Noncommutative products of Euclidean spaces

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Abstract

- noncommutative generalizations of $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$

- noncommutative generalizations of $\mathbb{S}^{N_1+N_2-1}$ and $\mathbb{S}^{N_1+N_2}$

- noncommutative generalizations of $\mathbb{S}^{N_1-1} \times \mathbb{S}^{N_2-1}$

- a quaternionic noncommutative torus $\mathbb{S}^3 \times_{\mathbf{u}} \mathbb{S}^3$, $\mathbf{u} \in \mathbb{S}^2 = \mathrm{SU}(2)/\,\mathrm{U}(1)$

- spherical manifolds : volume forms from top Chern–Connes characters

- spectral triples

- noncommutative principal bundles

The θ-deformation

$\mathbb{C}^2_\theta$ = Noncommutative space dual to the
$*$-algebra $\mathcal{A}_\theta$ generated by normal elements z_1, z_2 with relations

$$z_1 z_2 = e^{i\theta} z_2 z_1, \qquad z_1 z_2^* = e^{-i\theta} z_2^* z_1 \tag{1}$$

$\Rightarrow$ center generated by $z_1 z_1^* = \|z_1\|^2$ and $z_2 z_2^* = \|z_2\|^2$

Real version $\mathbb{C}^2_\theta = (\mathbb{R}^2)^2_\theta = \mathbb{R}^2 \times_\theta \mathbb{R}^2$

$$z_1 = x_1^1 + \mathrm{i} x_1^2, \quad z_2 = x_2^1 + \mathrm{i} x_2^2, \qquad (x_k^\lambda)^* = x_k^\lambda$$

and $\{(1) +$ normality of $z_k\} \Leftrightarrow (2)$

$$\begin{cases} x_1^\lambda x_1^\mu = x_1^\mu x_1^\lambda \quad ; \qquad x_2^\lambda x_2^\mu = x_2^\mu x_2^\lambda \\ x_1^\lambda x_2^\mu = R^{\lambda\mu}_{\nu\rho} x_2^\nu x_1^\rho \end{cases} \tag{2}$$

with

$$R^{\lambda\mu}_{\nu\rho} = \cos(\theta)\, \delta^\lambda_\rho \delta^\mu_\nu + \mathrm{i}\sin(\theta)\, C^\lambda_\rho D^\mu_\nu \qquad C = -D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

2-torus and 3-sphere

$$\mathbb{R}^2 \times_\theta \mathbb{R}^2/(\|x_1\|^2 - 1, \|x_2\|^2 - 1) = \mathbb{T}^2_\theta$$

$$\mathbb{R}^2 \times_\theta \mathbb{R}^2/(\|x_1\|^2 + \|x_2\|^2 - 1) = \mathbb{S}^3_\theta$$

nice examples of singular spaces

The quadratic $*$-algebra $\mathcal{A}_R$

The $*$-algebra $\mathcal{A}_R$ is generated by two sets of hermitian elements x_1^λ with $\lambda \in \{1,\dots,N_1\}$ and x_2^α with $\alpha \in \{1,\dots,N_2\}$ with relations

$$\begin{cases} x_1^\lambda x_1^\mu = x_1^\mu x_1^\lambda \quad ; \qquad x_2^\alpha x_2^\beta = x_2^\beta x_2^\alpha \\ x_1^\lambda x_2^\alpha = R^{\lambda\alpha}_{\beta\mu} x_2^\beta x_1^\mu \end{cases} \tag{3}$$

for suitable $R^{\lambda\alpha}_{\beta\mu} \in \mathbb{C}$. In view of the hermiticity of the x_1^λ, x_2^α we impose

$$\overline{R}^{\lambda\alpha}_{\beta\mu} R^{\mu\beta}_{\gamma\nu} = \delta^\lambda_\nu \delta^\alpha_\gamma \tag{4}$$

Thus $\mathcal{A}_R$ is a graded quadratic algebra $\mathcal{A}_R = \oplus_n \mathcal{A}_R^n$

which is connected: $\mathcal{A}_R^0 = \mathbb{C}\mathbb{1}$;

the elements x_1^λ, x_2^α form a basis of $\mathcal{A}_R^1$;

by the requirement (4):
the elements $x_1^\lambda x_1^\mu$ with $\lambda \leq \mu$, $x_2^\alpha x_2^\beta$ with $\alpha \leq \beta$ and $x_1^\lambda x_2^\alpha$ form a basis of $\mathcal{A}_R^2$.

The Koszul dual $\mathcal{A}_R^!$

A quadratic algebra is an algebra $\mathcal{A}$ of the form

$$\mathcal{A} = T(E)/(\mathcal{R})$$

where E is a finite-dimensional vector space and $(\mathcal{R})$ denotes the ideal of the tensor algebra generated by $\mathcal{R} \subset E \otimes E$.

The Koszul dual $\mathcal{A}^!$ of $\mathcal{A}$ is the quadratic algebra

$$\mathcal{A}^! = T(E^*)/(\mathcal{R}^\perp)$$

where $\mathcal{R}^\perp \subset E^* \otimes E^*$ is the orthogonal of $\mathcal{R}$.

Our $\mathcal{A}_R^!$ is generated by the dual bases $\theta^1_\lambda, \theta^2_\alpha$ of the x^λ_1, x^α_2 with relations

$$\begin{cases} \theta^1_\lambda \theta^2_\mu = -\theta^2_\mu \theta^1_\lambda \quad ; \qquad \theta^2_\alpha \theta^2_\beta = -\theta^2_\beta \theta^2_\alpha \\ \theta^2_\beta \theta^1_\mu = -R^{\lambda\alpha}_{\beta\mu} \theta^1_\lambda \theta^2_\alpha \end{cases} \tag{5}$$

Yang-Baxter condition

Define x^a for $a \in \{1, \ldots, N_1 + N_2\}$ by $x^\lambda = x_1^\lambda, x^{\alpha+N_1} = x_2^\alpha$.

Then the relations (3) together with $x_2^\alpha x_1^\lambda = \overline{R}^{\lambda\alpha}_{\beta\mu} x_1^\mu x_2^\alpha$ reads

$$x^a x^b = \mathcal{R}^{ab}_{cd} x^c x^d \tag{6}$$

where the matrice $\mathcal{R}$ is involutive, i.e. $\mathcal{R}^2 = \mathbb{1} \otimes \mathbb{1}$ or

$$\mathcal{R}^{ab}_{cd} \mathcal{R}^{cd}_{ef} = \delta^a_e \delta^b_f$$

We next impose the Yang-Baxter condition for $\mathcal{R}$

$$(\mathcal{R} \otimes \mathbb{1})(\mathbb{1} \otimes \mathcal{R})(\mathcal{R} \otimes \mathbb{1})^{abc} = (\mathbb{1} \otimes \mathcal{R})(\mathcal{R} \otimes \mathbb{1})(\mathbb{1} \otimes \mathcal{R})^{abc} \tag{7}$$

This breaks in a series of conditions:

Yang-Baxter condition - cont.d

This is equivalent to

$$\begin{cases} R^{\lambda\alpha}_{\gamma\rho} R^{\rho\beta}_{\delta\mu} = R^{\lambda\beta}_{\delta\rho} R^{\rho\alpha}_{\gamma\mu} & \qquad (\lambda\alpha\beta) \\ \overline{R}^{\lambda\alpha}_{\gamma\rho} \overline{R}^{\rho\beta}_{\delta\mu} = \overline{R}^{\lambda\beta}_{\delta\rho} \overline{R}^{\rho\alpha}_{\gamma\mu} & \qquad (\alpha\beta\lambda) \\ \overline{R}^{\lambda\alpha}_{\gamma\rho} R^{\rho\beta}_{\delta\mu} = R^{\lambda\beta}_{\delta\rho} \overline{R}^{\rho\alpha}_{\gamma\mu} & \qquad (\alpha\lambda\beta) \end{cases} \tag{8}$$

and

$$\begin{cases} R^{\lambda\alpha}_{\gamma\nu} R^{\mu\gamma}_{\beta\rho} = R^{\mu\alpha}_{\gamma\rho} R^{\lambda\gamma}_{\beta\nu} & \qquad (\lambda\mu\alpha) \\ \overline{R}^{\lambda\alpha}_{\gamma\nu} \overline{R}^{\mu\gamma}_{\beta\rho} = \overline{R}^{\mu\alpha}_{\gamma\rho} \overline{R}^{\lambda\gamma}_{\beta\nu} & \qquad (\alpha\lambda\mu) \\ R^{\lambda\alpha}_{\gamma\nu} \overline{R}^{\mu\gamma}_{\beta\rho} = \overline{R}^{\mu\alpha}_{\gamma\rho} R^{\lambda\gamma}_{\beta\nu} & \qquad (\lambda\alpha\mu) \end{cases} \tag{9}$$

for the $R^{\lambda\alpha}_{\beta\mu}$ and $\overline{R}^{\lambda\alpha}_{\beta\mu}$ (the components abc of (7) are in the $(\ldots)$).

Noncommutative product of $\mathbb{R}^{N_1}$ and $\mathbb{R}^{N_2}$

From now on it is assumed that the matrix R of relations (3) for the algebra $\mathcal{A}_R$ satisfies conditions (4), (8) and (9)

The classical (commutative) solution $R = R_0$ is

$$(R_0)^{\lambda\alpha}_{\beta\mu} = \delta^{\lambda}_{\mu}\delta^{\alpha}_{\beta}$$

and the corresponding algebra $\mathcal{A}_{R_0}$ is the algebra of polynomial functions on the product $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$

This is the reason we define the noncommutative product of $\mathbb{R}^{N_1}$ and $\mathbb{R}^{N_2}$

$$\mathbb{R}^{N_1} \times_R \mathbb{R}^{N_2}$$

to be the "dual" of the algebra $\mathcal{A}_R$ for general R.

Regularity properties of $\mathcal{A}_R$ and $\mathcal{A}^!_R$

Since the relations of $\mathcal{A}_R$ can be written in the form (6) with $\mathcal{R}$ involutive and satisfying the Yang-Baxter equation,

it follows from general results (Gurevich, Wambst) that $\mathcal{A}_R$ is very regular.

In particular $\mathcal{A}_R$ is a Koszul algebra of global dimension $N_1 + N_2$ having the Gorenstein property (an appropriate version of the Poincaré duality property)

In our case, this implies that, in terms of the x^a and the dual basis θ_a:

the $x^{a_1}\dots x^{a_p}$ for $a_1 \leq \dots \leq a_p$ and $p \in \mathbb{N}$ is a basis of $\mathcal{A}_R$

while the $\theta_{a_1}\dots\theta_{a_p}$ for $a_1 < \dots < a_p$ and $p \in \{1,\dots,N_1+N_2\}$ is a basis of $\mathcal{A}^!_R$.

As a consequence the Poincaré series are classical :

$$P_{\mathcal{A}_R}(t) := \sum_n \dim(\mathcal{A}^n_R)t^n = \left(\frac{1}{1-t}\right)^{N_1+N_2} \quad \text{and} \quad P_{\mathcal{A}^!_R}(t) = (1+t)^{N_1+N_2} \quad (10)$$

Regularity properties cont.d

There is no twist, and one gets a cyclic preregular multilinear form

$$W_{a_1,\dots,a_{N_1},b_1\dots b_{N_2}} := \sum_{r\,s} \varepsilon_{a_1,\dots,a_{N_1-1},r,b_1\dots,b_{N_2-1}s}(\delta^r_{a_{N_1}}\delta^s_{b_{N_2}} - \mathcal{R}^{r\,s}_{a_{N_1}\,b_{N_2}})$$

for ε the completely antisymmetric tensor with $\varepsilon_{1,2,\cdots,N_1+N_2} = 1$

the algebra $\mathcal{A}_R$ is in fact Calabi–Yau algebra (Ginsburg)

The element

$$\mathbb{1}\otimes W = W_{a_1,\dots,a_{N_1},b_1\dots b_{N_2}}\mathbb{1}\otimes x^{a_1}\dots x^{N_2}$$

is a non-trivial Hochschild cycle:

$$b(\mathbb{1}\otimes W) = 0$$

Noncommutative product of Euclidean spaces

Theorem. The following conditions (i) (ii) and (iii) are equivalent :

(i) $\sum_{a=1}^{N_1+N_2}(x^a)^2 = \sum_{\lambda=1}^{N_1}(x_1^\lambda)^2 + \sum_{\alpha=1}^{N_2}(x_2^\alpha)^2$ is central in $\mathcal{A}_R$,

(ii) $\sum_{\lambda=1}^{N_1}(x_1^\lambda)^2$ and $\sum_{\alpha=1}^{N_2}(x_2^\alpha)^2$ are in the center of $\mathcal{A}_R$, (11)

(iii) $\sum_{\lambda=1}^{N_1} R_{\beta\nu}^{\lambda\gamma} R_{\alpha\mu}^{\lambda\beta} = \delta_\alpha^\gamma \delta_{\mu\nu}$ and $\sum_{\alpha=1}^{N_2} R_{\beta\rho}^{\lambda\alpha} R_{\gamma\mu}^{\rho\alpha} = \delta_\mu^\lambda \delta_{\beta\gamma}$

We take R to satisfies also (11) and define the the noncommutative product of the Euclidean space $\mathbb{R}^{N_1}$ with the Euclidean space $\mathbb{R}^{N_2}$ to be dual of $\mathcal{A}_R$.

Clearly, the relations (11) are satisfied by the classical $R = R_0$;

$\mathcal{A}_R$ generalizes the algebra of polynomial functions on the product $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$

Restrictions on the structure of R

By using (4) and (11) one obtains

$$R^{\lambda\beta}_{\alpha\mu} = R^{\mu\alpha}_{\beta\lambda} = \overline{R}^{\mu\beta}_{\alpha\lambda} = (R^{-1})^{\beta\mu}_{\lambda\alpha} \tag{12}$$

In turn this implies that relations (8) and (9) reduce to

$$R^{\lambda\alpha}_{\beta\rho} R^{\rho\delta}_{\gamma\mu} = R^{\lambda\delta}_{\gamma\rho} R^{\rho\alpha}_{\beta\mu} \tag{13}$$

$$R^{\lambda\alpha}_{\gamma\nu} R^{\mu\gamma}_{\beta\rho} = R^{\mu\alpha}_{\gamma\rho} R^{\lambda\gamma}_{\beta\nu} \tag{14}$$

that is the first relation of (8) and the first relation of (9).

Corollary. Relations (3) define (the algebra of) a noncommutative product of a N_1-dimensional with a N_2-dimensional Euclidean spaces if and only if the $R^{\lambda\alpha}_{\beta\mu}$ satisfy Relations (12), (13) and (14).

Noncommutative product of spheres

The elements $\sum_{\lambda=1}^{N_1}(x_1^\lambda)^2 = \|x_1\|^2$ and $\sum_{\alpha=1}^{N_2}(x_2^\alpha)^2 = \|x_2\|^2$ of $\mathcal{A}_R$ being central one may consider the quotient algebra

$$\mathcal{A}_R/(\{\|x_1\|^2 - \mathbb{1}, \|x_2\|^2 - \mathbb{1}\}) \qquad \leftrightarrow \qquad \mathbb{S}^{N_1-1} \times_R \mathbb{S}^{N_2-1}$$

This defines by duality the noncommutative product of the classical spheres $\mathbb{S}^{N_1-1}$ and $\mathbb{S}^{N_2-1}$.

Indeed, for $R = R_0$, the above quotient is the restriction to $\mathbb{S}^{N_1-1} \times \mathbb{S}^{N_2-1}$ of the polynomial functions on $\mathbb{R}^{N_1+N_2}$.

Noncommutative spheres

With $\|x\|^2$ denoting the central element $\sum_{a=1}^{N_1+N_2}(x^a)^2 = \|x_1\|^2 + \|x_2\|^2$, one may also consider the quotient of $\mathcal{A}_R$

$$\mathcal{A}_R/(\|x\|^2 - \mathbb{1}) \qquad \leftrightarrow \qquad (\mathbb{S}^{N_1+N_2-1})_R$$

This defines (dualy) the noncommutative $(N_1 + N_2 - 1)$-sphere $(\mathbb{S}^{N_1+N_2-1})_R$ (a subspace of the noncommutative product of $\mathbb{R}^{N_1}$ with $\mathbb{R}^{N_2}$)

This is a noncommutative spherical manifold in the sense of Connes–Landi and Connes–Dubois-Violette (see below).

The (generalized) Clifford algebra $C\ell(\mathcal{A}_R)$

The $*$-algebra $C\ell(\mathcal{A}_R)$ is generated by two sets of hermitian elements Γ^1_λ with $\lambda \in \{1, \dots, N_1\}$ and Γ^2_α with $\alpha \in \{1, \dots, N_2\}$ with relations

$$\begin{cases} \Gamma^1_\lambda \Gamma^1_\mu + \Gamma^1_\mu \Gamma^1_\lambda = 2\delta_{\lambda\mu} \mathbb{1} \\ \Gamma^2_\alpha \Gamma^2_\beta + \Gamma^2_\beta \Gamma^2_\alpha = 2\delta_{\alpha\beta} \mathbb{1} \\ \Gamma^2_\beta \Gamma^1_\mu + R^{\lambda\alpha}_{\beta\mu} \Gamma^1_\lambda \Gamma^2_\alpha = 0 \end{cases} \tag{15}$$

Proposition In the algebra $C\ell(\mathcal{A}_R) \otimes \mathcal{A}_R$ one has :

$$(\Gamma^1_\lambda \otimes x^\lambda_1)^2 = \mathbb{1} \otimes \|x_1\|^2, \qquad (\Gamma^2_\alpha \otimes x^\alpha_2)^2 = \mathbb{1} \otimes \|x_2\|^2$$

and

$$(\Gamma^1_\lambda \otimes x^\lambda_1)(\Gamma^2_\alpha \otimes x^\alpha_2) + (\Gamma^2_\alpha \otimes x^\alpha_2)(\Gamma^1_\lambda \otimes x^\lambda_1) = 0$$

Structure of $C\ell(\mathcal{A}_R)$

Last proposition is equivalent to

$$(\Gamma(x))^2 = \mathbb{1} \otimes \|x\|^2 \tag{16}$$

with $\Gamma(x) = \Gamma_a \otimes x^a = \Gamma^1_\lambda \otimes x^\lambda_1 + \Gamma^2_\alpha \otimes x^\alpha_2$ and $\|x\|^2 = \sum_{a=1}^{N_1+N_2} (x^a)^2$.

The algebra $C\ell(\mathcal{A}_R)$ is nonhomogeneous quadratic with $\mathcal{A}^!_R$ as homogeneous part. It is not $\mathbb{N}$-graded but only $\mathbb{Z}_2$-graded and filtered with

$$\mathcal{F}^n = F^n(C\ell(\mathcal{A}_R)) = \{\text{elements of degree in } \Gamma \leq n\}$$

One has a surjective canonical homomorphism of graded algebra

$$\text{can} : \mathcal{A}^!_R \to \ \text{gr}(C\ell(\mathcal{A}^!_R)) = \oplus_{n\in\mathbb{N}} \mathcal{F}^n/\mathcal{F}^{n-1} \tag{17}$$

which induce the isomorphism of vector spaces

$$(\mathcal{A}^!_R)^1 \simeq \mathcal{F}^1/\mathcal{F}^0$$

Structure of $C\ell(\mathcal{A}_R)$ - cont.d

The fact that $\|x\|^2 = \sum (x^a)^2$ is central in $\mathcal{A}_R$ and the Koszulity of $\mathcal{A}_R^!$ imply the following PBW property (via the duality of Positselski).

Proposition The homomorphism (17) is an isomorphism of graded algebras.

Thus $C\ell(\mathcal{A}_R)$ is a Koszul nonhomogeneous quadratic algebra since $\mathcal{A}_R^!$ is Koszul, (cf. Dubois-Violette). This implies

$$\dim C\ell(\mathcal{A}_R) = \dim \mathcal{A}_R^! = 2^{N_1+N_2}$$

One has the following isomorphisms :

$$\begin{cases} C\ell(N_1) \simeq \text{ subalgebra of } C\ell(\mathcal{A}) \text{ generated by the } \Gamma^1_\lambda \\ C\ell(N_2) \simeq \text{ subalgebra of } C\ell(\mathcal{A}) \text{ generated by the } \Gamma^2_\alpha \end{cases}$$

and

$$C\ell(\mathcal{A}_R) \simeq C\ell(N_1 + N_2) \tag{18}$$

The ansatz A B C D

Nontrivial realizations of the $R^{\lambda\alpha}_{\beta\mu}$ are given by the following.

Theorem Let A and C be two commuting real $N_1 \times N_1$-matrices with A symmetric and C antisymmetric and let B and D be two commuting real $N_2 \times N_2$-matrices with B symmetric and D antisymmetric. Assume that

$$A^2 \otimes B^2 + C^2 \otimes D^2 = \mathbb{1}_{N_1} \otimes \mathbb{1}_{N_2}$$

then the $R^{\lambda\alpha}_{\beta\mu}$ given by

$$R^{\lambda\alpha}_{\beta\mu} = A^{\lambda}_{\mu} B^{\alpha}_{\beta} + \mathrm{i}\, C^{\lambda}_{\mu} D^{\alpha}_{\beta} \tag{19}$$

satisfy the assumptions (12), (13) and (14).

Enter the quaternions

An $SO(4)$-invariant decomposition of $M_4(\mathbb{R})$

$$q = x^0 1 + x^k e_k \in \mathbb{H} \quad \longleftrightarrow \quad x = (x^0, x^1, x^2, x^3) \in \mathbb{R}^4 \qquad \text{Euclidean}$$

a right and a left action

$$e_k q \leftrightarrow J_k^{(+)} x, \qquad q e_k \leftrightarrow -J_k^{(-)} x$$

$$J^{(\pm)}_{k\mu\nu} = \mp(\delta_{0\mu}\delta_{k\nu} - \delta_{0\nu}\delta_{k\mu}) - \varepsilon_{k\ell m}\delta^{\ell}_{\mu}\delta^{m}_{\nu}$$

$$J_k^{(\pm)} J_\ell^{(\pm)} = -\delta_{k\ell}\mathbb{1} + \sum_m \varepsilon_{k\ell m} J_m^{(\pm)}, \; J_k^{(+)} J_\ell^{(-)} = J_\ell^{(-)} J_k^{(+)}$$

$$M_4(\mathbb{R}) = \mathbb{R}^4 \otimes \mathbb{R}^4 = \mathbb{R}\mathbb{1} \oplus \wedge^2_{(+)}\mathbb{R}^4 \oplus \wedge^2_{(-)}\mathbb{R}^4 \oplus \mathbb{S}^2_0\mathbb{R}^4$$

Orthornormal basis $\mathbb{1}$, $J_k^{(+)}$, $J_\ell^{(-)}$, $J_r^{(+)} J_s^{(-)}$

$(+)$ = antisymmetric self-dual, $\qquad (-)$ = antisymmetric anti-self dual.

Noncommutative quaternionic planes

Use last theorem for $N_1 = N_2 = 4$

$A = \mathbb{1}$, $\quad B = u^0 \mathbb{1}$, $\quad C = J_1^{(\pm)}$, $\quad D = u^1 J_1^{(\pm)} + u^2 J_2^{(\pm)}$

with $(u^0)^2 + (u^1)^2 + (u^2)^2 = 1$. This gives:

$$R^{\lambda\alpha}_{\beta\mu} = u^0 \delta^\lambda_\mu \delta^\alpha_\beta + \mathrm{i}(J_1^{(\pm)})^\lambda_\mu (u^1 J_1^{(\pm)} + u^2 J_2^{(\pm)})^\alpha_\beta \tag{20}$$

By using the $J_k^{(\mp)}$ one defines an action of $\mathbb{H}$.

The choice of the direction 1 and of the plane (1 2) is immaterial since one can change them into an arbitrary direction $\vec{n}$ and an arbitrary plane which contains $\vec{n}$ by a rotation of $SO_3^{(\pm)}$.

The exchange $(+) \leftrightarrow (-)$ is induced for instance by the exchange $x^0 \leftrightarrow -x^0$ and therefore does not change the algebra $\mathcal{A}_R$ for R given by (20).

Noncommutative quaternionic planes cont.d

The solution given by (20) generalizes $\mathbb{C}^2_\theta$ for $\mathbb{C} \to \mathbb{H}$;

For the θ-deformation the parameter is in fact

$$\mathbb{S}^1/\mathbb{S}^0 = U_1(\mathbb{C})/U_1(\mathbb{R}) = P_1(\mathbb{R}).$$

The parameter here

$$\mathbf{u} \in \mathbb{S}^2 = \mathbb{S}^3/\mathbb{S}^1 = U_1(\mathbb{H})/U_1(\mathbb{C}) = P_1(\mathbb{C})$$

and for $u^0 = 1$ ($\Rightarrow u^1 = u^2 = 0$), this gives the classical $\mathbb{H}^2$.

Noncommutative quaternionic tori

The N.C. product of $\mathbb{H}$ by $\mathbb{H}$ corresponding to $\mathcal{A}_R$ is denoted $\mathbb{H}^2_{\mathbf{u}}$.

Tori obtained by the quotient by the ideal generated by $\{\|x_1\|^2 - 1, \|x_2\|^2 - 1\}$:

$$\mathcal{A}(\mathbb{T}^{\mathbb{H}}_{\mathbf{u}}) = \mathcal{A}(\mathbb{H}^2_{\mathbf{u}}) / < ||x_1||^2 - 1, ||x_2||^2 - 1 >$$

$$\mathbb{T}^{\mathbb{H}}_{\mathbf{u}} \simeq \mathbb{S}^3 \times_{\mathbf{u}} \mathbb{S}^3$$

an $SU(2) \times SU(2)$ action

Additional strata: other N.C. products of 4-dim. Euclidean spaces

Other $\mathcal{A}_R$ with $N_1 = N_2 = 4$ using the ansatz $A\ B\ C\ D$ with $J_k^{(\pm)}$

1. $A = \mathbb{1}, \quad B = \cos(\theta)\mathbb{1}, \quad C = J_1^{(\pm)}, \quad D = \sin(\theta) J_1^{(\mp)}.$

   The direction 1 for $J_\cdot^{(\pm)}$ (resp $J_\cdot^{\mp)}$) can be changed by acting with $SO_3^{(\pm)}$ (resp. $SO_3^{(\mp)}$). For $\theta = 0$ it corresponds to the classical $\mathbb{R}^4 \times \mathbb{R}^4$.

2. $A = C \cdot (v^k J_k^{(\mp)}), \quad B = D \cdot (w^k J_k^{(\mp)}),$
   $C = J_1^{\pm}, \quad D = u^1 J_1^{\pm} + u^2 J_2^{\pm}$ with $((u^1)^2 + (u^2)^2)(1 + \vec{v}^2 \vec{w}^2) = 1$.

3. $A = C \cdot (v^k J_k^{(\mp)}), \quad B = D \cdot (w^k J_k^{\pm}),$
   $C = J_1^{(\pm)}, \quad D = u J_1^{(\mp)}$ with $u^2 + \vec{v}^2 \vec{w}^2 = 1$.

Solutions 2 and 3 do not contain the classical $\mathbb{R}^4 \times \mathbb{R}^4$.

These solutions are noncommutative for any parameters and cannot be connected with solution 1 and with $\mathbb{H}^2_{\mathbf{u}}$.

Spherical conditions

A projection $p \in M_{2^n}(\mathcal{A}(S_R^{2n}))$,

$$p = \frac{1}{2}(\mathbb{1} + \Gamma_a x^a + \Gamma x)$$

$$ch_k(p) = 0, \qquad 0 \le k \le n-1 \qquad\qquad ch_n(p) \quad \text{the volume form}$$

A unitary $U \in M_{2^{n-1}}(\mathcal{A}(S_R^{2n-1}))$,

$$U = \mathbb{1}\, x^0 + \Sigma_j\, x^j$$

$$ch_{k-\frac{1}{2}}(U) = 0, \qquad 0 \le k \le n-1 \qquad\qquad ch_{n-\frac{1}{2}}(U) \quad \text{the volume form}$$

$$\Gamma_\mu = \begin{pmatrix} 0 & \Sigma_\mu \\ \Sigma_\mu, & 0 \end{pmatrix}$$

A matter of computation (finished in Toulouse !!) for the present examples

Spectral geometry

Principal bundles

coming up

thank you

A (homogeneous) *quadratic algebra* is an associative algebra $\mathcal{A}$ of the form

$$\mathcal{A} = A(E, R) = T(E)/(R)$$

Here E is a finite-dimensional vector space (the space of generators), R a subspace of $E \otimes E$ (the space of relations of $\mathcal{A}$)

The algebra $\mathcal{A} = A(E, R)$ is a connected (i.e. $\mathcal{A}_0 = \mathbb{C}\mathbb{1}$) graded algebra $\mathcal{A} = \bigoplus_{n \in \mathbb{N}} \mathcal{A}_n$ generated by the degree 1 part, $\mathcal{A}_1 = E$.

Its *Koszul dual* $\mathcal{A}^!$, is the quadratic algebra defined by

$$\mathcal{A}^! = A(E^*, R^\perp)$$

where E^* denotes the dual vector space of E and $R^\perp \subset E^* \otimes E^*$ is the orthogonal of the space of relations $R \subset E \otimes E$: $\quad R^\perp = \{\omega \in E^* \otimes E^* ;\ \langle \omega, r \rangle = 0\}$.

The dual vector spaces $\mathcal{A}^{!*}_n$ of the homogeneous components $\mathcal{A}^!_n$ of $\mathcal{A}^!$ are

$$\mathcal{A}^{!*}_1 = E \qquad \text{and} \qquad \mathcal{A}^{!*}_n = \bigcap_{r+s+2=n} E^{\otimes^r} \otimes R \otimes E^{\otimes^s}$$

In particular $\mathcal{A}^{!*}_2 = R$ and $\mathcal{A}^{!*}_n \subset E^{\otimes^n}$ for any $n \in \mathbb{N}$.

The sequence of free left $\mathcal{A}$-modules

$$\cdots \stackrel{b}{\to} \mathcal{A} \otimes \mathcal{A}^{!*}_{n+1} \stackrel{b}{\to} \mathcal{A} \otimes \mathcal{A}^{!*}_{n} \to \cdots \to \mathcal{A} \otimes \mathcal{A}^{!*}_{2} \stackrel{b}{\to} \mathcal{A} \otimes E \stackrel{b}{\to} \mathcal{A} \to 0 \qquad (21)$$

where $b : \mathcal{A} \otimes \mathcal{A}^{!*}_{n+1} \to \mathcal{A} \otimes \mathcal{A}^{!*}_{n}$ defined by

$$b(a \otimes (x_0 \otimes x_1 \otimes \cdots \otimes x_n)) = ax_0 \otimes (x_1 \otimes \cdots \otimes x_n)$$

is such that $b^2 = 0$, is a chain complex $K(\mathcal{A})$ of free left $\mathcal{A}$-modules called the *Koszul complex* of the quadratic algebra $\mathcal{A}$.

The quadratic algebra $\mathcal{A}$ is said to be a *Koszul algebra* whenever its Koszul complex is acyclic in positive degrees, that is, $H_n(K(\mathcal{A})) = 0$ for $n \geq 1$.

The presentation of $\mathcal{A}$ by generators and relations is equivalent to the exactness of the sequence

$$\mathcal{A} \otimes R \stackrel{b}{\to} \mathcal{A} \otimes E \stackrel{b}{\to} \mathcal{A} \stackrel{\varepsilon}{\to} \mathbb{C} \to 0 \qquad (22)$$

so one always has

$$H_1(K(\mathcal{A})) = 0 \qquad \text{and} \qquad H_0(K(\mathcal{A})) = \mathbb{C}$$

and, whenever $\mathcal{A}$ is Koszul, the sequence

$$K(\mathcal{A}) \stackrel{\varepsilon}{\to} \mathbb{C} \to 0, \qquad (23)$$

with ε induced by the projection onto degree 0, is a free resolution of the trivial module $\mathbb{C}$; a minimal projective resolution of $\mathbb{C}$ of graded modules.

If $\mathcal{A} = A(E, R)$ quadratic Koszul is such that $\mathcal{A}^!_D \neq 0$ and $\mathcal{A}^!_n = 0$ for $n > D$,

then the trivial (left) module $\mathbb{C}$ has projective dimension D which implies that $\mathcal{A}$ has global dimension D. In turn the Hochschild dimension of $\mathcal{A}$ is D.

By applying the functor $Hom_{\mathcal{A}}(\,\cdot\,, \mathcal{A})$ to the Koszul chain complex $K(\mathcal{A})$ of left $\mathcal{A}$-modules one obtains the cochain complex $L(\mathcal{A})$ of right $\mathcal{A}$-modules

$$0 \to \mathcal{A} \stackrel{b'}{\to} \cdots \stackrel{b'}{\to} \mathcal{A}^!_n \otimes \mathcal{A} \stackrel{b'}{\to} \mathcal{A}^!_{n+1} \otimes \mathcal{A} \stackrel{b'}{\to} \cdots . \tag{24}$$

with b' left multiplication by $\sum_k \theta^k \otimes e_k$ in $\mathcal{A}^! \otimes \mathcal{A}$ and (e_k, θ^k) are dual bases.

The algebra $\mathcal{A}$ is said to be *Koszul-Gorenstein* if it is Koszul of finite global dimension D as above and if $H^n(L(\mathcal{A})) = \mathbb{C}\,\delta^n_D$.

This implies that $\mathcal{A}^!_n \simeq \mathcal{A}^{!*}_{D-n}$ as vector spaces (a version of Poincaré duality).

Finally, a graded algebra $\mathcal{A} = \oplus_n \mathcal{A}_n$ is said to have *polynomial growth* whenever there are a positive C and $N \in \mathbb{N}$ such that, for any $n \in \mathbb{N}$,

$$\dim(\mathcal{A}_n) \leq C n^{N-1}.$$