Fredholm Determinants from Schrödinger-type Equations, and Deformation of Tracy-Widom Distribution

Taro Kimura ♠ 木村太郎

Institut de Mathématiques de Bourgogne, Université de Bourgogne

Joint work with X. Navand (IMB) [2408.06888]

Introduction: Random Matrix Theory

- **Gaussian Unitary Ensemble (GUE)**: Gaussian distribution of Hermitian random matrices

$$\mathsf{P}_{\text{matrix}}(M) = \frac{1}{Z_N} \mathrm{e}^{-\frac{1}{2}\operatorname{tr} M^2} \xrightarrow{\text{diagonalization}} \mathsf{P}_{\text{ev}}(x) = \frac{1}{\tilde{Z}_N} \prod_{i=1}^{N} \mathrm{e}^{-\frac{1}{2}x_i^2} \prod_{1 \le i < j \le N} (x_i - x_j)^2$$

with $M = (M_{ij})_{1 \le i,j \le N}$, $M_{ji} = \overline{M}_{ij}$, and eigenvalues $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$

- **Determinantal point process**: Correlation functions given by determinants of the kernel

$$\rho_k(x_1, \ldots, x_k) = \det_{1 \le i,j \le k} K_N(x_i, x_j) , \qquad K_N(x,y) = \sum_{n=0}^{N-1} \frac{\mathrm{e}^{-\frac{1}{4}(x^2+y^2)}}{\sqrt{2\pi} n!} H_n(x) H_n(y)$$

with $\{H_n\}_{n \in \mathbb{Z}_{\ge 0}}$ Hermite polynomials, $\int_{\mathbb{R}} \mathrm{e}^{-\frac{1}{2}x^2} H_n(x) H_m(y) \, \mathrm{d}x = \sqrt{2\pi} n! \delta_{n,m}$

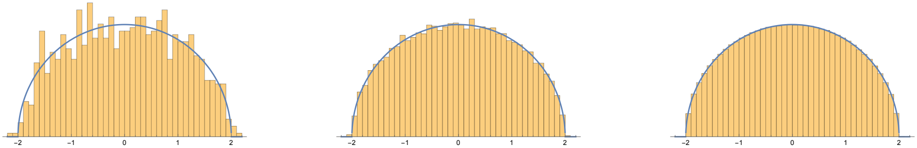
- **Gap probability**: Probability to find no points in the interval $I \subseteq \mathbb{R}$

$$\mathbb{P}[\text{no points in } I] = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_I \cdots \int_I \det_{1 \le i,j \le k} K_N(x_i, x_j) \, \mathrm{d}x_1 \cdots \mathrm{d}x_k = \det(1 - \mathcal{K}_N)_{L^2(I)}$$

with an integral operator acting on $L^2(I)$ defined by $\mathcal{K}_N : f \mapsto \int_I K_N(\cdot, x) f(x) \, \mathrm{d}x$

Large N limit

- **Wigner's semi-circle law**: Eigenvalue density $\frac{\rho_1(\sqrt{N}x)}{\sqrt{N}} \xrightarrow{N\to\infty} \frac{1}{2\pi}\sqrt{4-x^2}\,\mathbb{1}_{[-2,2]}$



$N = 10$ $N = 100$ $N = 1000$

- **Edge scaling limit**: Airy function $(\partial_x^2 - x)\,\mathrm{Ai}(x) = 0$

$$\lim_{N\to\infty} \frac{1}{N^{\frac{1}{6}}} K_N\left(2\sqrt{N} + \frac{x}{N^{\frac{1}{6}}}, 2\sqrt{N} + \frac{y}{N^{\frac{1}{6}}}\right) = \int_{\mathbb{R}_+} \mathrm{Ai}(x+z)\,\mathrm{Ai}(y+z)\,\mathrm{d}z =: K_{\mathsf{Ai}}(x,y)$$

- **Tracy–Widom distribution**: Largest eigenvalue distribution

$$\lim_{N\to\infty} \mathbb{P}\left[(x_{\mathsf{max}} - 2\sqrt{N})N^{\frac{1}{6}} \le s\right] = \det(1-\mathcal{K}_{\mathsf{Ai}})_{L^2([s,\infty))} = \exp\left(-\int_s^\infty (x-s)\mathsf{q}(x)^2\,\mathrm{d}x\right)$$

where $\mathsf{q}'' = x\mathsf{q} + 2\mathsf{q}^3$ with $\mathsf{q}(x) \xrightarrow{x\to\infty} \mathrm{Ai}(x)$ **(Hastings–McLeod solution to Painlevé II)**

Tracy–Widom distribution

- **Universality**: Tracy–Widom distribution appears in...
  - Random Matrix Theory [Tracy–Widom]
  - Random permutations [Baik–Daift–Johansson]
  - (T)ASEP [Johansson]
  - Growing turbulent interfaces [Prähofer–Spohn]
  - Quantum (JT) gravity [Johnson]
  - and more
- **Integrability**: Fredholm determinant = Isomonodromic τ-function [Jimbo–Miwa–Ueno]
  - Hamiltonian system
  - Lax formalism
  - Riemann–Hilbert problem
  - Painlevé equation
  - Schlesinger system
  - and more
- **This work**: Deformation of Tracy–Widom distribution
  - Fredholm determinant beyond Airy kernel, $(\partial_x^2 - x)\,\mathrm{Ai}(x) = 0 \longrightarrow (\partial_x^2 - v(x))\varphi_\xi(x) = \xi\varphi_\xi(x)$
  - Are there still Tracy–Widom-type formulas? Integrabilities?
  - Other deformations: GOE/GSE, finite temperature, higher order version...

Main result

Theorem [K–Navand]

Let $\mathcal{K}$ be an integral operator with the kernel K associated with a **Schrödinger-type equation**,

$$K(\xi,\zeta) = \int_{\mathbb{R}_+} \varphi_\xi(x)\varphi_\zeta(x)\,\mathrm{d}x\ , \qquad (\partial_x^2 - v(x;\xi))\varphi_\xi(x) = \xi\varphi_\xi(x).$$

We define $\psi(\xi) = \varphi_\xi(0)$ and the auxiliary wave function, $\mathsf{q} = (1-\mathcal{K})^{-1}\psi$. Under the assumption on the form of the wave function, $\varphi_\xi(x) = \phi(u(x)+\xi)$ with $\dot{u}_0 = \partial_x u(x)\big|_{x=0}$, $\ddot{u}_0 = \partial_x^2 u(x)\big|_{x=0}$, we have the following **functional formula of the Fredholm determinant**,

$$\begin{aligned} F([s,\infty)) &= \det(1-\mathcal{K})_{L^2([s,\infty))} \\ &= \exp\left(-\frac{1}{\dot{u}_0}\int_s^\infty (x-s)\left(\mathsf{q}^2 + \ddot{u}_0\left(\frac{2\mathsf{q}}{\psi}-1\right)\left[\mathsf{q}'^2 - \mathsf{q}\left(\mathsf{q}'' - \frac{\mathsf{q}^3}{\dot{u}_0} + \frac{\ddot{u}_0}{\dot{u}_0^2}\mathsf{q}\left(\frac{\mathsf{q}'}{\psi} - \frac{\psi'}{\psi^2}\mathsf{q}\right)\right)\right]\right)\mathrm{d}x\right), \end{aligned}$$

where the auxiliary wave function obeys the **integro-differential equation**,

$$\mathsf{q}'' = \frac{v_0 + x}{\dot{u}_0^2}\mathsf{q} + \frac{2}{\dot{u}_0}\left(\mathsf{q}^3 - \frac{\ddot{u}_0}{\dot{u}_0^2}\mathsf{q}\int_x^\infty \mathsf{q}^2\,\mathrm{d}x\right) - \frac{2\ddot{u}_0^2}{\dot{u}_0^4}\left(\frac{\mathsf{q}^3}{\psi^2} - \frac{\mathsf{q}^2}{\psi}\right) + \frac{\ddot{u}_0}{\dot{u}_0^2}\left(\mathsf{q}' + 2\frac{\psi'}{\psi^2}\mathsf{q}^2 - \frac{4\mathsf{q}\mathsf{q}'}{\psi}\right)$$

with $v_0 = v(0;\xi)$ and the boundary condition, $\mathsf{q} \xrightarrow{x\to\infty} \psi$.

Remarks

- If $\ddot{u}_0 = 0$, the formula is reduced to the Tracy–Widom formula, but not necessarily, $\psi \xrightarrow{\ddot{u}_0=0} \mathrm{Ai}$
- We may write the kernel in the **Christoffel–Darboux form**: $K(\xi,\zeta) = \dot{u}_0 \dfrac{\psi(\xi)\psi'(\zeta) - \psi'(\xi)\psi(\zeta)}{\xi - \zeta}$
- We may rewrite the kernel,

$$K(\xi,\zeta) = \int_{\mathbb{R}_+} \varphi_\xi(x)\varphi_\zeta(x)\,\mathrm{d}x = \int_{\mathbb{R}_+} \phi(u(x)+\xi)\phi(u(x)+\zeta)\,\mathrm{d}x$$

which is analogous to the additive version of **Hankel composition kernel**. [Bothner]

- We may also rewrite the kernel,

$$K(\xi,\zeta) = \int_{u(0)}^{u(\infty)} \frac{\phi(u+\xi)\phi(u+\zeta)}{f(u)}\,\mathrm{d}u \quad \text{with} \quad f(u) = \frac{\mathrm{d}u}{\mathrm{d}x},$$

which agrees with the **finite-temperature Airy kernel** under the following choice,

$$\phi = \mathrm{Ai}\,, \qquad -\frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^{-u(x)} + 1\,, \qquad v(x;\xi) = \frac{\partial_x^2\,\mathrm{Ai}(u(x)+\xi)}{\mathrm{Ai}(u(x)+\xi)} - \xi$$

Relations to the previous studies to be clarified.

[Amir–Corwin–Quastel] [Cafasso–Claeys–Ruzza] [Bothner–Cafasso–Tarricone] [Charlier–Claeys–Ruzza]...

Contents

Kernel from Schrödinger-type equation

- **Schrödinger-type equation**: $(\partial_x^2 - v(x;\xi))\varphi_\xi(x) = \xi\varphi_\xi(x)$ with a real-valued potential $v(x;\xi)$ with the assumption on the form of the wave function, $\varphi_\xi(x) = \phi(u(x)+\xi)$.

Definition

For an interval $I = \bigsqcup_{j=1}^m [\tau_{2j-1}, \tau_{2j}] \subset \mathbb{R}$, define an integral operator, $\mathcal{K} : \ f \mapsto \int_I K(\cdot,\xi) f(\xi)\, \mathrm{d}\xi$ with the kernel

$$K(\xi,\zeta) = \int_{\mathbb{R}_+} \varphi_\xi(x)\varphi_\zeta(x)\,\mathrm{d}x = \int_{\mathbb{R}_+} \phi(u(x)+\xi)\phi(u(x)+\zeta)\,\mathrm{d}x$$

Proposition

Let $\psi(\xi) = \varphi_\xi(0)$, and $\dot{u}_0 = \partial_x u(x)|_{x=0}$. Then, we have the **Christoffel–Darboux formula**,

$$K(\xi,\zeta) = \dot{u}_0 \frac{\psi(\xi)\psi'(\zeta) - \psi'(\xi)\psi(\zeta)}{\xi - \zeta}$$

- The kernel K used for quantum gravity microstate counting [Johnson]
- Airy kernel: $K_{\mathsf{Ai}}(x,y) = \int_{\mathbb{R}_+} \mathrm{Ai}(z+x)\,\mathrm{Ai}(z+y)\,\mathrm{d}z = \dfrac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\,\mathrm{Ai}(y)}{x-y}$

Resolvent kernels

Definition

- Define an integral operator, called the **resolvent**, $\mathcal{R} : f \mapsto \mathcal{K}(1-\mathcal{K})^{-1} f$. Denoting n-th power of $\mathcal{R}$ by $\mathcal{R}^n$, we write
$$\mathcal{R}^n : f \longmapsto \int_I R_n(\cdot, \xi) f(\xi) \, \mathrm{d}\xi \,,$$
where we call $R_n(\cdot, \cdot)$ the **n-th order resolvent kernel**.
- Define the **auxiliary wave functions**: $\chi_{n,k} = \mathcal{R}^n (1-\mathcal{K})^{-1} \psi^{(k)}$.

- **Christoffel–Darboux formula** for the n-th resolvent kernel:
$$R_n(\xi, \zeta) = \frac{\dot{u}_0}{\xi - \zeta} \sum_{k=1}^{n} \left(\chi_{n-k,0}(\xi) \chi_{k-1,1}(\zeta) - \chi_{n-k,1}(\xi) \chi_{k-1,0}(\zeta) \right) .$$
- In particular, for $n = 1$, we have
$$R(\xi, \zeta) = R_1(\xi, \zeta) = \dot{u}_0 \frac{\chi_{0,0}(\xi) \chi_{0,1}(\zeta) - \chi_{0,1}(\xi) \chi_{0,0}(\zeta)}{\xi - \zeta}$$

Fredholm determinant and Hamiltonian system

- **Fredholm determinant** associated with the operator $\mathcal{K}$ and the interval $I = \bigsqcup_{j=1}^{m} [\tau_{2j-1}, \tau_{2j}]$ is given by

$$F(I) := \det(1 - \mathcal{K})_{L^2(I)} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_I \cdots \int_I \det_{1 \le i,j \le k} K(x_i, x_j) \, \mathrm{d}x_1 \cdots \mathrm{d}x_k$$

and $\log F(I) = -\sum_{n=1}^{\infty} \frac{1}{n} \operatorname{tr} \mathcal{K}^n = -\sum_{n=1}^{\infty} \frac{1}{n} \int_I \cdots \int_I K(x_1, x_2) K(x_2, x_3) \cdots K(x_n, x_1) \, \mathrm{d}x_1 \cdots \mathrm{d}x_n$,

which leads to

$$\frac{\partial}{\partial \tau_j} \log F(I) = (-1)^{j-1} R_1(\tau_j, \tau_j)$$

Proposition

We have the hierarchical Hamiltonians, $H_n(\tau_j) = \frac{1}{\dot{u}_0} R_n(\tau_j, \tau_j)$, yielding **Hamilton equations**,

$$\mathsf{q}'_{n,k} = \frac{\partial H_n}{\partial \mathsf{p}_k}, \quad \mathsf{p}'_k = -\frac{\partial H_n}{\partial \mathsf{q}_{n,k}} \qquad \text{with} \qquad (\mathsf{q}_{n,k}, \mathsf{p}_k) = (\chi_{n-k,0}, \chi_{k-1,1})$$

together with **Poisson structures**: $\{f, g\}_n = \sum_{k=1}^{n} \left(\frac{\partial f}{\partial \mathsf{q}_{n,k}} \frac{\partial g}{\partial \mathsf{p}_k} - \frac{\partial f}{\partial \mathsf{p}_k} \frac{\partial g}{\partial \mathsf{q}_{n,k}} \right) \longrightarrow \{H_{r<n}, H_n\}_n = 0$

Hierarchical Lax formalism

- **∞-dim Fuchsian system**: Let $\mathcal{X} = (\chi_{0,0}\ \chi_{0,1}\ \chi_{1,0}\ \chi_{1,1}\ \cdots)^{\mathsf{T}}$. Denoting $\partial_j = \partial_{\tau_j}$, we have infinite size matrices obeying

$$\partial_j \mathcal{X}(\xi) = -\frac{A_j}{\xi - \tau_j}\mathcal{X}(\xi)\,, \qquad \partial_\xi \mathcal{X}(\xi) = \left(B_\xi + \sum_{j=1}^{2m} \frac{A_j}{\xi - \tau_j} \right) \mathcal{X}(\xi)$$

where A_j is (formally) traceless and block triangular (B_ξ is not unless $\ddot{u}_0 = 0$),

$$A_j = (-1)^{j-1}\dot{u}_0 \begin{pmatrix} \chi_{0,1}\chi_{0,0} & -\chi_{0,0}^2 & 0 & 0 & 0 & \cdots \\ \chi_{0,1}^2 & -\chi_{0,1}\chi_{0,0} & 0 & 0 & 0 & \cdots \\ \chi_{0,1}\chi_{0,0} + \chi_{0,1}\chi_{1,0} + \chi_{0,0}\chi_{1,1} & -\chi_{0,0}^2 - 2\chi_{0,0}\chi_{1,0} & \chi_{0,1}\chi_{0,0} & -\chi_{0,0}^2 & 0 & \cdots \\ \chi_{0,1}^2 + 2\chi_{0,1}\chi_{1,1} & -\chi_{0,1}\chi_{0,0} - \chi_{0,1}\chi_{1,0} - \chi_{0,0}\chi_{1,1} & \chi_{0,1}^2 & -\chi_{0,1}\chi_{0,0} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Proposition

Compatibility condition gives rise to **Schlesinger system** (explicit check up to certain order):

$$\partial_j A_i = \frac{[A_i, A_j]}{\tau_i - \tau_j}\,, \qquad \partial_j A_j = \sum_{\substack{i=1 \\ i \neq j}}^{2m} \frac{[A_i, A_j]}{\tau_j - \tau_i} - [A_j, B_j]$$

Contents

Tracy–Widom-type functional formula

- For $I = [s, \infty)$, the Fredholm determinant depends only on s: $\dfrac{\mathrm{d}}{\mathrm{d}s} \log F([s,\infty)) = \dot{u}_0 H_1$

Lemma

Let $\mathsf{q} = \chi_{0,0}$. For $I = [s, \infty)$, the Hamiltonian behaves as

$$\frac{\mathrm{d}H_1}{\mathrm{d}s} = -\frac{1}{\dot{u}_0^2}\mathsf{q}^2 - \frac{\ddot{u}_0}{\dot{u}_0^2}(H_1 + H_2) = -\frac{1}{\dot{u}_0^2}\mathsf{q}^2 - \frac{\ddot{u}_0}{\dot{u}_0^2}\left(\frac{2\mathsf{q}}{\psi} - 1\right) H_1,$$

which involves higher Hamiltonians via $\ddot{u}_0$. Integrating above, we have the Fredholm determinant in terms of the auxiliary wave function,

$$\begin{aligned} &F([s,\infty)) \\ &= \exp\left(-\frac{1}{\dot{u}_0}\int_s^\infty (x-s)\left(\mathsf{q}^2 + \ddot{u}_0\left(\frac{2\mathsf{q}}{\psi} - 1\right)\left[\mathsf{q}'^2 - \mathsf{q}\left(\mathsf{q}'' - \frac{\mathsf{q}^3}{\dot{u}_0} + \frac{\ddot{u}_0}{\dot{u}_0^2}\mathsf{q}\left(\frac{\mathsf{q}'}{\psi} - \frac{\psi'}{\psi^2}\mathsf{q}\right)\right)\right]\right)\mathrm{d}x\right) \end{aligned}$$

which can be also written via integral by parts as follows,

$$F([s,\infty)) = \exp\left(\int_s^\infty \left[\mathsf{q}\left(\dot{u}_0\mathsf{q}'' - \mathsf{q}^3 + \frac{\ddot{u}_0}{\dot{u}_0}\mathsf{q}\left(\frac{\mathsf{q}'}{\psi} - \frac{\psi'}{\psi}\mathsf{q}\right) - \dot{u}_0\mathsf{q}'^2\right)\right]\mathrm{d}x\right)$$

Non-linear integro-differential equation for auxiliary wave functions

- Recall $\mathsf{q} = \chi_{0,0}$ and the closure relation:

$$\chi_{0,2} = \frac{v_0 + x}{\dot{u}_0^2}\chi_{0,0} - \frac{\ddot{u}_0}{\dot{u}_0^2}\chi_{0,1} + \frac{1}{\dot{u}_0}\left(\mu_{0,0}\chi_{0,1} - \mu_{0,1}\chi_{0,0}\right) , \qquad \mu_{n,k} = \int_I \psi(x)\chi_{n,k}(x)\,\mathrm{d}x \,.$$

- Identities: $\mu_{0,0} + \mu_{1,0} = \int_I \mathsf{q}^2\,\mathrm{d}x \xrightarrow{I=[s,\infty)} \int_s^\infty \mathsf{q}^2\,\mathrm{d}x, \quad \chi_{n,k} = \left(\frac{\mathsf{q}}{\psi} - 1\right)^{n-p}\chi_{p,k},$ etc.

Proposition

The auxiliary wave function q obeys the **non-linear integro-differential equation**,

$$\mathsf{q}'' = \frac{v_0 + x}{\dot{u}_0^2}\mathsf{q} + \frac{2}{\dot{u}_0}\left(\mathsf{q}^3 - \frac{\ddot{u}_0}{\dot{u}_0^2}\mathsf{q}\int_x^\infty \mathsf{q}^2\,\mathrm{d}x\right) - \frac{2\ddot{u}_0^2}{\dot{u}_0^4}\left(\frac{\mathsf{q}^3}{\psi^2} - \frac{\mathsf{q}^2}{\psi}\right) + \frac{\ddot{u}_0}{\dot{u}_0^2}\left(\mathsf{q}' + 2\frac{\psi'}{\psi^2}\mathsf{q}^2 - \frac{4\mathsf{q}\mathsf{q}'}{\psi}\right)$$

with the boundary condition $\mathsf{q} \xrightarrow{x\to\infty} \psi$. This equation is reduced to **Painlevé II equation**,

$$\mathsf{q}'' = x\mathsf{q} + 2\mathsf{q}^3,$$

if $\ddot{u}_0 = 0$ under the shift $x + v_0 \to x$ and the rescaling, $x \to \dot{u}_0^{\frac{2}{3}}x$, $\mathsf{q} \to \dot{u}_0^{-\frac{1}{6}}\mathsf{q}$.

Main result

Theorem [K–Navand]

Let $\mathcal{K}$ be an integral operator with the kernel K associated with a **Schrödinger-type equation**,

$$K(\xi,\zeta) = \int_{\mathbb{R}_+} \varphi_\xi(x)\varphi_\zeta(x)\,\mathrm{d}x\ , \qquad (\partial_x^2 - v(x;\xi))\varphi_\xi(x) = \xi\varphi_\xi(x).$$

We define $\psi(\xi) = \varphi_\xi(0)$ and the auxiliary wave function, $\mathsf{q} = (1-\mathcal{K})^{-1}\psi$. Under the assumption on the form of the wave function, $\varphi_\xi(x) = \phi(u(x)+\xi)$ with $\dot{u}_0 = \partial_x u(x)\big|_{x=0}$, $\ddot{u}_0 = \partial_x^2 u(x)\big|_{x=0}$, we have the following **functional formula of the Fredholm determinant**,

$$\begin{aligned} F([s,\infty)) &= \det(1-\mathcal{K})_{L^2([s,\infty))} \\ &= \exp\left(-\frac{1}{\dot{u}_0}\int_s^\infty (x-s)\left(\mathsf{q}^2 + \ddot{u}_0\left(\frac{2\mathsf{q}}{\psi}-1\right)\left[\mathsf{q}'^2 - \mathsf{q}\left(\mathsf{q}'' - \frac{\mathsf{q}^3}{\dot{u}_0} + \frac{\ddot{u}_0}{\dot{u}_0^2}\mathsf{q}\left(\frac{\mathsf{q}'}{\psi} - \frac{\psi'}{\psi^2}\mathsf{q}\right)\right)\right]\right)\mathrm{d}x\right), \end{aligned}$$

where the auxiliary wave function obeys the **integro-differential equation**,

$$\mathsf{q}'' = \frac{v_0 + x}{\dot{u}_0^2}\mathsf{q} + \frac{2}{\dot{u}_0}\left(\mathsf{q}^3 - \frac{\ddot{u}_0}{\dot{u}_0^2}\mathsf{q}\int_x^\infty \mathsf{q}^2\,\mathrm{d}x\right) - \frac{2\ddot{u}_0^2}{\dot{u}_0^4}\left(\frac{\mathsf{q}^3}{\psi^2} - \frac{\mathsf{q}^2}{\psi}\right) + \frac{\ddot{u}_0}{\dot{u}_0^2}\left(\mathsf{q}' + 2\frac{\psi'}{\psi^2}\mathsf{q}^2 - \frac{4\mathsf{q}\mathsf{q}'}{\psi}\right)$$

with $v_0 = v(0;\xi)$ and the boundary condition, $\mathsf{q} \xrightarrow{x\to\infty} \psi$.

Contents

Conclusion

- **Summary**:
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  - Hierarchical Lax formalism and Poisson structure
  - Tracy–Widom-type functional formula and the integro-differential equation
- **Perspectives**:
  - GOE/GSE, finite temperature, higher order version...
  - Riemann–Hilbert approach and the asymptotic behavior

Thank you for your attention!