An index theorem for the Dirac operator on Lorentzian manifolds

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Index theory in Lorentzian signature?

Index theory requires (some kind of) compactness of the underlying manifold.

Problem 1: Fredholm operators on closed manifolds are elliptic. Hyperbolic PDE theory does not work on such spacetimes
$\Rightarrow$ no Lorentzian analog to Atiyah-Singer index theorem

Problem 2: Closed Lorentzian manifolds violate causality conditions
$\Rightarrow$ useless as models in General Relativity

But: There is one for the Atiyah-Patodi-Singer index theorem!

Globally hyperbolic spacetimes

A subset $\Sigma \subset M$ is called Cauchy hypersurface if each inextendible timelike curve in M meets Σ exactly once.

If M has a Cauchy hypersurface then M is called globally hyperbolic.

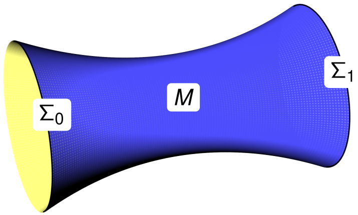
Examples:

- Minkowski spacetime (Special Relativity)
- Schwarzschild Model (Black Hole)
- Friedmann cosmos (Big Bang, cosmic expansion)
- deSitter spacetime
- $\dots$

Globally hyperbolic spacetimes

Theorem (Bernal-Sánchez 2005)

Every globally hyperbolic Lorentzian manifold is isometric to $M = I \times \Sigma$ with metric $-N^2 dt^2 + g_t$ such that each $\{t\} \times \Sigma$ is a smooth spacelike Cauchy hypersurface.



Let M be a globally hyperbolic Lorentzian manifold **with boundary** $\partial M = \Sigma_0 \sqcup \Sigma_1$

Σ_j compact smooth spacelike Cauchy hypersurfaces

Geometric setup

- M as above
- spin structure $\rightsquigarrow$ spinor bundle $SM \to M$
- $n = \dim(M)$ even $\rightsquigarrow$ splitting $SM = S_R M \oplus S_L M$
- Hermitian vector bundle $E \to M$ with connection $\rightsquigarrow$ twisted Dirac operator $D : C^\infty(M, V_R) \to C^\infty(M, V_L)$ where $V_{R/L} = S_{R/L} M \otimes E$

The Cauchy problem

Well-posedness of Cauchy problem

The map $D \oplus \mathrm{res}_\Sigma : C^\infty(M; V_R) \to C^\infty(M; V_L) \oplus C^\infty(\Sigma; V_R)$ is an isomorphism of topological vector spaces.

Wave propagator U:

$$\begin{array}{ccc} & \{v \in C^\infty(M; V_R) \mid Dv = 0\} & \\ {}^{\mathrm{res}_{\Sigma_0}}_{\cong}\swarrow & & \searrow^{\mathrm{res}_{\Sigma_1}}_{\cong} \\ C^\infty(\Sigma_0, V_R) & \xrightarrow{\quad U \quad} & C^\infty(\Sigma_1, V_R) \end{array}$$

U extends to **unitary** operator $L^2(\Sigma_0; V_R) \to L^2(\Sigma_1; V_R)$.

Fredholm pairs

Definition

Let H be a Hilbert space and $B_0, B_1 \subset H$ closed linear subspaces. Then (B_0, B_1) is called a Fredholm pair if $B_0 \cap B_1$ is finite dimensional and $B_0 + B_1$ is closed and has finite codimension. The number

$$\mathrm{ind}(B_0, B_1) = \dim(B_0 \cap B_1) - \dim(H/(B_0 + B_1))$$

is called the index of the pair (B_0, B_1).

Elementary properties:

1.) $\mathrm{ind}(B_0, B_1) = \mathrm{ind}(B_1, B_0)$

2.) $\mathrm{ind}(B_0, B_1) = -\,\mathrm{ind}(B_0^\perp, B_1^\perp)$

3.) Let $B_0 \subset B_0'$ with $\dim(B_0'/B_0) < \infty$. Then

$$\mathrm{ind}(B_0', B_1) = \mathrm{ind}(B_0, B_1) + \dim(B_0'/B_0).$$

Fredholm pairs and the Dirac operator

Let $B_0 \subset L^2(\Sigma_0, V_R)$ and $B_1 \subset L^2(\Sigma_1, V_R)$ be closed subspaces.

Observation (B.-Hannes 2017)

The following are equivalent:

(i) The pair $(B_0, U^{-1}B_1)$ is Fredholm of index k;

(ii) The pair (UB_0, B_1) is Fredholm of index k;

(iii) The restriction

$$D : \ker(\pi_{B_0^\perp} \circ \mathrm{res}_{\Sigma_0}) \cap \ker(\pi_{B_1^\perp} \circ \mathrm{res}_{\Sigma_1}) \to L^2(M_0, V_L)$$

is a Fredholm operator of index k.

Trivial example

Let $\dim(B_0) < \infty$ and $\operatorname{codim}(B_1) < \infty$.

Then D with these boundary conditions is Fredholm with index

$$\dim(B_0) - \operatorname{codim}(B_1)$$

Atiyah-Patodi-Singer boundary conditions

In Bernal-Sánchez splitting the Dirac operator takes the form

$$D = \sigma \cdot \left(\frac{\partial}{\partial t} + iA_t - \frac{n-1}{2}H\right)$$

A_t is a selfadjoint elliptic Dirac operator on Σ_t.
Spectral projector:

$$P_+ = \begin{cases} \chi_{[0,\infty)}(A_0) \text{ on } \Sigma_0 \\ \chi_{(-\infty,0]}(A_1) \text{ on } \Sigma_1 \end{cases}$$

APS-boundary conditions:

$$P_+(v|_{\partial M}) = 0$$

Anti-APS conditions

Anti-APS-boundary conditions:

$$(I - P_+)(v|_{\partial M}) = 0$$

Do not give rise to a Fredholm operator in the Riemannian case.

The Lorentzian index theorem

Theorem (B.-Strohmaier 2015)
Under APS-boundary conditions D is a Fredholm operator. The kernel consists of smooth spinor fields and

$$\mathrm{ind}(D_{\mathrm{APS}}) = \int_M \widehat{\mathrm{A}}(M) \wedge \mathrm{ch}(E) + \int_{\partial M} T(\widehat{\mathrm{A}}(M) \wedge \mathrm{ch}(E)) - \frac{h(A_0) + h(A_1) + \eta(A_0) - \eta(A_1)}{2}$$

Here $h(A) = \dim\ker(A)$ and $\eta(A) =$ eta-invariant of A.

The Lorentzian index theorem

Moreover,

$$\begin{aligned}\mathrm{ind}(D_{\mathrm{APS}}) = {} & \dim\ker[D : C^\infty_{\mathrm{APS}}(M; V_R) \to C^\infty(M; V_L)] \\ & - \dim\ker[D : C^\infty_{\mathrm{aAPS}}(M; V_R) \to C^\infty(M; V_L)]\end{aligned}$$

aAPS conditions are as good as APS-boundary conditions

Proof of the regularity statement

- If v is a distributional spinor solving $Dv = 0$ then $\mathrm{WF}(v) \subset \{\text{lightlike covectors}\}$
- v restricts to distributions along $\Sigma_{0/1}$
- APS conditions along $\Sigma_0 \Rightarrow$ $\mathrm{WF}(v) \subset \{\text{future-directed lightlike covectors}\}$ along Σ_0
- propagation of singularities $\Rightarrow$ $\mathrm{WF}(v) \subset \{\text{future-directed lightlike covectors}\}$ on all of M
- similarly, APS along $\Sigma_1 \Rightarrow$ $\mathrm{WF}(v) \subset \{\text{past-directed lightlike covectors}\}$
- $\Rightarrow \mathrm{WF}(v) = \emptyset$, i.e. v is smooth

Proof of the index theorem

Decompose the wave propagator

$$U = \begin{pmatrix} U_{++} & U_{+-} \\ U_{-+} & U_{--} \end{pmatrix}$$

w.r.t. decomposition

$$\begin{aligned} L^2(\Sigma_0; V_R) &= P_+ L^2(\Sigma_0; V_R) & \oplus \quad & (I - P_+) L^2(\Sigma_0; V_R) \\ &= L^2_{[0,\infty)}(\Sigma_0; V_R) & \oplus \quad & L^2_{(-\infty,0)}(\Sigma_0; V_R), \\ L^2(\Sigma_1; V_R) &= (I - P_+) L^2(\Sigma_1; V_R) & \oplus \quad & P_+ L^2(\Sigma_1; V_R) \\ &= L^2_{(0,\infty)}(\Sigma_1; V_R) & \oplus \quad & L^2_{(-\infty,0]}(\Sigma_1; V_R) \end{aligned}$$

Then

$$\mathrm{ind}(D_{\mathrm{APS}}) = \dim\ker(U_{--}) - \dim\ker(U_{++})$$

Proof of the index theorem

Step 1: Show that D_{APS} is Fredholm
$\Leftrightarrow$ U_{++} and U_{--} are Fredholm
$\Leftarrow$ U_{+-} and U_{-+} are compact (uses microlocal analysis)

Step 2: Compute the index
Introduce auxiliary *Riemannian* metric $\hat{g}$ on M ("Wick rotation")

$$\mathrm{sf}(A_t) = \mathrm{ind}(\hat{D}_{\mathrm{APS}}) = \text{geometric expression}(\hat{g})$$

$$\mathrm{sf}(A_t) = \mathrm{ind}(D_{\mathrm{APS}})$$

$$\text{geometric expression}(g) = \text{geometric expression}(\hat{g})$$

The chiral anomaly

No natural physical interpretation of APS boundary conditions in the Riemannian case.
But the Lorentzian version allows to compute the chiral anomaly in QFT.

1.) Quantize harmonic spinor field (constr. field operators Ψ, $\bar{\Psi}$)
2.) Want to quantize classical Dirac current

$$J(X) = \langle \psi, X \cdot \psi \rangle$$

Fix a Cauchy hypersurface Σ and try

$$J_\mu^\Sigma(p) = \omega_\Sigma(\bar{\Psi}^{\dot{A}}(p)(\gamma_\mu)_{\dot{A}}^{B}\Psi_B(p))$$

Here ω_Σ is the state associated with Σ.
Problem: singularities of two-point function. Need regularization procedure (renormalization).
3.) **But:** relative current does exist

$$J^{\Sigma_0,\Sigma_1} = J^{\Sigma_0} - J^{\Sigma_1}$$

Charge creation and index

Theorem (B.-Strohmaier 2015)

For any two Cauchy hypersurfaces with product structure near them, the relative current J^{Σ_0,Σ_1} is coclosed and its integral Q_R over any Cauchy hypersurface equals

$$\mathrm{ind}(U_{--}) = \mathrm{ind}(D_{\mathrm{APS}}).$$

Hence

$$Q_R = \int_M \widehat{\mathrm{A}}(M) \wedge \mathrm{ch}(E) - \frac{h(D_{\Sigma_0}) - h(D_{\Sigma_1}) + \eta(D_{\Sigma_0}) - \eta(D_{\Sigma_1})}{2}.$$

Similarly

$$Q_L = -\int_M \widehat{\mathrm{A}}(M) \wedge \mathrm{ch}(E) + \frac{h(D_{\Sigma_0}) - h(D_{\Sigma_1}) + \eta(D_{\Sigma_0}) - \eta(D_{\Sigma_1})}{2}.$$

Total charge $Q = Q_R + Q_L$ is zero.
Chiral charge $Q_{\mathrm{chir}} = Q_R - Q_L$ is not!

Example

- Spacetime $M = \mathbb{R} \times S^{4k-1}$ with metric $-dt^2 + g_t$ where g_t are Berger metrics.
- Flat connection on trivial bundle E.
- Chiral anomaly:

$$Q_{\mathrm{chir}}^{\Sigma_0,\Sigma_1} = (-1)^k 2\binom{2k}{k}$$

- See Gibbons 1979 for $k = 1$.

Generalized APS boundary conditions

For $a \in \mathbb{R}$ define

$$\mathrm{APS}_0(a) := L^2_{(-\infty,a)}(A_0),$$
$$\mathrm{APS}_1(a) := L^2_{(a,\infty)}(A_1).$$

Then

$$\begin{aligned}\mathrm{ind}(\mathrm{APS}_0(a_0),&\mathrm{APS}_1(a_1)) \\ &= \mathrm{ind}(\mathrm{APS}_0(0), \mathrm{APS}_1(0)) \\ &\quad + \mathrm{sgn}(a_0) \cdot \#\{A_0\text{-ev's between 0 and } a_0\} \\ &\quad - \mathrm{sgn}(a_1) \cdot \#\{A_1\text{-ev's between 0 and } a_1\}\end{aligned}$$

Boundary conditions in graph form

A pair (B_0, B_1) of closed subspaces $B_i \subset L^2(\Sigma_i, V^R)$ is called **boundary conditions in graph form** if there are L^2-orthogonal decompositions

$$L^2(\Sigma_i, V^R) = V_i^- \oplus W_i^- \oplus V_i^+ \oplus W_i^+, \qquad i = 0, 1,$$

such that

(i) W_i^+, W_i^- are finite dimensional;

(ii) $W_i^- \oplus V_i^- = L^2_{(-\infty, a_i)}(A_i)$ and $W_i^+ \oplus V_i^+ = L^2_{[a_i, \infty)}(A_i)$ for some $a_i \in \mathbb{R}$;

(iii) There are bounded linear maps $g_0 : V_0^- \to V_0^+$ and $g_1 : V_1^+ \to V_1^-$ such that

$$B_0 = W_0^+ \oplus \Gamma(g_0),$$
$$B_1 = W_1^- \oplus \Gamma(g_1),$$

where $\Gamma(g_{0/1}) = \{v + g_{0/1} v \mid v \in V^{\mp}_{0/1}\}$.

Boundary conditions in graph form

Theorem (B.-Hannes 2017)

Let $a_0, a_1 \in \mathbb{R}$. Then the pair $(\Gamma(g_0), \Gamma(g_1))$ is Fredholm of the same index as $(\mathrm{APS}_0(a_0), \mathrm{APS}_1(a_1))$ provided

(A) g_0 or g_1 is compact **or**

(B) $\|g_0\| \cdot \|g_1\|$ is small enough.

1.) Applies if $g_0 = 0$ or $g_1 = 0$.
2.) Contains local elliptic boundary conditions in the sense of Lopatinski-Schapiro.
3.) Conditions (A) and (B) cannot both be dropped (counterexamples).

References

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Thank you for your attention!