The Dimer Model & Geometric Dynamics

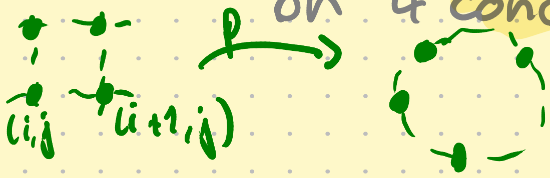
with Niklas Affolter and Béatrice de Tilière

Model space Ω with w weight $\longrightarrow$ $Z = \sum_{\sigma \in \Omega} w(\sigma)$ | partition function gener. $\longrightarrow$ Functional equation $F(Z, Z', \dots) = 0$ $\longrightarrow$ Analysis geometry

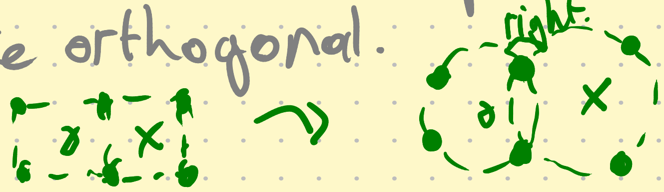
I- A geometric dynamics: Orthogonal Circle Patterns

Schramm '97: An OCP is a map $p : \mathbb{Z}^2 \to \mathbb{C}$ s.t.

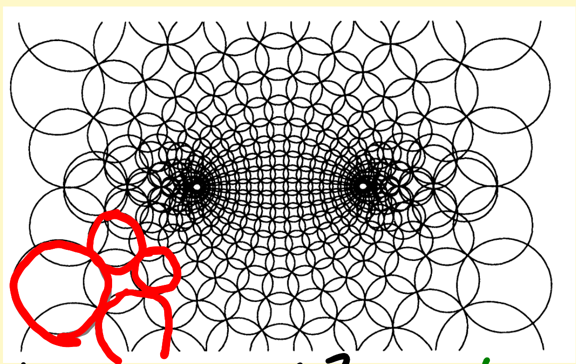
- $\forall$ unit square in $\mathbb{Z}^2$, the four vertices are sent on 4 concyclic points in $\mathbb{C}$



- Circles that correspond to adjacent squares are orthogonal.



From Schramm

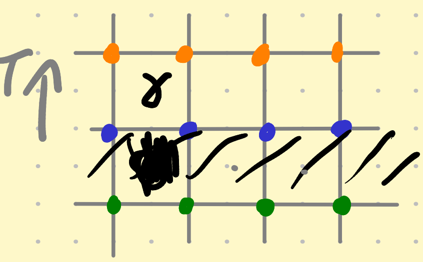


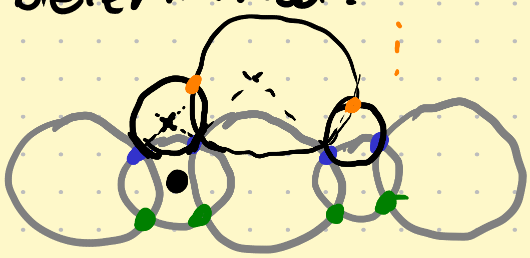
A discrete version of holomorphic functions

"Analytic continuation"? $p((1,0))$

Suppose that $P_0 = (p_{i,0})_{i \in \mathbb{Z}}$, $P_1 = (p_{i,1})_{i \in \mathbb{Z}}$.

Then the rest is determined.



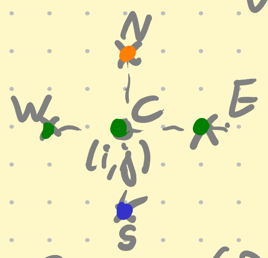


let $T : (p_0, p_1) \mapsto (p_1, p_2)$ is an operator that describes the evolution:

Dynamical properties?

II - Algebraic dynamics & P-nets
(Rational)

Def: A P-net is a map $p : \mathbb{Z}^2 \to \mathbb{C}$ s.t. $\forall (i,j) \in \mathbb{Z}^2$,

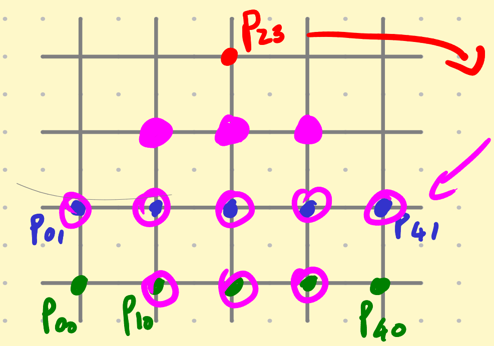


$$\frac{1}{p_N - p_C} - \frac{1}{p_W - p_C} + \frac{1}{p_S - p_C} - \frac{1}{p_E - p_C} = 0.$$

Prop: (Bobenko - Pinkall '99) If p is an OCP, then it is a P-net.

So T can be seen as a rational transfo (a rational map on $\mathbb{C}^{\mathbb{Z}} \times \mathbb{C}^{\mathbb{Z}}$)

(If p is m-periodic horizontally: $p_{i+m,j} = p_{i,j}$ then T can be seen as a map on $\mathbb{C}^m \times \mathbb{C}^m$)

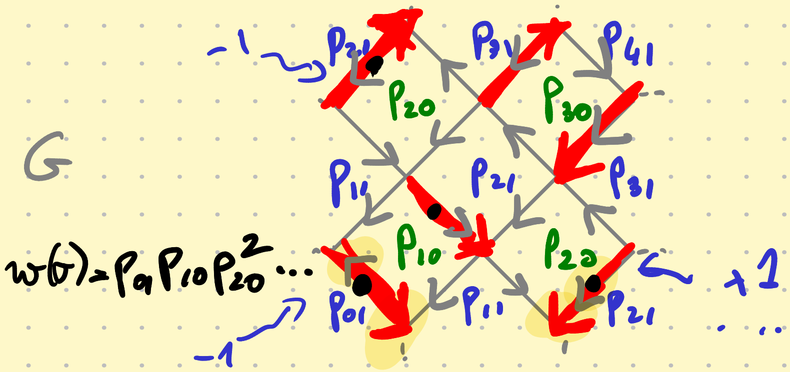


is a rational Function of

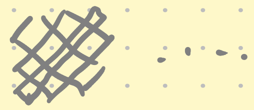
Initial data

We express now this rational function.

G



G is an Aztec Diamond of size 2 here



$\Omega = \{$oriented dimer configs on $G\}$

For $\sigma \in \Omega$, let

$$w(\sigma) := \prod_{\vec{e} \in \sigma} \varepsilon_{\vec{e}} \, P_{\text{right of } \vec{e}}$$

$\uparrow$ ± 1 depending if $\vec{e}$ agrees with a certain fixed orientation called Kasteleyn orientation

$$Z(G, P_0, P_1) := \sum_{\sigma \in \Omega} w(\sigma).$$

<u>Theo</u> (A. dT-M '23)

$\forall j \geq 2$, $\quad P_{ij} = \left(\prod_{f \in \text{Faces}(G)} P_f\right) \dfrac{Z(G, P_0^{-1}, P_1^{-1})}{Z(G, P_0, P_1)}$

where G is an A.D. of size $j-1$.

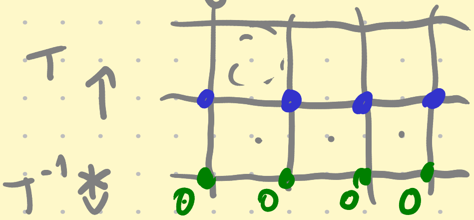
Z (for oriented dimers) is also Kasteleyn determinant: $\pm Z = \det \tilde{K}$ where

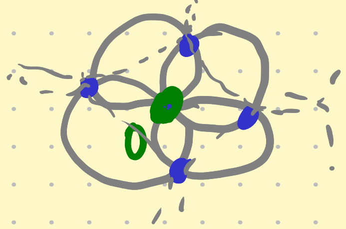
$$\tilde{K}_{w,b} = \begin{cases} P_{right} - P_{left} & \text{(w —— b, } P_{left} \text{ above, } P_{right} \text{ below)} \\ 0 & \text{if not neighb.} \end{cases}$$

III - Consequences

- Study singularities of the OCP evolution

Imagine that we want an OCP s.t. $P_0 \equiv 0$.





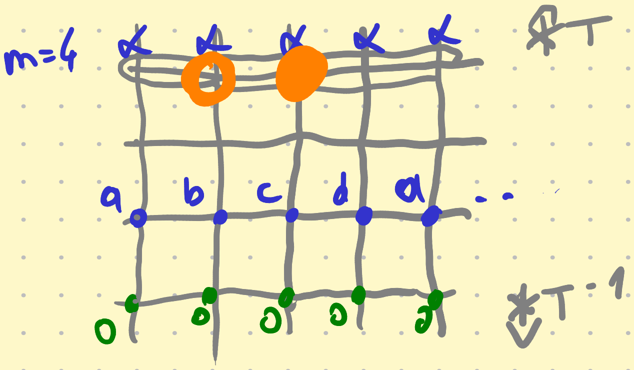
Cor $(A, \alpha T, m)$ OCP have the "Devron property":

if p is an OCP / p-net st $p_0 \equiv 0$, and

P is m-hor. periodic $(p_{i+m,j} = p_{i,j})$

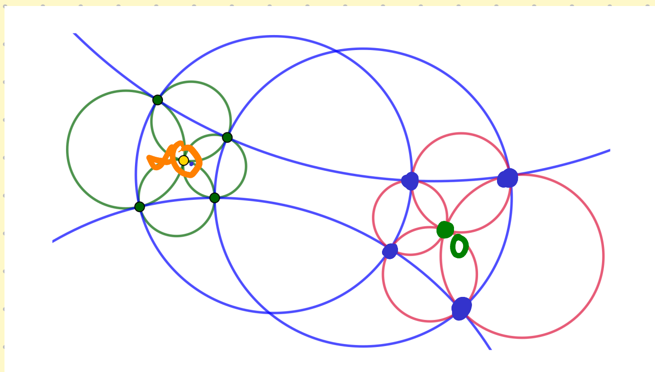
Then P_{m-1} is constant

$(\exists \alpha \,/\, \forall i,\ p_{i,m-1} = \alpha)$



Glick '15 calls "Devron prop" for a system st initial data singular for T^{-1} becomes singular for a certain T^k.

Idea: this should mean that T is integrable ...?



$$\alpha = \frac{m}{\sum_{i=0}^{m-1} p_{i,1}^{-1}}$$

- **Degree of iterates of T**

$i \geq 2$ $\quad \underline{p_{ij}} \equiv T^{j}(p_0, p_1)_i$

In general, for T rational, quantify the complexity of T using algebraic entropy:

$$\frac{1}{n} \log(\underbrace{\deg(T^n)}_{\text{max exponent of the numerator / denominator in } T^n}) \xrightarrow[n \to \infty]{} e(T)$$

That is, $\deg(T^n) = \exp(e(T)\, n + o(n))$

In our case, $\deg(T^n)$ is the number of dimers in an A.D of size $n \pm 1$?
$\sim n^2$.

There has to be many cancellations.
(in particular, $e(T) = 0$)

$T : (x, y) \mapsto \left(\frac{P(x,y)}{Q(x,y)}, \frac{R}{S} \right)$ $\deg T^n \sim n^2$.