action groupoid $E \rtimes_{r-s} E/F$.

Remarks

1. This construction can be done with any field.
2. If $r \neq s$, every orbit meets $F \setminus \{0\}$: the restriction $\mathring{\mathcal{E}}$ of $\mathcal{E}$ to $F \setminus \{0\}$ is Morita equivalent to $\mathcal{E}$.

The case Γ =point: linear groupoids

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Projective and spherical groupoids

Assume $F \neq \{0\}$. The group $\mathbb{R}^*$ acts freely on $\mathcal{E} \setminus (\ker r \cup \ker s) \rightrightarrows F \setminus V$ and leads to the projective groupoid: $\mathcal{P}E \rightrightarrows \mathbb{P}(F)$.

$$\mathcal{P}E = \mathbb{P}(E) \setminus \mathbb{P}(\ker r) \cup \mathbb{P}(\ker s)$$

Source and range are induced by s and r. For composable x, $y \in \mathcal{P}E$:
$x \cdot y = \{u + v - s(u)\ ;\ u \in x,\ v \in y;\ s(u) = r(v)\}$ and the inverse of x is $(s + r - id)(x)$.

The same for spherical...

Remark

If F is just a line, $\mathcal{P}E$ is a group:

- If $r = s$, then $\mathcal{P}E$ is isomorphic to the abelian group $\ker(r) = \ker(s)$.
- If $r \neq s$, then $\mathcal{P}E \simeq (\ker(r) \cap \ker(s)) \rtimes \mathbb{R}^*$.

The case $\Gamma = \Gamma^{(0)}$: Families of linear, projective and spherical groupoids

Same constructions for $E \to V$ a (real) vector-bundle, $F \subset E$ a subbundle and $r, s : E \to F$ bundle-maps, sections of $F \subset E$. It gives:

- A groupoid structure on E: $\mathcal{E} \rightrightarrows F$.
- $\mathcal{E} \simeq F \rtimes_\alpha E/F$ where $\alpha = r - s : E/F \to F$.
- Associated families of projective and spherical groupoids.

Example important for us. $G \overset{r,s}{\rightrightarrows} M$ Lie groupoid, $V \subset M$ (locally) closed submanifold considered as a groupoid (only space).
$E = N_V^G \to V$, $F = N_V^M$ and $\overline{dr}, \overline{ds} : N_V^G \to N_V^M$.
We get groupoids $\mathcal{N}_V^G \rightrightarrows N_V^M$, $\mathcal{P}(N_V^G) \rightrightarrows \mathbb{P}(N_V^M)$ and $\mathcal{S}(N_V^G) \rightrightarrows \mathbb{S}(N_V^M)$.

$$DNC(G, V) = G \times \mathbb{R}^* \sqcup \mathcal{N}_V^G \times \{0\} \rightrightarrows M \times \mathbb{R}^* \sqcup N_V^M \times \{0\}$$

$$Blup(G, V) = \mathring{G} \sqcup \mathcal{P}(N_V^G) \rightrightarrows M \setminus V \sqcup \mathbb{P}(N_V^M).$$

$$SBlup(G, V) = \mathring{G} \sqcup \mathcal{P}(N_V^G) \rightrightarrows M \setminus V \sqcup \mathbb{S}(N_V^M).$$

Deformations, blowups and exact sequences

Let $\Gamma \rightrightarrows V$ be a closed Lie subgroupoid of a Lie groupoid $G \overset{r,s}{\rightrightarrows} M$.
Let $\mathring{M} = M \setminus V$.
Let $\mathring{\mathcal{N}}_\Gamma^G$ restriction of the groupoid $\mathcal{N}_\Gamma^G \rightrightarrows N_V^M$ to $N_V^M \setminus V$.
Writing

$$DNC_+(G,\Gamma) = G \times \mathbb{R}_+^* \sqcup \mathcal{N}_\Gamma^G \times \{0\} \rightrightarrows M \times \mathbb{R}_+^* \sqcup N_V^M$$

$$SBlup_{r,s}(G,\Gamma) = D N\widetilde{C_+(G},\Gamma)/\mathbb{R}_+^* = G_{\mathring{M}}^{\mathring{M}} \sqcup \mathcal{S}\mathcal{N}_\Gamma^G \rightrightarrows \mathring{M} \sqcup \mathbb{S}(N_V^M)$$

we obtain exact sequences (assume that Γ is amenable)

$$0 \longrightarrow C^*(G \times \mathbb{R}_+^*) \longrightarrow C^*(DNC_+(G,\Gamma)) \longrightarrow C^*(\mathcal{N}_\Gamma^G) \longrightarrow 0$$

$$0 \longrightarrow C^*(G_{\mathring{M}}^{\mathring{M}}) \longrightarrow C^*(SBlup_{r,s}(G,\Gamma)) \longrightarrow C^*(\mathcal{S}\mathcal{N}_\Gamma^G) \longrightarrow 0.$$

Also

$$0 \longrightarrow C^*(G_{\mathring{M}}^{\mathring{M}} \times \mathbb{R}_+^*) \longrightarrow C^*(D N\widetilde{C_+(G},\Gamma)) \longrightarrow C^*(\mathring{\mathcal{N}}_\Gamma^G) \longrightarrow 0.$$

Connecting KK-elements

(Γ amenable)

$$0 \longrightarrow C^*(G \times \mathbb{R}_+^*) \longrightarrow C^*(DNC_+(G,\Gamma)) \longrightarrow C^*(\mathcal{N}_\Gamma^G) \longrightarrow 0 \quad \partial_{DNC_+}$$

$$0 \longrightarrow C^*(G_{\mathring{M}}^{\mathring{M}} \times \mathbb{R}_+^*) \longrightarrow C^*(D\widetilde{NC_+(G},\Gamma)) \longrightarrow C^*(\mathring{\mathcal{N}}_\Gamma^G) \longrightarrow 0 \quad \partial_{\widetilde{DNC_+}}$$

$$0 \longrightarrow C^*(G_{\mathring{M}}^{\mathring{M}}) \longrightarrow C^*(SBlup_{r,s}(G,\Gamma)) \longrightarrow C^*(\mathcal{S}\mathcal{N}_\Gamma^G) \longrightarrow 0 \quad \partial_{SBlup}$$

Connecting elements:
$\partial_{DNC_+} \in KK^1(C^*(\mathcal{N}_\Gamma^G), C^*(G \times \mathbb{R}_+^*))$,
$\partial_{\widetilde{DNC_+}} \in KK^1(C^*(\mathring{\mathcal{N}}_\Gamma^G), C^*(G_{\mathring{M}}^{\mathring{M}} \times \mathbb{R}_+^*))$ and
$\partial_{SBlup} \in KK^1(C^*(\mathcal{S}\mathcal{N}_\Gamma^G), C^*(G_{\mathring{M}}^{\mathring{M}}))$.

Connecting KK-elements

$$0 \longrightarrow C^*(G \times \mathbb{R}_+^*) \longrightarrow C^*(DNC_+(G,\Gamma)) \longrightarrow C^*(\mathcal{N}_\Gamma^G) \longrightarrow 0 \quad \partial_{DNC_+}$$

$$\begin{array}{ccccccccc}
0 \longrightarrow & C^*(G_{\mathring{M}}^{\mathring{M}} \times \mathbb{R}_+^*) & \longrightarrow & C^*(\widetilde{DNC_+}(G,\Gamma)) & \longrightarrow & C^*(\mathring{\mathcal{N}}_\Gamma^G) & \longrightarrow 0 & \partial_{\widetilde{DNC_+}} \\
 & \mathring{\beta} \Big| & & \beta \Big| & & \beta^\partial \Big| & & \\
0 \longrightarrow & C^*(G_{\mathring{M}}^{\mathring{M}}) & \longrightarrow & C^*(SBlup_{r,s}(G,\Gamma)) & \longrightarrow & C^*(\mathcal{S}\mathcal{N}_\Gamma^G) & \longrightarrow 0 & \partial_{SBlup}
\end{array}$$

KK-equivalences: Connes-Thom elements β.

Connecting KK-elements

$$
\begin{array}{ccccccccc}
0 \longrightarrow & C^*(G\times\mathbb{R}_+^*) & \longrightarrow & C^*(DNC_+(G,\Gamma)) & \longrightarrow & C^*(\mathcal{N}_\Gamma^G) & \longrightarrow 0 & \partial_{DNC_+} \\
& \uparrow \mathring{j} & & \uparrow j & & \uparrow j^\partial & & \\
0 \longrightarrow & C^*(G_{\mathring{M}}^{\mathring{M}}\times\mathbb{R}_+^*) & \longrightarrow & C^*(\widetilde{DNC_+}(G,\Gamma)) & \longrightarrow & C^*(\mathring{\mathcal{N}}_\Gamma^G) & \longrightarrow 0 & \partial_{\widetilde{DNC_+}} \\
& \mid \mathring{\beta} & & \mid \beta & & \mid \beta^\partial & & \\
0 \longrightarrow & C^*(G_{\mathring{M}}^{\mathring{M}}) & \longrightarrow & C^*(SBlup_{r,s}(G,\Gamma)) & \longrightarrow & C^*(\mathcal{S}\mathcal{N}_\Gamma^G) & \longrightarrow 0 & \partial_{SBlup}
\end{array}
$$

The j's coming from inclusion.

Connecting KK-elements

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C^*(G \times \mathbb{R}_+^*) & \longrightarrow & C^*(DNC_+(G,\Gamma)) & \longrightarrow & C^*(\mathcal{N}_\Gamma^G) & \longrightarrow 0 & \partial_{DNC_+} \\
 & & \uparrow \mathring{j} & & \uparrow j & & \uparrow j^\partial & & \\
0 & \longrightarrow & C^*(G_{\mathring{M}}^{\mathring{M}} \times \mathbb{R}_+^*) & \longrightarrow & C^*(\widetilde{DNC_+}(G,\Gamma)) & \longrightarrow & C^*(\mathring{\mathcal{N}}_\Gamma^G) & \longrightarrow 0 & \partial_{\widetilde{DNC}_+} \\
 & & \Big| \mathring{\beta} & & \Big| \beta & & \Big| \beta^\partial & & \\
0 & \longrightarrow & C^*(G_{\mathring{M}}^{\mathring{M}}) & \longrightarrow & C^*(SBlup_{r,s}(G,\Gamma)) & \longrightarrow & C^*(\mathcal{S}\mathcal{N}_\Gamma^G) & \longrightarrow 0 & \partial_{SBlup}
\end{array}
$$

Proposition

1. $\partial_{\widetilde{DNC}_+} \otimes [\mathring{j}] = [j^\partial] \otimes \partial_{DNC_+} \in KK(C^*(\mathring{\mathcal{N}}_\Gamma^G), C^*(G))$.
2. $\partial_{SBlup} \otimes \mathring{\beta} = \pm \beta^\partial \otimes \partial_{\widetilde{DNC}_+} \in KK^1(C^*(\mathcal{S}\mathcal{N}_\Gamma^G), C^*(\mathring{G}))$.

Connecting KK-elements

$$
\begin{array}{ccccccccc}
0 \longrightarrow & C^*(G\times\mathbb{R}_+^*) & \longrightarrow & C^*(DNC_+(G,\Gamma)) & \longrightarrow & C^*(\mathcal{N}_\Gamma^G) & \longrightarrow 0 & \partial_{DNC_+} \\
 & \uparrow{\scriptstyle \mathring{j}} & & \uparrow{\scriptstyle j} & & \uparrow{\scriptstyle j^\partial} & & \\
0 \longrightarrow & C^*(G_{\mathring{M}}^{\mathring{M}}\times\mathbb{R}_+^*) & \longrightarrow & C^*(\widetilde{DNC_+}(G,\Gamma)) & \longrightarrow & C^*(\mathring{\mathcal{N}}_\Gamma^G) & \longrightarrow 0 & \partial_{\widetilde{DNC_+}} \\
 & \Big|{\scriptstyle \mathring{\beta}} & & \Big|{\scriptstyle \beta} & & \Big|{\scriptstyle \beta^\partial} & & \\
0 \longrightarrow & C^*(G_{\mathring{M}}^{\mathring{M}}) & \longrightarrow & C^*(SBlup_{r,s}(G,\Gamma)) & \longrightarrow & C^*(\mathcal{SN}_\Gamma^G) & \longrightarrow 0 & \partial_{SBlup}
\end{array}
$$

Proposition

$\partial_{SBlup}\otimes\mathring{\beta}\otimes[\mathring{j}] = \pm\beta^\partial\otimes[j^\partial]\otimes\partial_{DNC_+}\in KK^1(C^*(\mathcal{SN}_\Gamma^G), C^*(G))$.

Proposition

If $\mathring{M}$ meets all the G-orbits, $\mathring{j}$ is a Morita equivalence - and therefore ∂_{DNC_+} determines ∂_{SBlup}.

Connecting KK-elements

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & C^*(G \times \mathbb{R}_+^*) & \longrightarrow & C^*(DNC_+(G,\Gamma)) & \longrightarrow & C^*(\mathcal{N}_\Gamma^G) & \longrightarrow & 0 & & \partial_{DNC_+} \\
 & & \mathring{j} \big\uparrow & & j \big\uparrow & & j^\partial \big\uparrow & & & & \\
0 & \longrightarrow & C^*(G_{\mathring{M}}^{\mathring{M}} \times \mathbb{R}_+^*) & \longrightarrow & C^*(\widetilde{DNC_+}(G,\Gamma)) & \longrightarrow & C^*(\mathring{\mathcal{N}}_\Gamma^G) & \longrightarrow & 0 & & \partial_{\widetilde{DNC_+}} \\
 & & \mathring{\beta} \big| & & \beta \big| & & \beta^\partial \big| & & & & \\
0 & \longrightarrow & C^*(G_{\mathring{M}}^{\mathring{M}}) & \longrightarrow & C^*(SBlup_{r,s}(G,\Gamma)) & \longrightarrow & C^*(\mathcal{SN}_\Gamma^G) & \longrightarrow & 0 & & \partial_{SBlup}
\end{array}
$$

Proposition

$\partial_{SBlup} \otimes \mathring{\beta} \otimes [\mathring{j}] = \pm \beta^\partial \otimes [j^\partial] \otimes \partial_{DNC_+} \in KK^1(C^*(\mathcal{SN}_\Gamma^G), C^*(G)).$

Proposition

If $\mathfrak{A}_x \to (N_V^M)_x$ is nonzero for every $x \in V$, then $\mathring{j}, j, j^\partial$ are isomorphisms. ♣

Full symbol extension → known or new index theorems

Diagrams with full index instead of connecting maps

$$
\begin{array}{ccccccccc}
0 \longrightarrow & C^*(G\times\mathbb{R}_+^*) & \longrightarrow & \Psi^*(DNC_+(G,\Gamma)) & \longrightarrow & \Sigma_{DNC_+} & \longrightarrow 0 & & \widetilde{\mathrm{Ind}}_{DNC_+}\\
 & \uparrow{\scriptstyle \mathring{j}} & & \uparrow{\scriptstyle j^{\Psi}} & & \uparrow{\scriptstyle j^{\Sigma}} & & & \\
0 \longrightarrow & C^*(G_{\mathring{M}}^{\mathring{M}}\times\mathbb{R}_+^*) & \longrightarrow & \Psi^*(D\widetilde{NC_+(G},\Gamma)) & \longrightarrow & \Sigma_{\widetilde{DNC_+}} & \longrightarrow 0 & & \widetilde{\mathrm{Ind}}_{\widetilde{DNC_+}}\\
 & \Big|{\scriptstyle \mathring{\beta}} & & \Big|{\scriptstyle \beta^{\Psi}} & & \Big|{\scriptstyle \beta^{\Sigma}} & & & \\
0 \longrightarrow & C^*(G_{\mathring{M}}^{\mathring{M}}) & \longrightarrow & \Psi^*(SBlup_{r,s}(G,\Gamma)) & \longrightarrow & \Sigma_{SBlup} & \longrightarrow 0 & & \widetilde{\mathrm{Ind}}_{SBlup}
\end{array}
$$

Proposition

$$\widetilde{\mathrm{Ind}}_{SBlup}\otimes\mathring{\beta}\otimes[\mathring{j}] = \pm\beta^{\Sigma}\otimes[j^{\partial}]\otimes\widetilde{\mathrm{Ind}}_{DNC_+}\in KK^1(C^*(\Sigma_{SBlup}),C^*(G)).$$

Proposition

- The β's are K-equivalences.
- If $\mathfrak{A}_x\to(N_V^M)_x$ is nonzero for every $x\in V$, the j's are K-equivalences.

When $\Gamma = V$

In that case... Thom isomorphism, the index map of the groupoid $\mathcal{N}_V^G$ is invertible in $KK(C_0((N_V^G)^*), C^*(\mathcal{N}_V^G))$.

Naturality of the index

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_0(\mathfrak{A}^*G \times \mathbb{R}_+^*) & \longrightarrow & C_0(\mathfrak{A}^*(DNC_+(G,\Gamma))) & \longrightarrow & C_0((N_V^G)^*) & \longrightarrow & 0 \\ & & \downarrow \mathrm{Ind} & & \downarrow \mathrm{Ind} & & \simeq \downarrow \mathrm{Ind} & & \\ 0 & \longrightarrow & C^*(G \times \mathbb{R}_+^*) & \longrightarrow & C^*(DNC_+(G,\Gamma)) & \longrightarrow & C^*(\mathcal{N}_\Gamma^G) & \longrightarrow & 0 \end{array}$$

Computation of ∂ - and in a similar way - computation of the index maps.

Boutet de Monvel type constructions

We now on assume that V is transverse to G.

By transversality, the groupoid G_V^V is a submanifold.

(Sub)-Morita equivalence of groupoids

$$SBlup_{r,s}(G_V^V \times (\mathbb{R}\times\mathbb{R}), V) \underset{\subset}{\sim} SBlup_{r,s}(G, V).$$

(equivalence if V meets all the G-orbits).

The groupoid

$$SBlup_{r,s}(G_{M\sqcup(V\times\mathbb{R})}^{M\sqcup(V\times\mathbb{R})}, V\times(\{0,1\}\times\{0,1\}))$$

is written as the union of

- the subgroupoids $SBlup_{r,s}(G, V)$ and $SBlup_{r,s}(G_V^V \times (\mathbb{R}\times\mathbb{R}), V)$
- the linking spaces $SBlup_{r,s}(G_V \times \mathbb{R}, V)$ and $SBlup_{r,s}(G^V \times \mathbb{R}, V)$

The gauge adiabatic groupoid

$SBlup_{r,s}(G_V^V \times (\mathbb{R} \times \mathbb{R}), V) \rightrightarrows SBlup(V \times \mathbb{R}, V) \simeq V \times (\mathbb{R}_+ \sqcup \mathbb{R}_-)$, restricted to $V \times \mathbb{R}_+$, is the "gauge adiabatic groupoid" $(G_V^V)_{ga}$ of G_V^V:

Recall:

- Connes tangent (or adiabatic) groupoid of H is $DNC(H, H^{(0)})$
- The gauge adiabatic groupoid H_{ga} is $DNC_+(H, H^{(0)}) \rtimes \mathbb{R}_+^*$.

Exact sequences of C^*-algebras

$$0 \to C^*(H) \otimes C_0(\mathbb{R}_+^*) \to C_+^*(DNC(H, H^{(0)})) \to C_0(\mathfrak{A}^* H) \to 0$$

taking crossed product by $\mathbb{R}_+^*$:

$$0 \to C^*(H) \otimes \mathcal{K} \to C^*(H_{ga}) \to C_0(\mathfrak{A}^* H) \rtimes \mathbb{R}_+^* \to 0$$

Compare with the pseudodifferential exact sequence

$$0 \to C^*(H) \to \Psi^*(H) \to C(\mathbb{S}^* \mathfrak{A} H) \to 0.$$

The PT bimodule

[Debord S. (2014)] $C^*(SBlup_{r,s}(H_{ga}, H^{(0)})) - \Psi^*(H)$-bimodule $\mathscr{E}_H$ relating the exact sequences

$$0 \to C^*(H) \otimes \mathcal{K} \to C^*(SBlup_{r,s}(H \times (\mathbb{R} \times \mathbb{R}), H^{(0)})) \to C_0(\mathfrak{A}^* H) \rtimes \mathbb{R}_+^* \to 0$$

and

$$0 \to C^*(H) \to \Psi^*(H) \to C(\mathbb{S}^* \mathfrak{A} H) \to 0.$$

Putting together with the (sub)-Morita equivalence, we find a Poisson-trace

bimodule $\mathscr{E}_{PT}(G, V)$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^*(\mathring{G}) & \longrightarrow & C^*(SBlup_{r,s}(G,V)) & \longrightarrow & C^*(\mathcal{S}(\mathcal{N}_V^G)) & \longrightarrow & 0 \\ & & \Big| \mathscr{E}'_{PT}(G,V) & & \Big| \mathscr{E}_{PT}(G,V) & & \Big| \mathscr{E}''_{PT}(G,V) & & \\ 0 & \longrightarrow & C^*(G_V^V) & \longrightarrow & \Psi^*(G_V^V) & \longrightarrow & C(\mathbb{S}^* \mathfrak{A} G_V^V) & \longrightarrow & 0 \end{array}$$

(where $\mathring{M} = M \setminus V$ and $\mathring{G} = G$ restricted to $\mathring{M}$).

Boutet de Monvel type constructions

The C^*-algebra $C^*_{BM}(G,V) = \mathcal{K}\Big(C^*(SBlup_{r,s}(G,V)) \oplus \mathscr{E}_{PT}(G,V)^*\Big)$ algebra of matrices of the form $\begin{pmatrix} K & P \\ T & Q \end{pmatrix}$ where

$K \in C^*(SBlup_{r,s}(G,V)),\ P \in \mathscr{E}_{PT}(G,V),\ T \in \mathscr{E}_{PT}(G,V)^*,\ Q \in \Psi^*(G_V^V)$.

Exact sequence $0 \to C^*(G_{\mathring{M} \coprod V}^{\mathring{M} \coprod V}) \to C^*_{BM}(G,V) \xrightarrow{r_V^{C^*}} \Sigma^{C^*}_{bound}(G,V) \to 0$, where $\Sigma^{C^*}_{bound}(G,V)$ = algebra of *Boutet de Monvel type boundary symbols:*

Matrices $\begin{pmatrix} k & p \\ t & q \end{pmatrix}$ where $k \in C^*(\mathcal{S}N_V^G)$, $q \in C(\mathbb{S}^*\mathfrak{A}G_V^V)$, $p, t^* \in \mathscr{E}''_{PT}$.

$r_V^{C^*}$: *zero order symbol map of the Boutet de Monvel type calculus.*

$$r_V^{C^*}\begin{pmatrix} K & P \\ T & Q \end{pmatrix} = \begin{pmatrix} r_V^{\asymp}(K) & r_V^{\propto}(P) \\ r_V^{\backpropto}(T) & \sigma_V(Q) \end{pmatrix}$$

- σ_V: ordinary order 0 principal symbol on the groupoid G_V^V;
- $r_V^{\asymp}, r_V^{\propto}, r_V^{\backpropto}$: restrictions to the boundary.

A Boutet de Monvel type pseudodifferential algebra

$\Psi^*_{BM}(G,V)$: algebra of matrices $R = \begin{pmatrix} \Phi & P \\ T & Q \end{pmatrix}$ with

$\Phi \in \Psi^*(SBlup_{r,s}(G,V)),\ P \in \mathscr{E}_{PT}(G,V),\ T \in \mathscr{E}_{PT}(G,V)^*$ and $Q \in \Psi^*(G_V^V)$.

Two symbols:

- *classical symbol* $\sigma_c(R) = \sigma_0(\Phi)$;
- *boundary symbol* $r_V^{BM} : \Psi^*_{BM}(G,V) \to \Sigma^{\Psi^*}_{bound}(G,V)$ defined by

$$r_V \begin{pmatrix} \Phi & P \\ T & Q \end{pmatrix} = \begin{pmatrix} r_V^{\psi}(\Phi) & r_V^{\propto}(P) \\ r_V^{\propto}(T) & \sigma_V(Q) \end{pmatrix}$$

where $r_V^{\psi} : \Psi^*(SBlup_{r,s}(G,V)) \to \Psi^*(\mathcal{S}N_V^G)$ is the restriction.

Computation of all kinds of connecting maps and index maps...

Work in progress...just posted ! (joint with Claire Debord).

Thank you!