

A Baum-Connes conjecture for singular foliations

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I.A. and Georges Skandalis / [arXiv:1509.05862](https://arxiv.org/abs/1509.05862)

Toulouse, June 2016

Summary

- 1 Formulation of BC using DNC
- 2 A hierarchy of singular foliations using dimension
- 3 Calculation of K-theory for action of $SO(3)$ on \mathbb{R}^3
- 4 Height 1 foliations: Mapping cone and Lhs
- 5 BC theorem
- 6 Appendix: Assembly map via double DNC

Assembly map as analytic index

$\mathcal{G} \rightrightarrows M$ acting on $p : E \rightarrow M$: Pullback \mathcal{G}_p^p Morita equivalent to \mathcal{G} .

Have $\mathcal{G} \ltimes E \hookrightarrow \mathcal{G}_p^p$. Consider $\text{DNC}(\mathcal{G}_p^p, \mathcal{G} \ltimes E)$ and get:

$$\mu_E = [\text{ev}_0]^{-1} \otimes [\text{ev}_1] \otimes \mathcal{E} \quad : \quad K_0(C_0(\ker dp^*) \rtimes \mathcal{G}) \rightarrow K_0(C^*(\mathcal{G}))$$

- ▶ Proper action \Rightarrow amenable
- ▶ $\mathcal{G} \curvearrowright \ker dp$ proper $\Rightarrow C_0(\ker dp^*) \rtimes \mathcal{G}$ type-1 C^* -algebra \Rightarrow **nuclear**
- ▶ If $E = \underline{E\mathcal{G}}$ then $\mu_{\underline{E\mathcal{G}}}$ is **Baum-Connes assembly map**

Singular foliations (Stefan-Sussmann)

Definition - Proposition

Singular foliation (M, \mathcal{F}) is a $C^\infty(M)$ -submodule of $\mathcal{X}_c(M)$ such that:

- 1 \mathcal{F} locally finitely generated;
- 2 $[\mathcal{F}, \mathcal{F}] \subseteq \mathcal{F}$.

- ▶ **Stefan-Sussmann thm:** \mathcal{F} integrates to immersed submanifolds.
- ▶ Dimension of leaves may jump!

Example: Lie group G acting on M . Module \mathcal{F} generated by image of infinitesimal action $\text{Lie}(G) \rightarrow \mathfrak{X}(M)$.

A., Skandalis

- ▶ Every (M, \mathcal{F}) has a holonomy groupoid $H(\mathcal{F}) \rightrightarrows M$.
- ▶ $H(\mathcal{F})$ topological gpd (very bad!). Each $H(\mathcal{F})_x$ smooth, $x \in M$.
- ▶ There are always $C^*(\mathcal{F})$, $C^*_r(\mathcal{F})$.

Careful look at action $SO(3) \curvearrowright \mathbb{R}^3$ (I)

$\dim(\text{Lie}(SO(3))) = 3$, so $\mathcal{F} = \text{span}_{C^\infty(M)} \langle X, Y, Z \rangle$.

Take any (M, \mathcal{F}) . At $x \in M$ put $\mathcal{F}_x = \mathcal{F}/I_x\mathcal{F}$. Get exact sequence

$$0 \rightarrow \mathfrak{g}_x \rightarrow \mathcal{F}_x \xrightarrow{\text{ev}_x} T_x L_x \rightarrow 0$$

- ▶ L_x regular $\Rightarrow \mathcal{F}_x = T_x L_x$
- ▶ L_x singular $\Rightarrow \dim(\mathcal{F}_x) > \dim(L_x)$.
- ▶ $\dim(\mathcal{F}_x)$ (upper) **semicontinuous**

For $(\mathbb{R}^3, \mathcal{F})$ we have:

- ▶ $\mathcal{F}_0 = \mathfrak{g}_x = \text{Lie}(SO(3))$, so $\dim(\mathcal{F}_0) = 3$
- ▶ For $x \neq 0$, $\dim(\mathcal{F}_x) = 2$

$$H(\mathcal{F}) = (S^2 \times S^2 \times \mathbb{R}_*^+) \cup SO(3) \times \{0\}$$

Careful look at action $SO(3) \curvearrowright \mathbb{R}^3$ (II)

$H(\mathcal{F}) = (S^2 \times S^2 \times \mathbb{R}_*^+) \cup SO(3) \times \{0\}$ decomposes \mathbb{R}^3 :

- ▶ $\Omega_1 = \{x \in \mathbb{R}^3 : \dim(\mathcal{F}_x) \leq 3\} = \mathbb{R}^3$
- ▶ $\Omega_0 = \{x \in \mathbb{R}^3 : \dim(\mathcal{F}_x) \leq 2\} = \mathbb{R}^3 \setminus \{0\}$

Generalize to arbitrary (M, \mathcal{F}) :

- ▶ $\dim(\mathcal{F}_x)$ upper semicontinuous $\Rightarrow \Omega_i = \{x \in M : \dim(\mathcal{F}_x) \leq i\}$ open
- ▶ Also, $Y_i = \Omega_i \setminus \Omega_{i-1}$ closed and saturated.

Definition

1 **Decomposition sequence** of (M, \mathcal{F}) :

$$\Omega_0 \subseteq \Omega_1 \subseteq \dots \subseteq \Omega_{k-1} \subseteq \Omega_k = M$$

2 We say that (M, \mathcal{F}) has **height k** . ($k = +\infty$ allowed and possible!)

Careful look at action $SO(3) \curvearrowright \mathbb{R}^3$ (II)

So foliation $(\mathbb{R}^3, \mathcal{F})$ has **height** $\mathbf{k} = 1$:

$$\Omega_0 = \mathbb{R}^3 \setminus \{0\}, \quad \Omega_1 = \mathbb{R}^3, \quad Y_0 = \Omega_0, \quad Y_1 = \{0\}.$$

- ▶ $C^*(M, \mathcal{F})|_{\Omega_0} = C_0(\Omega_0) \cdot C^*(M, \mathcal{F}) = C_0(\mathbb{R}_*^+) \otimes \mathcal{K}(L^2(S^2))$
- ▶ $C^*(M, \mathcal{F})|_{Y_1} = C^*(M, \mathcal{F})/C^*(M, \mathcal{F})|_{\mathbb{R}^2 \setminus Y_1} = C^*(SO(3))$

Exact sequence of (full) C^* -algebras:

$$0 \longrightarrow C_0(\mathbb{R}_*^+) \otimes \mathcal{K}(L^2(S^2)) \longrightarrow C^*(M, \mathcal{F}) \xrightarrow{\pi_{\mathcal{F}}} C^*(SO(3)) \longrightarrow 0$$

Action groupoid $\mathcal{G} = \mathbb{R}^2 \rtimes SO(3) \rightrightarrows \mathbb{R}^3$:

- ▶ $\mathcal{G}|_{Y_1} = H(\mathcal{F})|_{Y_1} = SO(3) \times \{0\}$
- ▶ Exact sequence:

$$0 \longrightarrow C_0(\mathbb{R}_*^+) \otimes (C(S^2) \rtimes SO(3)) \longrightarrow C_0(\mathbb{R}^3) \rtimes SO(3) \longrightarrow C^*(SO(3)) \longrightarrow 0$$

Nicely decomposable foliations

Definition

Let (M, \mathcal{F}) singular foliation, decomposition sequence

$$\Omega_0 \subseteq \Omega_1 \subseteq \dots \subseteq \Omega_j \dots \subseteq M$$

Put $Y_0 = \Omega_0$ $Y_j = \Omega_j \setminus \Omega_{j-1}$.

A **nice decomposition** is

1 sequence $(W_j)_{0 \leq j \leq k}$ of open sets such that

$$Y_j \subset W_j \subset \Omega_j \quad W_j \cap \Omega_{j-1} \subset W_{j-1}$$

2 Lie groupoids $\mathcal{G}_j \rightrightarrows W_j$ which define $\mathcal{F}|_{W_j}$ and

$$\mathcal{G}_j|_{Y_j} = \mathbf{H}(\mathcal{F})|_{Y_j}$$

3 morphisms $q_j : \mathcal{G}_j|_{\Omega_{j-1} \cap W_j} \rightarrow \mathcal{G}_{j-1}$ (for $j > 0$) which are **submersions**

Examples

- 1 $\mathbb{R} \curvearrowright M$ periodic pts **fixed**: Period bounding lemma \Rightarrow

$$\Omega_0 = \{\text{continuity pts of } \dim(\mathcal{F}_x)\} \quad Y_1 = \partial(\text{Int}(\text{fixed points}))$$

Nice decomposition: $(H(\mathcal{F})|_{\Omega_0} \rightrightarrows \Omega_0, M \times \mathbb{R} \rightrightarrows M)$

- 2 $\mathbb{R} \curvearrowright M$ stably periodic pts: Bochner's linearization thm \Rightarrow

Put $\widehat{\text{Per}} = \text{Per} \setminus \text{Int}(\text{fixed points})$. Then:

$$H(\mathcal{F}) = \text{Int}(\text{fixed points}) \coprod \widehat{\text{Per}} \times S^1 \coprod (M \setminus \text{Per}) \times \mathbb{R}$$

$SO(3) \curvearrowright \mathbb{R}^3$: calculation (I)

$SO(3)$ compact, whence amenable. So $C^*(\mathcal{F}) = C_r^*(\mathcal{F})$.

$$\pi : \mathbb{R}^3 \rtimes SO(3) \rightarrow H(\mathcal{F}) = (S^2 \times S^2 \times \mathbb{R}_*^+) \cup SO(3) \times \{0\}$$

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & J & \xlongequal{\quad} & J & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & C_0(\mathbb{R}_*^+) \otimes (C(S^2) \rtimes SO(3)) & \xrightarrow{i} & C_0(\mathbb{R}^3) \rtimes SO(3) & \longrightarrow & C^*(SO(3)) \longrightarrow 0 & \text{(ES4)} \\
 & & \downarrow \hat{q} & & \downarrow \pi & & \parallel & \\
 0 & \longrightarrow & C_0(\mathbb{R}_*^+) \otimes \mathcal{K}(L^2(S^2)) & \longrightarrow & C_r^*(\mathbb{R}^3, \mathcal{F}) & \longrightarrow & C^*(SO(3)) \longrightarrow 0 & \text{(ES5)} \\
 & & \downarrow & & \downarrow & & & \\
 & & 0 & & 0 & & &
 \end{array}$$

where q : integration along fibers of $(s, t) : S^2 \rtimes SO(3) \rightarrow S^2 \times S^2$.

Height 1 foliations

Proposition

Given a diagram of exact sequences of C^* -algebras and morphisms:

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & I & \xrightarrow{i} & B_1 & \longrightarrow & Q & \longrightarrow & 0 \\
 & & \downarrow \pi & & \downarrow & & \parallel & & \\
 0 & \longrightarrow & B_0 & \xrightarrow{i'} & A & \longrightarrow & Q & \longrightarrow & 0
 \end{array}$$

the mapping cone $\mathcal{C}_{(\pi, i)}$ of the map $(\pi, i) : I \rightarrow B_0 \oplus B_1$ is canonically E^1 -equivalent to A (KK-equivalent).

Conclusion: Need to formulate the Baum-Connes conjecture for mapping cones!

$SO(3) \curvearrowright \mathbb{R}^3$: calculation (II)

$\rho : C^*(SO(3)) \rightarrow \mathcal{K}(L^2(S^2))$ natural reprn of $SO(3)$ on $L^2(S^2)$.

$j : C^*(SO(3)) \rightarrow C(S^2) \rtimes SO(3)$ induced by unital inclusion $\mathbb{C} \rightarrow C(S^2)$.

$$\begin{array}{ccc}
 C^*(SO(3)) & \xrightarrow{j} & C(S^2) \rtimes SO(3) \\
 & \searrow \rho & \downarrow q \\
 & & \mathcal{K}(L^2(S^2))
 \end{array}$$

$SO(3) \curvearrowright \mathbb{R}^3$: calculation (III)

$C_0(\mathbb{R}^3) =$ mapping cone of $\mathbb{C} \rightarrow C(S^2)$. Taking crossed products by the action of $SO(3)$ and using the first diagram, we find:

- ▶ $C_0(\mathbb{R}^3) \rtimes SO(3)$ in (EC5) is mapping cone \mathcal{C}_j , where

$$j : C^*(SO(3)) \rightarrow C(S^2) \rtimes SO(3)$$

- ▶ Foliation algebra $C^*(\mathcal{F})$ in (EC6) is mapping cone \mathcal{C}_ρ .

$SO(3) \curvearrowright \mathbb{R}^3$: calculation (IV)

To describe $C^*(\mathcal{F})$ it suffices to describe the representation

$$\rho : C^*(SO(3)) \rightarrow \mathcal{K}(L^2(S^2)).$$

- ▶ Peter-Weyl: $C^*(SO(3)) = \bigoplus_{m \in \mathbb{N}} M_{2m+1}(\mathbb{C})$ and $K_0(C^*(SO(3))) = \mathbb{Z}^{(\mathbb{N})}$ (and $K_1(C^*(SO(3))) = \{0\}$).
- ▶ In order to compute the map $\rho_* : K_0(C^*(SO(3))) \rightarrow \mathbb{Z}$, we have to understand how many times the repn σ_m ($\dim(\sigma_m) = 2m + 1$) appears in ρ , *i.e.* count dimension of $\text{Hom}_{SO(3)}(\sigma_m, \rho)$.
- ▶ Since $S^2 = SO(3)/S^1$, $\rho = \text{Ind}_{S^1}^{SO(3)}(\varepsilon)$ where ε **trivial repn** of S^1 .
- ▶ Frobenius reciprocity thm:
 $\dim(\text{Hom}_{SO(3)}(\sigma_m, \rho)) = \dim(\text{Hom}_{S^1}(\sigma_m, \varepsilon)) = 1$.
- ▶ So $\rho_* : K_0(C^*(SO(3))) \rightarrow \mathbb{Z}$ maps each generator $[\sigma_m]$ of $K_0(C^*(SO(3)))$ to 1.

$$K_0(C^*(\mathcal{F})) = \ker \rho_* \simeq \mathbb{Z}^{(\mathbb{N})} \quad K_1(C^*(\mathcal{F})) = 0$$

Other height 1 foliations:

- ▶ $GL(n) \curvearrowright \mathbb{R}^n$:

$$H(\mathcal{F}) = (\mathbb{R}^2 \setminus \{0\}) \times (\mathbb{R}^2 \setminus \{0\}) \coprod \{0\} \times GL(2, \mathbb{R})^+$$

- ▶ Actions of $SL(2, \mathbb{R})$, $SL(2, \mathbb{C})$, etc:

$$H(\mathcal{F}) = (\mathbb{R}^2 \setminus \{0\}) \times (\mathbb{R}^2 \setminus \{0\}) \coprod \{0\} \times SL(2, \mathbb{R})$$

\mathcal{G} action groupoid

Height $k > 1$ foliations

Proposition

All this extends to foliations (M, \mathcal{F}) of **any** height: The foliation C^* -algebra is “K”-equivalent (E-equivalent) to a mapping **telescope**.

Examples of higher height arise looking at flag manifolds... For instance:

- 1
 - G minimal parabolic subgp of $GL(n, \mathbb{R})$ (upper triangular matrices).
 - $G \times G \hookrightarrow M(n, \mathbb{R})$ by left and right multiplication.
 - Orbits labeled by symmetric group S_n (**Bruhat decomposition**)

- 2 $PG = G/\mathbb{R}^* \hookrightarrow \mathbb{R}P^{n-1} \dots$

Abstract LHS for height 1 foliations (I)

Step 0: Construct equivariant KK-theory for **longitudinally smooth groupoids**, e.g. $H(\mathcal{F})$. (With Kasparov product and descent map.)

Step 1: Need to construct the left-hand side K^{top} for the mapping cone.

Let $G_i \rightrightarrows G_i^{(0)}$ Hausdorff Lie groupoids, $i = 1, 2$.

- ▶ Inclusion $i : G_0 \rightarrow G_1$ easy...
- ▶ **Difficulty**: $\pi : G_0 \rightarrow G_1$ **surjective submersion** and $G_0^{(0)} \subseteq G_1^{(0)}$.
- ▶ Strategy: First, we identify π with a KK-element \hat{q}_1 .

$$\text{Lhs} = \mathcal{C}_{\hat{q}_1}$$

Step 2: Explicit constructions of $\mathcal{C}_{\hat{q}_1}$ and **assembly map**:

Double deformation to normal cone...

Abstract LHS for height 1 foliations (II)

Let $M_0 = \underline{EG}_0$ and $M_1 = \underline{EG}_1$. Assume they are smooth manifolds and $p_i : M_i \rightarrow G_i^0$ are K -oriented, $i = 0, 1$. Dimensions of fibers even.

- ▶ $G_0 \curvearrowright M_0 \times_{p_0, t_1} G_1$ properly.
 - quotient $M_0 \times_{G_0} G_1$ is Hausdorff + locally compact.
 - map $M_0 \rightarrow M_0 \times_{p_0, t_1} G_1$, $x \mapsto (x, \pi(p_0(x)))$ continuous so defines $M_0 \rightarrow M_0 \times_{G_0} G_1$.

- ▶ $G_1 \curvearrowright M_0 \times_{G_0} G_1$ properly.
 - M_1 universal, so get $M_0 \times_{G_0} G_1 \rightarrow M_1$, G_1 -equivariant.
 - Composition: $q : M_0 \rightarrow M_1$ G_0 -equivariant.
 - $p_1 \circ q = p_0$, so get morphism of proper groupoids

$$q : M_0 \rtimes G_0 \rightarrow M_1 \rtimes G_1, \quad q(x, \gamma) = (q(x), \pi(\gamma))$$

Abstract LHS for height 1 foliations (III)

q is K -oriented, so induces elt $q! \in \text{KK}_{G_0}(C_0(M_0), C_0(M_1))$.

- LeGall's descent map j_{G_0} gives

$$\hat{q}! = \tilde{\pi}_*(j_{G_0}(q!)) \in \text{KK}(C_0(M_0) \rtimes G_0, C_0(M_1) \rtimes G_1)$$

where $\tilde{\pi}$ the morphism $C_0(M_1) \rtimes G_0 \rightarrow C_0(M_1) \rtimes G_1$ induced by π .

Proposition

Morphism $\pi : C^*(G_0) \rightarrow C^*(G_1)$ corresponds to element $\hat{q}!$. More precisely,

$$\pi_*((\widehat{p_0})!) = \hat{q}! \otimes (\widehat{p_1})!$$

Abstract LHS for height 1 foliations (IV)

Now we need to construct the **cone** of $\hat{q}_!$. Abstractly:

Theorem (V. Lafforgue, G. Skandalis)

A KK-element $x \in \text{KK}(A, B)$ can be given as a composition

$$x = [f]^{-1} \otimes [g] \quad (\spadesuit)$$

of a morphism $g : D \rightarrow B$ with an element $[f]^{-1}$ which is the KK-inverse of a morphism $f : D \rightarrow A$ which is invertible in KK-theory

Define (up to KK-equivalence) the cone of x as being the cone of g :

Lemma

Given an element $x \in \text{KK}(A, B)$, the mapping cone C_g does not depend on the decomposition (\spadesuit) up to KK-equivalence.

Abstract LHS for height 1 foliations (V)

Both $\mathcal{C}_{\hat{q}_1}$ and **assembly map** can be constructed **explicitly**.

- ▶ Double adiabatic deformation (geometric) construction...
- ▶ Works when M_0, M_1 not smooth manifolds? Take direct limits...
- ▶ Constructions extend to telescopes instead of cones.

BC for singular foliations

Theorem (I.A. and G. Skandalis)

(M, \mathcal{F}) nicely decomposable foliation, groupoids $\mathcal{G}_k \rightrightarrows W_k$ such that:

- ▶ Classifying spaces of $\underline{E\mathcal{G}_k}$ are manifolds
- ▶ The *full* Baum-Connes conjecture holds for all of them,

Then the *full* Baum-Connes map is an isomorphism.

Corollary

Moreover, if the groupoids \mathcal{G}_k are *amenable* then the assembly map is an isomorphism.

Appendix: Double DNC I

Recall $\pi : G_0 \rightarrow G_1$ surjective submersion and $q : M_0 \rightarrow M_1$.

Put $M'_1 = q(M_0)$. Then $G_0 \hookrightarrow M'_1$ by $\gamma \cdot q(x) = q(\gamma x) = \pi(\gamma \cdot q(x))$.

Groupoids: $X = M_0 \rtimes G_0$, $Y = (M'_1 \rtimes G_0)_q^q$, $Z = (G_0)_{p_0}^{p_0}$

Inclusion $X \subset Y \subset Z$. Put

$$H_0 = \text{DNC}^2(X, Y, Z) = \text{DNC}(\text{DNC}(Z, X), \text{DNC}(Y, X))$$

objects: $\text{DNC}(Y, X) \times [0, 1] \times [0, 1]$

Let

- $q' : M_0 \amalg M_1 \rightarrow M_1$, $q' = q \amalg \text{id}$
- $p' : M_0 \amalg M_1 \rightarrow G_1^{(0)}$ $p' = p_1 \circ q'$

Put

$$H_1 = \text{DNC}((G_1)_{p'}^{p'}, (M_1 \rtimes G_1)_{q'}^{q'} \times \mathbb{R}^*)$$

objects: $M_0 \amalg M_1 \times \mathbb{R}^* \times \mathbb{R}$

Appendix: Double DNC II

Functoriality of DNC:

$$\begin{array}{ccc}
 (M'_1 \times G_0)_q^q & \longrightarrow & (G_0)_{p_0}^{p_0} \\
 \downarrow & & \downarrow \\
 (M_1 \times G_1)_q^q & \longrightarrow & (G_1)_{p_0}^{p_0}
 \end{array}$$

gives gpd morphism

$$\pi : (H_0)_{\mathbb{R}^* \times \mathbb{R}} \rightarrow H_1$$

Longitudinally smooth groupoid

$$\mathbb{H} = (H_0)_Q \cup (H_1)_{Z_1}$$

glued by $\pi : (H_0)_{Z_1} \rightarrow (H_1)_{Z_1}$, where

$$Q = \text{upper half triangle of } [0, \frac{1}{2}] \times [0, \frac{1}{2}), \quad Z_0 = [0, 1) \times [0, \frac{1}{2}], \quad Z_1 = Z_0 \setminus Q$$

Appendix: Double DNC III

$$F_0 = [0, 1) \times \{0\} \cup \{0\} \times [0, \frac{1}{2}], \quad F_1 = \{\frac{1}{2}\} \times [\frac{1}{2}, 1)$$

- ▶ Semi-split sequence:

$$C^*(\mathbb{H}_{F_0}) \xleftarrow{\sigma_0} C^*(\mathbb{H}) \xrightarrow{\sigma_1} C^*(\mathbb{H}_Q) \xrightarrow{\mathcal{E}} \mathcal{C}_{\pi^*}$$

where $\pi^* : C^*(G_0) \rightarrow C^*(G_1)$ (integration along fibers)

- ▶ σ_0 is KK-equivalence

Foliated assembly map:

$$\mu = [\sigma_0]^{-1} \otimes [\sigma_1] \otimes \mathcal{E}$$

Appendix: Double DNC IV

- ▶ KK-element of $\pi^* : C^*(G_0) \rightarrow C^*(G_1)$: Semi-split sequence

$$0 \longrightarrow C^*(\mathbb{H}_{Z_1}) \longrightarrow C^*(\mathbb{H}) \longrightarrow C^*(\mathbb{H}_Q) \longrightarrow 0$$

- ▶ Assembly map of $G_1 \times (0, 1)$: $\mu_1 = [\sigma_{1,0}]^{-1} \otimes [\sigma_{1,1}] \otimes \varepsilon_1$

$$C^*(\mathbb{H}_{Z_1 \cap F_0}) \xleftarrow{\sigma_{1,0}} C^*(\mathbb{H}_{Z_1}) \xrightarrow{\sigma_{1,1}} C^*(\mathbb{H}_{Z_1 \cap F_1}) \xrightarrow{\varepsilon_1} C^*(G_1)(0, 1)$$

- ▶ Assembly map of G_0 : $\mu_0 = [\sigma_{0,0}]^{-1} \otimes [\sigma_{0,1}] \otimes \varepsilon_0$

$$C^*(\mathbb{H}_{Q \cap F_0}) \xleftarrow{\sigma_{0,0}} C^*(\mathbb{H}_Q) \xrightarrow{\sigma_{0,1}} C^*(\mathbb{H}_{Q \cap F_1}) \xrightarrow{\varepsilon_1} C^*(G_0)$$