Memory effects in a gravitational plane wave

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In collaboration with Jean-Philippe Uzan

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- Useful toy model for studying memory effects outside the asymptotically flat framework

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- Need a careful classification of memories in pp-wave geometries

Generalities on vacuum pp-wave

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- BJR coordinates are not global. Need to switch to Brinkman coordinates which play the role of null Fermi coordinates when studying memories ...

A new symmetry of vacuum gravitational plane wave

Symmetries of Einstein equation

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- First point: the Raychaudhuri equation admits a $\mathrm{SL}(2,\mathbb{R})$ symmetry

Conformal symmetry of the Raychaudhuri equation

- Non-linear field equation

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- Invariant under Möbius reparametrization : new solution generating map [**JBA**, Uzan '24]

$$\mathrm{Sch}[M \circ f] = \mathrm{Sch}[f] \qquad \text{where} \qquad M(u) = \frac{au+b}{cu+d} \qquad ad - bc \neq 0\,, \tag{9}$$

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$$A_{11} = \frac{1}{\dot{F}} \qquad A_{22} = \frac{1}{\dot{G}}\,. \tag{6}$$

- Einstein equation recast into

$$\boxed{\mathrm{Sch}[F] + \mathrm{Sch}[G] = 0} \tag{7}$$

wehre $\mathrm{Sch}[f]$ denotes the Schwarzian derivative of the function f

$$\mathrm{Sch}[f] = \frac{\dddot{f}}{\dot{f}} - \frac{3}{2}\frac{\ddot{f}^2}{\dot{f}^2}\,. \tag{8}$$

- Invariant under Möbius reparametrization : new solution generating map [**JBA**, Uzan '24]

$$\mathrm{Sch}[M \circ f] = \mathrm{Sch}[f] \qquad \text{where} \qquad M(u) = \frac{au+b}{cu+d} \qquad ad - bc \neq 0\,, \tag{9}$$

- Simplest solutions:

$$F(u) = \frac{au+b}{cu+d}\,, \quad G(u) = \frac{\tilde{a}u+\tilde{b}}{\tilde{c}u+\tilde{d}} \qquad \rightarrow \qquad A_{11}(u) = (cu+d)^2\,, \quad A_{22}(u) = (\tilde{c}u+\tilde{d})^2\,. \tag{10}$$

Conformal symmetry of the Raychaudhuri equation

- Non-linear field equation

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- For polarized vacuum GPW, the Raychaudhuri equation recasts into a balance equation between the Schwarzian derivatives of the two field (F, G) related to the components of the wave-profile
- Provide a solution generating map for the wave profiles
- Large freedom in the choice of wave-profile solving the Einstein equation
- Symmetry reduced version of a more general structure found later for any null hypersurfaces [Ciambelli, Leigh, Freidel '24]

Explicit and hidden symmetries of pp-waves
(for the geodesic motion)

Conformal isometries of pp-waves

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- Consider first the conformal killing equations:

$$\mathcal{L}_\xi g_{\mu\nu} = \Omega g_{\mu\nu} \,,$$

which splits into

$$\begin{aligned} \partial_v \xi^u &= 0 \,, \\ \partial_u \xi^v &= 0 \,, \\ \partial_i \xi^u + A_{ij} \partial_v \xi^j &= 0 \,, \\ \partial_i \xi^v + A_{ij} \partial_u \xi^j &= 0 \,, \\ \partial_v \xi^v + \partial_u \xi^u &= \Omega \,, \\ \xi^u \partial_u A_{ij} + A_{ik} \partial_j \xi^k + A_{jk} \partial_i \xi^k &= \Omega A_{ij} \,. \end{aligned}$$

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$$\mathcal{B}_+^\alpha \partial_\alpha = H^{xx}(u_0, u)\partial_x + H^{xy}(u_0, u)\partial_y - x\partial_v$$
$$\mathcal{B}_-^\alpha \partial_\alpha = H^{yy}(u_0, u)\partial_y + H^{yx}(u_0, u)\partial_x - y\partial_v$$

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$$H^{ij}(u_0, u) \equiv \int_{u_0}^{u} A^{ij}(w)\mathrm{d}w$$

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- In a more condensed form, the KV are given by

$$\mathcal{L}_\xi g_{\mu\nu} = 0 \qquad \xi^\alpha \partial_\alpha = \left\{ h + b_i x^i \right\} \partial_v + \left[\chi^i + b_j H^{ij}(u_0, u) \right] \partial_i \,.$$

with 5 parameters (h, χ^i, b_i)

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- Does the pp-wave admit more symmetries ? Hidden symmetries ?

Conformal isometries of pp-waves

- Consider the CVK: $\Omega \neq 0$
- If we look for a CKV for any A_{ij}, only one solution: HKV for $\Omega = 2$

$$Z^{\alpha}\partial_{\alpha} = 2v\partial_v + x^i\partial_i \,. \tag{12}$$

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Hidden symmetry of pp-waves

- Koutras theorem:
  if a spacetime admits both a gradient Killing vector and a HKV then it also admits a non-trivial rank-2 Killing tensor (KT) which generates an additional symmetry.
  With the gradient KV and the HKV given by

$$\xi_\alpha \mathrm{d}x^\alpha = (\partial_\alpha \Phi)\mathrm{d}x^\alpha \qquad Z_\alpha \mathrm{d}x^\alpha \tag{14}$$

  the KT is explicitely given by

$$K_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \left[Z_{(\mu}\xi_{\nu)} - \Phi g_{\mu\nu}\right]\mathrm{d}x^\mu \mathrm{d}x^\nu . \tag{15}$$

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Killing tensor for pp-waves

- For the pp-wave, we have a gradient KV and a HKV given by

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- All these symmetries hold for any wave-profile $A_{ij}(u)$!
- Can be use to integrate the geodesic motion ... and the geodesic deviation equation

Integrating the geodesic motion from the symmetries

- Lagrangian

$$L = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \dot{u}\dot{v} + \frac{1}{2} A_{ij} \dot{x}^i \dot{x}^j \tag{18}$$

gives the equations

$$\ddot{v} - \frac{1}{2} A'_{ij} \dot{x}^i \dot{x}^j = 0\,, \qquad A_{ij} \ddot{x}^j + A'_{ij} \dot{x}^j \dot{u} = 0\,, \qquad \ddot{u} = 0\,, \tag{19}$$

with a prime refering to a derivative w.r.t. u.

- Phase space of geodesic motion

$$\{v, p_v\} = \{u, p_u\} = 1\,, \qquad \{x^i, p_j\} = \delta^i{}_j\,. \tag{20}$$

with momenta

$$p_u = \frac{\delta \mathcal{L}}{\delta \dot{u}} = \dot{v}\,, \tag{21}$$

$$p_i = \frac{\delta \mathcal{L}}{\delta \dot{x}^i} = A_{ij} \dot{x}^j\,, \tag{22}$$

$$p_v = \frac{\delta \mathcal{L}}{\delta \dot{v}} = \dot{u}\,, \tag{23}$$

and hamltonian

$$H = p_u p_v + \frac{1}{2} A^{ij} p_i p_j = \frac{\epsilon}{2}\,. \quad \epsilon = \{0, -1\} \tag{24}$$

- What are the conserved charges ?

Algebraic integration of the geodesic flow

- Conserved charges

$$\xi^\alpha \partial_\alpha \to \mathcal{O} = \xi^\alpha p_\alpha \qquad K_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \to K = K^{\mu\nu} p_\mu p_\nu \tag{25}$$

- Translations: Since the Hamiltonian does not depend on neither v nor x^i, p_v and p_i are automatically conserved. We denote them as

$$\mathcal{N} = p_v\,, \qquad P_+ = p_x\,, \qquad P_- = p_y\,, \tag{26}$$

- Carrolian boost: The conserved charges generating the boosts are given by

$$\mathcal{B}_+ = H^{xx}(u)p_x + H^{xy}(u)p_y - p_v x(u)\,, \tag{27}$$

$$\mathcal{B}_- = H^{yy}(u)p_y + H^{yx}(u)p_x - p_v y(u)\,. \tag{28}$$

- Hidden Killing tensor charge: charge coming from the Killing tensor reads

$$\mathcal{K} = K^{\mu\nu} p_\mu p_\nu = p_v \mathcal{Z} - 2uH \qquad \text{with} \qquad \mathcal{Z} = 2p_v v + p_i x^i\,, \tag{29}$$

- Charge algebra

$$\{P_\pm, \mathcal{B}_\pm\} = \mathcal{N}\,. \tag{30}$$

$$\{P_\pm, \mathcal{K}\} = \mathcal{N} P_\pm\,, \qquad \{B_\pm, \mathcal{K}\} = \mathcal{N} B_\pm\,, \qquad \{\mathcal{N}, \mathcal{K}\} = 2\mathcal{N}^2\,, \tag{31}$$

- Geodesic motion integrable since $(\mathcal{N}, \mathcal{B}_\pm, H)$ are in involution

Algebraic integration of the geodesic flow

- Trivial equation for $\dot{u}$:

$$u = p_v \tau = \mathcal{N}\tau. \tag{32}$$

- Transverse motion: Carrolian boosts allows to write the x and y trajectories as

$$x(u) = \frac{1}{\mathcal{N}}\left[H^{xx}(u_0, u)P_+ + H^{xy}(u_0, u)P_-\right] - \frac{\mathcal{B}_+}{\mathcal{N}}\,, \tag{33}$$

$$y(u) = \frac{1}{\mathcal{N}}\left[H^{yy}(u_0, u)P_- + H^{yx}(u_0, u)P_+\right] - \frac{\mathcal{B}_-}{\mathcal{N}}\,, \tag{34}$$

- Longitudinal motion: Combining the HKV and the KT charge give v-trajectory

$$v(u) = \frac{1}{2\mathcal{N}^2}\left[\epsilon u - H^{ij}(u_0, u)p_i p_j + p_i \mathcal{B}^i + \mathcal{K}\right]. \tag{35}$$

- Relations between initial conditions and conserved charges

$$v_0 \equiv v(u_0) = \frac{1}{2\mathcal{N}^2}\left[\epsilon u_0 - \mathcal{N} p_i x_0^i + \mathcal{K}\right]. \qquad x_0^i \equiv x^i(u_0) = -\frac{\mathcal{B}^i}{\mathcal{N}}\,. \tag{36}$$

- More important for the memories, the x-trajectory reduces to

$$\boxed{x^i(u) = \frac{1}{\mathcal{N}}\left[H^{ij}(u_0, u)p_j - \mathcal{B}^i\right]} \tag{37}$$

Properties of a timelike geodesic congruence

- With the exact solution to the geodesic, we have the 4-velocity

$$u^u = \frac{\mathrm{d}u}{\mathrm{d}\tau} = \mathcal{N} ,$$
$$u^i = \frac{\mathrm{d}x^i}{\mathrm{d}\tau} = A^{ij} p_j ,$$
$$u^v = \frac{\mathrm{d}v}{\mathrm{d}\tau} = \frac{1}{2\mathcal{N}} \left(\epsilon - A^{ij} p_i p_j \right) . \tag{38}$$

- We can compute the invariant quantities: expansion, shear and rotation

$$\Theta = \nabla_\mu u^\mu = \mathcal{N} \varrho \tag{39}$$
$$\sigma = \sigma_{\mu\nu}\sigma^{\mu\nu} = -\mathcal{N}^2 \left[\dot{A}^{ij}\dot{A}_{ij} + \frac{2}{3}\varrho(\varrho + \dot{A}_{ij}p^i p^j) - \frac{1}{9}\varrho^2 (p_i p^i)^2 \right] , \tag{40}$$
$$\omega_{\alpha\beta} = \nabla_{[\alpha} u_{\beta]} = 0 \tag{41}$$

with $\varrho = A^{ij}\dot{A}_{ij}$

- $u^\alpha \partial_\alpha$ is hyper-surface orthogonal and the effect of the wave is to expand/contract and shear the congruence of geodesics

Constructing the Fermi coordinates

- Geodesic deviation cannot be analyzed in an arbitrary coordinates system
- First task, construct the adapted Fermi normal coordinates (either null or timelike)
- For null geodesic, the procedure is closely related to the Penrose limit [Penrose '76]

Adapted Fermi normal coordinates

- Pick up a null geodesic $\bar{\gamma}$ with tangent vector

$$\bar{u}^\mu \partial_\mu = \partial_u \qquad \bar{u}_\mu \mathrm{d}x^\mu = \mathrm{d}v\,. \tag{42}$$

- Introduce a set of adapted Fermi coordinates, X^I, with $I \in \{0, \ldots, 3\}$ related to the initial coordinates x^μ via

$$E^I{}_\mu \equiv \frac{\partial X^I}{\partial x^\mu}\,. \tag{43}$$

- Choose the "time-leg" such that the coordinate X^0 coincides with the affine parameter of the geodesic

$$\bar{E}^0{}_\mu \mathrm{d}x^\mu = \bar{u}_\mu \mathrm{d}x^\mu\,. \tag{44}$$

  Impose that i) the remaining legs be parallel transported along the null geodesic, and ii) the orthogonality relations

$$\bar{u}^\mu \nabla_\mu E^I{}_\nu = 0 \qquad g_{\mu\nu}|_{\bar{\gamma}} = \bar{E}^I{}_\mu \bar{E}^J{}_\nu \eta_{IJ} \tag{45}$$

- In our case, one obtains

$$\dot{E}^A{}_i = \frac{1}{2} A^{jk} \dot{A}_{ki} E^A_j\,, \qquad \dot{E}^i{}_A = -\frac{1}{2} A^{ik} \dot{A}_{kj} E^j{}_A \tag{46}$$

- Analyzing the effects in the Fermi coordinates requires to analytically solve this PT equation

- Fermi coordinates : $X^0 = U$, $X^1 = V$ and $X^A = \{X, Y\}$ such that the transverse space w.r.t. the reference geodesic admits the coordinates $X^a \in \{V, X^A\}$.
- Then, the Taylor expansion of the BJR coordinates $x^\mu(U, V, X^A)$ up to second order reads

$$x^\mu(U, V, X^A) = x^\mu(U) + \bar{E}^\mu{}_a(U) X^a - \frac{1}{2} \bar{E}^\alpha{}_a(U) \bar{E}^\beta{}_b(U) \Gamma^\mu{}_{\alpha\beta}(U) X^a X^b \,. \tag{47}$$

- It follows that the Fermi and BJR coordinates are related by

$$u = U \,, \tag{48}$$

$$v = V + \frac{1}{4} \dot{A}_{ij} \bar{E}^i{}_A \bar{E}^j{}_B X^A X^B \,, \tag{49}$$

$$x^i = \bar{E}^i{}_A X^A \,. \tag{50}$$

- Ssetting $V = 0$ and $X^A = 0$, one recovers the position of the reference geodesic $\bar{\gamma}$.
- The vacuum GPW metric becomes in the Fermi coordinates

$$\boxed{\mathrm{d}s^2 = 2\mathrm{d}U\mathrm{d}V + \delta_{AB} \mathrm{d}X^A \mathrm{d}X^B + H_{AB}(U) X^A X^B \mathrm{d}U^2 \,,} \tag{51}$$

with the wave-profile

$$H_{AB} = \frac{1}{2} E^i{}_A \, \partial_u \left(\dot{A}_{ij} E^j{}_B \right) \,. \tag{52}$$

- Known as the Brinkmann coordinates which are the one use to analyze the physical effects

A quick look at Einstein equation in the Brinkmann form

- The vacuum GPW metric becomes in the Fermi coordinates

$$\mathrm{d}s^2 = 2\mathrm{d}U\mathrm{d}V + \delta_{AB}\mathrm{d}X^A\mathrm{d}X^B + H_{AB}(U)X^AX^B\mathrm{d}U^2\,, \tag{53}$$

with the wave-profile

$$H_{AB} = \frac{1}{2}E^i{}_A\,\partial_u\left(\dot{A}_{ij}E^j{}_B\right)\,. \tag{54}$$

- Einstein equation translates into

$$H^A{}_A = 0 \qquad \to \qquad H_{AB} = \begin{pmatrix} H_+ & H_\times \\ H_\times & -H_+ \end{pmatrix} \tag{55}$$

- Any profile $(H_+, H_\times)$ is solution of Einstein equation
- Relation to initial description for the polarized wave-profile

$$H_+ = \frac{1}{2}\left[\frac{\ddot{A}_{11}}{A_{11}} - \frac{1}{2}\left(\frac{\dot{A}_{11}}{A_{11}}\right)^2\right] = -\frac{1}{2}\left[\frac{\ddot{A}_{22}}{A_{22}} - \frac{1}{2}\left(\frac{\dot{A}_{22}}{A_{22}}\right)^2\right]. \tag{56}$$

$$= \mathrm{Sch}[F] = -\mathrm{Sch}[G] \tag{57}$$

or with $A_{ii}(u) \equiv \mathcal{A}_i^2(u)$

$$H_+ = \frac{\ddot{\mathcal{A}}_1}{\mathcal{A}_1} = -\frac{\ddot{\mathcal{A}}_2}{\mathcal{A}_2} \tag{58}$$

- For a given example, either choose H_+ and determine (A_{11}, A_{22}) or the reverse: not so easy in practice

Geodesic deviation from the symmetries

Geodesic deviation

- Consider two nearby curves $\bar{X}^\mu(\tau) := X^\mu(\tau, \sigma = 0)$ and $X^\mu(\tau, \sigma)$ and their relative distance in the Fermi frame

$$\Delta X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma) - X^\mu(\tau,0)$$
$$= \zeta^\mu(\tau,\sigma) - \bar{\Gamma}^\mu{}_{\alpha\beta}\zeta^\alpha(\tau,\sigma)\zeta^\beta(\tau,\sigma) + \mathcal{O}(\sigma^3) \qquad \zeta^\mu = \sigma N^\mu + \mathcal{O}(\sigma^2)\,. \tag{59}$$

- $\zeta^\mu(\tau,\sigma)$ understood as a euclidean distance only in the Fermi coordinates where $\bar{\Gamma}^\mu{}_{\alpha\beta} = 0$
- Deviation vector satisfies the following dynamics

$$\boxed{\frac{D^2 N^\mu}{\mathrm{d}\tau^2} = \bar{R}^\mu{}_{\alpha\beta\gamma}\bar{u}^\alpha \bar{u}^\beta N^\gamma} \tag{60}$$

- Split parallel and orthogonal contributions to the deviation: same for memories

$$N = N_\parallel + N_\perp \qquad N_\parallel = f(\tau)\bar{u} \qquad f(\tau) = C_1\tau + C_2\,. \tag{61}$$

- The remaining equation is hard to solve in general
- The solution space related to the explicit and hidden symmetries of the spacetime: The Caviglia, Zordan, and Salmistraro theorem [Caviglia, Zordan, and Salmistraro '82]

Integrability of the geodesic deviation equation (GDE)

- Crucial link with hidden symmetries of spacetime: CSZ theorem
- Consider any affine tensor:

$$\nabla_{(\mu} K_{\alpha_1 \ldots \alpha_p)} = h_{(\alpha_1 \ldots \alpha_p)} \qquad \nabla_\rho h_{(\alpha_1 \ldots \alpha_p)} = 0 \tag{62}$$

- When $h_{(\alpha_1 \ldots \alpha_p)} = 0$, one has a killing tensor ... different only for Petrov type N [Collins '85]
- Any affine rank-p affine tensor provides an exact solution of the GDE

$$\boxed{N^\mu = \bar{K}^\mu{}_{\alpha_1 \ldots \alpha_p} \bar{u}^{\alpha_1} \ldots . \bar{u}^{\alpha_p}} \tag{63}$$

[Caviglia, Zordan, and Salmistraro '82]

- Suppose we have killing vectors and rank-2 KT, then the deviation vector solution space reads

$$N^\mu = \{\bar{\xi}^\mu, \bar{K}^\mu{}_\nu \bar{u}^\nu\} \tag{64}$$

- For radiative spacetime, provides another way to link memories to (hidden) symmetries
- Apply this for pp-wave geometry

Geodesic deviation vector

- With the Fermi coordinates $X^0 = U$, $X^1 = V$ and $X^A = \{X, Y\}$ at hand, we can analyze the geodesic deviation
- The position and the velocity of the test particle are given in Brinkmann coordinates by

$$X^A = E^A{}_i \left(H^{ij} p_j - \mathcal{B}^i \right) , \tag{65}$$

$$\dot{X}^A = \dot{E}^A{}_i \left(H^{ij} p_j - \mathcal{B}^i \right) + E^A{}_i A^{ij} p_j \, . \tag{66}$$

- Parametrized by the two charges $(p_i, \mathcal{B}^i)$: 2d translations and carrolian boosts
- Consider the null reference geodesic with trajectory $\bar{X}^\mu$ and a second arbitrary test particle X^μ

$$\boxed{\zeta^A \equiv Y^A - \bar{X}^A \, ,} \tag{67}$$

encodes their relative distance

- We want to analyze

$$\zeta^A(u) = \zeta^i(u) E^A{}_i(u_0) \qquad \text{with} \qquad \boxed{\zeta^i(u) = \mathcal{A}_i(u) \left[H^{ii}(u_0, u) p_i - \mathcal{B}^i \right] \, .} \tag{68}$$

- Its dynamics satisfies the geodesic deviation equation

$$\ddot{\zeta}_A = R_{AUUB} \zeta^B \tag{69}$$

$$= R_{iuuj} E^i{}_A E^j{}_B \zeta^B \tag{70}$$

$$= \frac{1}{2} \left(\ddot{A}_{ij} - \frac{1}{2} A^{km} \dot{A}_{ki} \dot{A}_{mj} \right) E^i{}_A E^j{}_B \zeta^B \, . \tag{71}$$

Classification of memory effects

Classification of memory effects

The three different types of memory effects

- Consider situations for which asymptotycally, i.e. for $u < u_0$ and $u > u_f$, one has

$$\zeta^i \neq 0 \qquad \text{and} \qquad \ddot{\zeta}^i = 0\,. \tag{72}$$

- **Velocity Memory (VM)**:
  When the relative velocity in the two asymptotic regions satisfies

$$\Delta\dot{\zeta} = \dot{\zeta}_f - \dot{\zeta}_0 \neq 0 \tag{73}$$

  $\rightarrow$ constant shift on the asymptotic value of the relative velocity.

- **Vanishing Velocity Memory (VM0)**: Subcase corresponding to

$$\Delta\dot{\zeta} = \dot{\zeta}_f - \dot{\zeta}_0 = 0 \tag{74}$$

  $\rightarrow$ no velocity memory but still interesting effects on the couple of test particles.

- **Displacement Memory (DM)**: subcase such that

$$\dot{\zeta}_f = 0 \qquad \text{and} \qquad \zeta_f \neq \zeta_0 \tag{75}$$

  $\rightarrow$ the relative velocity vanishes in the asymptotic future

Classification of memory effects

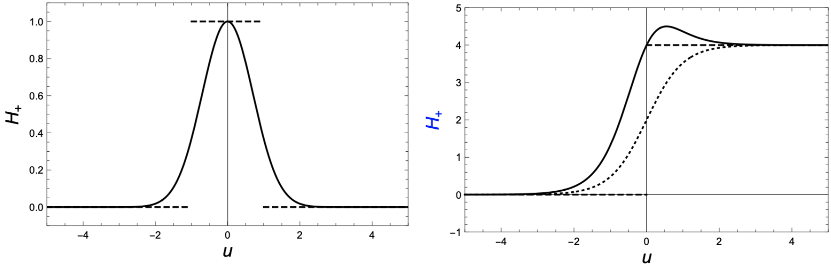
A brief look at the conclusions so far

- First work to analyze the memory effects in vacuum gravitational plane wave [Zhang, Duval, Gibbons, Horvathy '17 '18]
- VGPW only exhibits velocity memory effects / displacement memory effect can never occur

- Very recently, two numerical examples where a displacement occurs have been presented in [Zhang, Horvathy '24]
- No analytic conditions were provided leaving the question of the classification of the conditions to have a velocity versus a displacement memory open

$\rightarrow$ complement this finding by providing a full classification of the conditions for the different memories

Pulse versus step profiles



- So far, only pulse profiles have been studied
- Full classification holds for both:
  very different conditions to realize memories depending on the nature of the wave
- Step profiles are more adapted to model realistic memory contributions [Favata '19, '20]

Memory effects for pulse profiles

- Focus on the polarized case: $\mathcal{A}_{12} = 0$
- To analyze the memory, we need the dynamics of the geodesic deviation vector

$$\dot{\zeta}^i(u) = \frac{\dot{\mathcal{A}}_i(u)}{\mathcal{A}_i(u)}\zeta^i(u) + \frac{p_i}{\mathcal{A}_i(u)}\,, \tag{76}$$

$$\ddot{\zeta}^i(u) = \frac{\ddot{\mathcal{A}}_i(u)}{\mathcal{A}_i(u)}\zeta^i(u). \tag{77}$$

Translates into a general condition valid at all u

$$\boxed{p_i = \dot{\zeta}_i\mathcal{A}_i - \dot{\mathcal{A}}_i\zeta_i} \tag{78}$$

- Asymptotic behavior of the pulse profile:

$$H_+(u) = \frac{\ddot{\mathcal{A}}_1}{\mathcal{A}_1} = -\frac{\ddot{\mathcal{A}}_2}{\mathcal{A}_2} = 0 \qquad \text{for} \qquad u < u_0 \qquad u > u_f \tag{79}$$

- Asymptotic behavior of the relative acceleration

$$\ddot{\mathcal{A}}_i = 0 \qquad \rightarrow \qquad \ddot{\zeta}^i(u) = 0 \qquad \text{for} \qquad u < u_0 \qquad u > u_f \tag{80}$$

- Compute the asymptotic form of the relative distance and velocity in terms of the initial conditions $(p_i, \mathcal{B}_i)$ and the asymptotic wave form $(\mathcal{A}_0, \mathcal{A}_f)$.

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- **Asymptotic past:** for $u < u_0$, one has

$$\mathcal{A}(u) = \dot{\mathcal{A}}_0(u - u_0) + \mathcal{A}_0 \,, \tag{81}$$

$$H(u, u_0) = \int_{u_0}^{u} \frac{1}{\mathcal{A}^2(u)} = \frac{u - u_0}{\mathcal{A}_0\mathcal{A}(u)} \qquad \text{such that} \qquad \mathcal{A}H(u, u_0) = \frac{u - u_0}{\mathcal{A}_0} \,, \tag{82}$$

$$\zeta(u) = \dot{\zeta}_0(u - u_0) + \zeta_0 \tag{83}$$

- **Asymptotic future:** for $u > u_f$, one has

$$\mathcal{A}(u) = \dot{\mathcal{A}}_f(u - u_f) + \mathcal{A}_f \,, \tag{84}$$

$$H(u_0, u) = \int_{u_0}^{u_f} \frac{1}{\mathcal{A}^2(u)} + \int_{u_f}^{u} \frac{1}{\mathcal{A}^2(u)} = H_{0f} + \frac{u - u_f}{\mathcal{A}_f\mathcal{A}(u)} \,. \tag{85}$$

$$\zeta(u) = \dot{\zeta}_f(u - u_f) + \zeta_f \,. \tag{86}$$

- Express the different asymptotic quantities in terms of the initial conditions $(p_i, \mathcal{B}^i)$

$$\boxed{\zeta_0 = -\mathcal{B}\mathcal{A}_0 \,, \qquad \dot{\zeta}_0 = \frac{p}{\mathcal{A}_0} - \mathcal{B}\dot{\mathcal{A}}_0} \tag{87}$$

$$\boxed{\zeta_f = \mathcal{A}_f(H_{0f}\,p - \mathcal{B}) \qquad \dot{\zeta}_f = \frac{p}{\mathcal{A}_f} + \dot{\mathcal{A}}_f(H_{0f}\,p - \mathcal{B})} \,, \tag{88}$$

Classification of memories for pulse profiles

- A **VM** occurs under the condition $\dot{\zeta}_f \neq \dot{\zeta}_0$.
  Since pulses profiles enjoy a constant asymptotic velocities

$$\ddot{A}_i = 0 \qquad \text{for} \qquad u < u_0 \qquad u > u_f \tag{89}$$

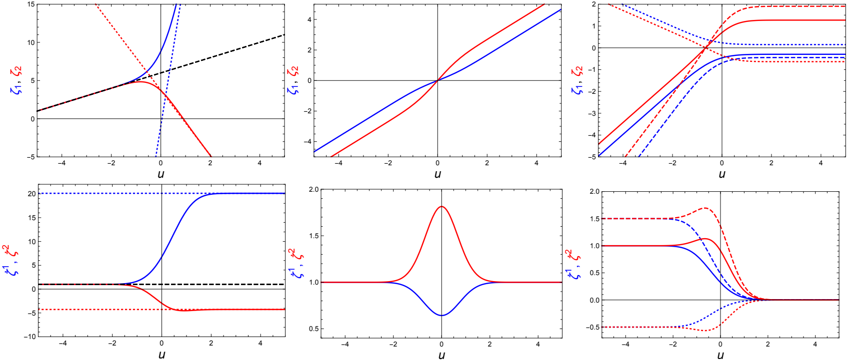
  they generically lead to a constant VM whatever the profile $H_+(u)$ and the constants of motions $(\mathcal{B}, p)$ or, similarly, whatever the initial conditions $(\zeta_0, \dot{\zeta}_0)$.
  This is a generic properties of any pulse.
- A **VM0** occurs in the special cases in which $\dot{\zeta}_f = \dot{\zeta}_0$. This translates into

$$\boxed{(\dot{\mathcal{A}}_f - \dot{\mathcal{A}}_0)\mathcal{B} = -p\left[\frac{\mathcal{A}_f - \mathcal{A}_0}{\mathcal{A}_f \mathcal{A}_0} - \dot{\mathcal{A}}_f H_{0f}\right]} \,. \tag{90}$$

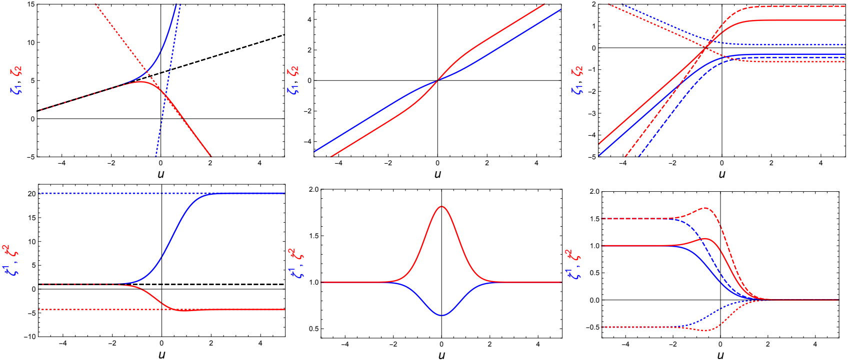
- A **DM** occurs when $\dot{\zeta}_f = 0$.
  This translates into a specific tuning between the two constants of motion,

$$\boxed{\dot{\mathcal{A}}_f \mathcal{B} = -p\left(\frac{1}{\mathcal{A}_f} + \dot{\mathcal{A}}_f H_{0f}\right)} \,. \tag{91}$$

- Lead to a finer classification depending on $\dot{\mathcal{A}}_f = \dot{\mathcal{A}}_0$ or $\dot{\mathcal{A}}_f \neq \dot{\mathcal{A}}_0$
- Let us see some examples.



- Profiles of the relative displacement (ζ_1, ζ_2) (upper line) and relative velocity $(\dot{\zeta}_1, \dot{\zeta}_2)$ (lower line) for $H_+ = \mathrm{e}^{-u^2}$ with initial conditions that ensures $\dot{\mathcal{A}}_f - \dot{\mathcal{A}}_0 \neq 0$.



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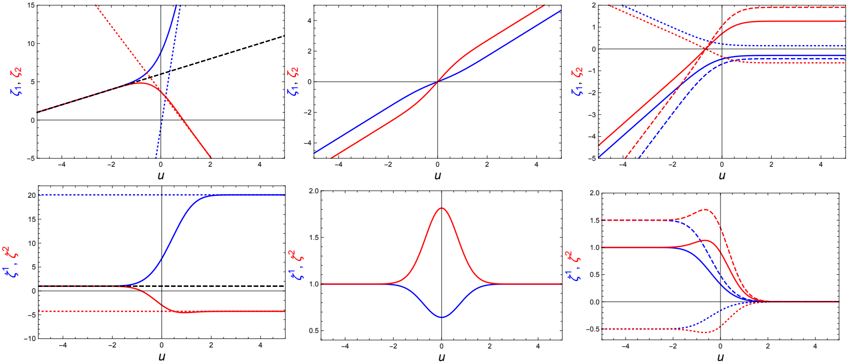
- Left panel: assume $p_i = (1, 1)$ and $\mathcal{B}_i = (-1, -1) \to$ clear non vanishing VM, so that $\dot{\zeta}_f \neq \dot{\zeta}_0$. Projected motion. Longitudinal position can be different so no colliding trajectories.



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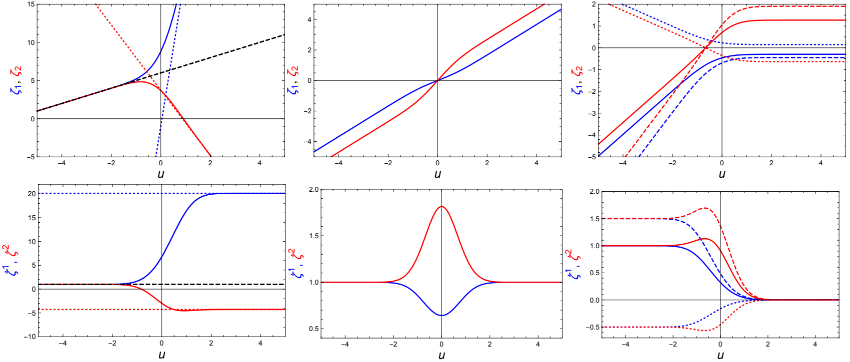

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- Middle: $p_i \neq (0,0)$ and B_i is tuned to get a VM0 $\to$ a vanishing VM for which $\dot{\zeta}_f = \dot{\zeta}_0$.
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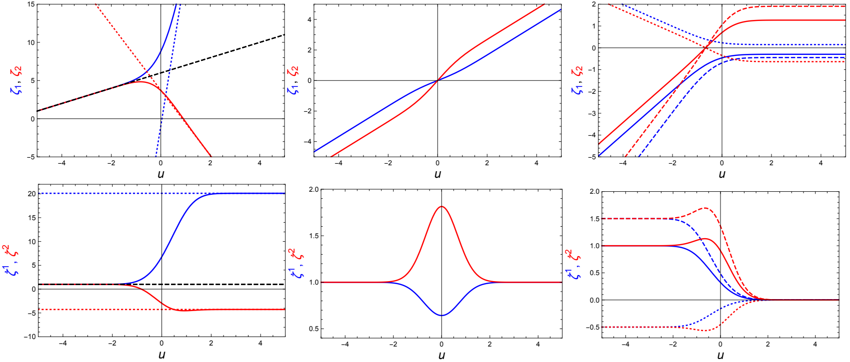

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New type of memory (middle column) simply switches the projected position of the two particles !

Memory effects for step profiles

Step profile

- **VM**: A wave with a step profile generically leads to a rather surprising velocity memory effect where

$$\dot{\zeta}_- \neq 0 \qquad \text{while} \qquad \dot{\zeta}_+ = 0$$

The wave cancels the relative motion, i.e. the relative velocity, between the two particles.

- **VM0**: It occurs when one further has $\dot{\zeta}_- = 0$, which requires $\frac{p}{\mathcal{A}_0} = \mathcal{B}\dot{\mathcal{A}}_0$. This means that a VM0 requires that

$$\boxed{\frac{1}{\mathcal{A}_0\dot{\mathcal{A}}_0} = H_{0f} + \frac{1}{\mathcal{A}_f(\dot{}_f + \lambda_f\mathcal{A}_f)},} \tag{92}$$

which is a non-trivial property of the spacetime geometry.

- **DM**: A displacement memory effect requires

$$\dot{\zeta}_+ = \dot{\zeta}_- = 0 \qquad \text{while} \qquad \zeta_+ \neq \zeta_-$$

This implies that one needs $\dot{\mathcal{A}}_+ \neq \dot{\mathcal{A}}_-$ together with the constraint (92).

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Next goals

- Study the memories of extended quadrupolar bodies described by Dixon's theory

$$\frac{Dp^\mu}{\mathrm{d}\tau} = -\frac{1}{2} R^\mu{}_{\nu\alpha\beta} v^\nu S^{\alpha\beta} - \frac{1}{6} J^{\alpha\beta\gamma\delta} \nabla^\mu R_{\alpha\beta\gamma\delta} \tag{93}$$

$$\frac{DS}{\mathrm{d}\tau} = 2p^{[\mu} v^{\nu]} + \frac{4}{3} R^{[\mu}{}_{\alpha\beta\gamma} J^{\nu]\alpha\beta\gamma} \tag{94}$$

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- Particle production form vacuum gravitational plane wave: quantum memory [Gibbons '75]

Thank you

Penrose limit

- Concretely, pick up a null geodesic γ and construct null Fermi coordinates $X^A = (U, V, X^i)$ with $i \in (1, 2)$ adapted to the region around the geodesic

$$X^A = E^A_a x^a + E^A_\mu \bar{\Gamma}^\mu{}_{ab} x^a x^b + \mathcal{O}((x^a)^3) \tag{95}$$

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- In the region around the geodesic γ, the gravitational field can be described as
$$\begin{aligned} \mathrm{d}s^2 &= 2\mathrm{d}U\mathrm{d}V + \delta_{ij}\mathrm{d}X^i\mathrm{d}X^j - \bar{R}_{\lambda i \lambda j}(U) X^i X^j \mathrm{d}U^2 \\ &- \frac{4}{3}\bar{R}_{\lambda jik}(U) X^j X^k \mathrm{d}U\mathrm{d}X^i - \frac{1}{3}\bar{R}_{ijk\ell}(U) X^k X^\ell \mathrm{d}X^i \mathrm{d}X^j \\ &+ \mathcal{O}(X^3) \end{aligned} \tag{96}$$

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- Organize this expansion in x^a using conformal transformation of the transverse space: $(U, V, X^i) \to (U, \lambda^2 V, \lambda X^i)$

Penrose limit

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$$\Psi_i = \mathcal{O}(\lambda^{4-i}) \qquad \text{for } i \in (0, \ldots, 4) \tag{97}$$

Penrose limit

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