Exercises on the lectures - Gibbs measures in hyperbolic geometry and dynamics

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The exercises use the notations of my personal notes of the lectures given at IHES in july 2025. They are available here
`https://imag.umontpellier.fr/ schapira/recherche/IHES2025-Barbara.pdf`

1 Construction of Gibbs measures

1.1 The geodesic flow on the hyperbolic disc

1.1.1 Hyperbolic plane / disc

The hyperbolic plane is defined as $\mathbb{H} = \mathbb{R} \times \mathbb{R}_+^*$ and endowed with the hyperbolic metric $ds^2 = \frac{dx^2+dy^2}{y^2}$. The geodesics are the curves which minimize the distance.

Exercise 1.1 Check these classical facts. The hyperbolic geodesics are the vertical half-lines and the half-circles orthogonal to the boundary $\mathbb{R} \times \{0\}$. The isometries preserving orientation are the homographies $z \to \frac{az+b}{cz+d}$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix with determinant 1.

1.1.2 Geodesic flow

Exercise 1.2 *If c_1, c_2 are two geodesic rays such that $d(c_1(t), c_2(t)) \to 0$ when $t \to +\infty$, then show that for every $t \geq 0$,*

$$d(c_1(t), c_2(t)) \leq e^{-t} d(c_1(0), c_2(0))$$

Hint *Use the upper half plane model and come back to two vertical rays.*

Denote by o the center of the disk $\mathbb{D}$. If $v \in T^1\mathbb{D}$, $\pi(v)$ is the basepoint of v. The Hopf coordinates are given by the homeomorphism

$$H : v \in T^1\mathbb{D} \mapsto \left(v^-, v^+, \beta_{v^+}(o, \pi(v)))\right) .$$

In these coordinates, the geodesic flow acts as follows. If $v \simeq (v^-, v^+, s)$, then

$$g_t(v) \simeq (v^-, v^+, s+t) .$$

Consider an isometry $\gamma \in PSL(2, \mathbb{R})$. In these coordinates it acts as follows

$$\gamma.v \simeq (\gamma v^-, \gamma v^+, s + \beta_{v^+}(\gamma^{-1} o, o) .$$

Exercise 1.3 *Check and prove the above formulas.*

Exercise 1.4 *There is a* $1-1$ *correspondance between Radon measures* m *invariant under the geodesic flow on* T^1S*, Radon measure* $\tilde{m}$ *invariant under the geodesic flow and the group* Γ *on* $T^1\mathbb{D}$*, and* geodesic currents*, i.e. Radon measures* $\mathcal{C}$ *on* $\partial^2\mathbb{D}$ *that are* Γ *invariant.*

1.2 Patterson Sullivan Gibbs construction

Exercise 1.5 *Show that if* Γ *contains at least two hyperbolic isometries with distinct axes, then there does not exist* Γ*-invariant probability measures on* S^1*.*

1.2.1 Hölder maps, Poincare series

Define the Poincaré series associated with (Γ, f) as

$$P_{(\Gamma,f)}(s) = \sum_{\gamma\in\Gamma} e^{-sd(o,\gamma o)+\int_o^\gamma \tilde{f}}$$

Set

$$\delta^f = \lim_{t\to\infty} \frac{1}{t} \log \sum_{\gamma\in\Gamma, d(o,\gamma o)\in[t,t+1]} e^{\int_o^{\gamma o} \tilde{f}}\,.$$

Exercise 1.6 *Show that this series converges for* $s > \delta^f$ *and diverges for* $s < \delta^f$*.*

Exercise 1.7 *Read Patterson's trick in [**?**].*

Exercise 1.8 *Deduce that* ν^f *is supported on* S^1

Exercise 1.9 *Define the* limit set $\Lambda_\Gamma = \overline{\Gamma o} \setminus \Gamma o$*.*

- *Show that it is the smallest* Γ*-invariant set on* S^1*.*
- *Show that* ν^f *gives full measure to* Λ_Γ*.*

Exercise 1.10 *Denote by* ρ^f *and* β^f *the cocycles on* $S^1 \times \mathbb{D} \times D$ *defined by*

$$\rho^f_\xi(x,y) = \lim_{t\to\infty} \int_x^\xi \tilde{f} - \int_y^\xi \tilde{f} \quad \text{and} \quad \beta^f = \delta^f\beta - \rho^f$$

Use the geodesic rays c_x *and* c_y *from* x *(resp* y*) to* ξ *to give a rigorous meaning to the above expression.*

Exercise 1.11 *Show that the measure* ν^f *is* Γ *quasi invariant and that for a.e.* ξ *and all* $\gamma\in\Gamma$*,*

$$\frac{d\gamma_*\nu^f}{d\nu^f}(\xi) = \exp = \exp(-\delta^f\beta_\xi(\gamma o, o) + \rho^f_\xi(\gamma o, o))\,.$$

Exercise 1.12 *Prove the Shadow lemma by following the steps below.*

1. *Use the conformality of* ν^f *to get*
$$\nu^f(\mathcal{O}_o(B(\gamma o,R)) = \gamma_*\nu^f(\gamma^{-1}(\mathcal{O}_o(B(\gamma o,R)) = \gamma_*\nu^f(\mathcal{O}_{\gamma^{-1}o}(B(o,R)))\,.$$
2. *Show that on* $\mathcal{O}_{\gamma^{-1}o}(B(o,R))$ *the Radon Nikodym derivative* $d\gamma_*\nu^f/d\nu^f$ *is uniformly close to* $\exp\left(-\delta^f d(o,\gamma o) + \int_o^{\gamma o} f\right)$*.*
3. *Check that the measure on the right is bounded from above*
4. *Use the fact that* ν^f *is not a single Dirac measure to show that there exists some* $\alpha > 0$*, such that for every* $y \in \mathbb{D}\cup S^1$*,* $\nu^f(\mathcal{O}_y(B(o,R)) \geq \alpha > 0$*.*

1.2.2 Product measure

Exercise 1.13 *Show that the measure $\mathcal{C}^f$ on $S^1 \times S^1$ defined by*

$$d\mathcal{C}^f(\xi,\eta) = \exp\left(\beta^f_\eta(o,x) + \beta^f_\xi(o,x)\right) d\nu^f(\xi)\, d\nu^f(\eta),$$

(with $x \in (\xi\eta)$ an arbitrary point= is a geodesic current, i.e. a Γ-invariant measure. We admit that it gives zero measure to the diagonal of $S^1 \times S^1$.

It allows to define a measure $\tilde{m}^f$ on $T^1\mathbb{D}$, as

$$\tilde{m}^f = (H^{-1})_*(\tilde{C}^f \otimes dt)\,.$$

This measure $\tilde{m}^f$ is Γ-invariant and (g^t) invariant.Therefore it induces a measure m^f on $T^1S = T^1\mathbb{D}/\Gamma$, that is a Radon measure, i.e. gives finite mass to compact sets.

Exercise 1.14 *Show that the measure m^f is supported on*

$$\Omega := \left(H^{-1}(\Lambda_\Gamma \times \Lambda_\Gamma \times \mathbb{R})\right)/\Gamma \subset T^1S$$

Exercise 1.15 *Show that the surface $S = \mathbb{D}/\Gamma$ is convex-cocompact if and only if $H^{-1}(\Lambda_\Gamma \times \Lambda_\Gamma \times \mathbb{R})$ is cocompact, i.e.*

$$\Omega := \left(H^{-1}(\Lambda_\Gamma \times \Lambda_\Gamma \times \mathbb{R})\right)/\Gamma$$

is compact.

Hint*: Recall that S is convex-cocompact, by definition, if the convex hull $\mathcal{C}^{core}$ of the limit set in $\mathbb{D}$ is cocompact. First observe that $\Omega \subset T^1\mathcal{C}^{core}/\Gamma$ and deduce that one direction of the equivalence is easy. For the other direction, use the fact that triangles are thin to show that any point of C^{core} is at uniformly bounded distance of a geodesic joining two points of Λ_Γ.*

Exercise 1.16 *Define a dynamical ball as the set*

$$B(v,T,\epsilon) = \{w \in T^1S, \forall 0 \le t \le T, d(g^tv, g^tw) \le \epsilon\}$$

*Show that $B(v,T,\epsilon)$ is comparable (in Hopf coordinates) to $\mathcal{O}_{\pi(v)}(B(\pi(g^Tv),r)) \times \mathcal{O}_{\pi(g^Tv)}(B(v,r)) \times [-\rho,\rho]$ for r,ρ suitable constants. See [**?**] (***Completer ref***)*

Exercise 1.17 *Prove that, when finite, the measure m^f satisfies the Gibbs property : for* **every** *$v \in T^1S$,*

$$m^f(B(v,T,\epsilon)) \asymp \exp\left(-\delta^f T + \int_0^T f(g^tv)\right) dt$$

In particular, check that

$$h(m^f) = \delta^f - \int f\, dm^f$$

thanks to the Shadow Lemma and the above exercise.

2 Regularity of entropy under perturbations

2.1 Boundaries

2.2 Identifying the sets of invariant measures

Exercise 2.1 *Show the following properties*

- $m_{\mathcal{C}}^{g_1}$ *has full support iff* $m_{\mathcal{C}}^{g_2}$ *has full support.*
- $m_{\mathcal{C}}^{g_1}$ *is ergodic iff* $m_{\mathcal{C}}^{g_2}$ *is ergodic.*
- $m_{\mathcal{C}}^{g_1}$ *is supported on a periodic orbit iff* $m_{\mathcal{C}}^{g_2}$ *is supported on a periodic orbit.*
- $m_{\mathcal{C}}^{g_1}$ *is a quasi product measure (i.e.* $\mathcal{C}$ *is equivalent to a product measure) iff* $m_{\mathcal{C}}^{g_2}$ *is a quasi product measure.*

2.3 Geodesic stretch

Exercise 2.2 *Use the fact that the* g_1 *and* g_2*- geodesics from* $\pi(v)$ *to* $v_+^{g_1}$ *are at bounded distance one another to prove that the ergodic average of the geodesic stretch satisfies*

$$\frac{1}{T}\int_0^T \mathcal{E}^{g_1\to g_2}(g_1^t v)dt \asymp \frac{d^{g_2}(\pi(v),\pi(g_1^T v))}{d^{g_1}(\pi(v),\pi(g_1^T v))}$$

2.4 Morse correspondance

Exercise 2.3 *Check from the definition that the Hopf coordinates, and therefore the homeomorphism* $\Phi^{g_1\to g_2}$ *commute with the geodesic flow :*

$$\Phi^{g_1\to g_2}\circ g_1^t = g_2^t\circ\Phi^{g_1\to g_2}$$

Exercise 2.4 *Show that* $\tilde{\Psi}^{g_1\to g_2}$ *is* Γ*-equivariant, and induces therefore an orbit equivalence from* $T^1_{g_1}S$ *to* $T^1_{g_2}S$*.*

Exercise 2.5 *Let* $G: T^1_{g_2}S\to\mathbb{R}$ *be a continuous map, and* $m_{\mathcal{C}}^{g_i}$ *be two invariant probability measures on* $T^1_{g_i}S$ *associated with the same geodesic current* $\mathcal{C}$ *on* $\partial^2\tilde{S}$*. Show that*

$$\int_{T^1_{g_2}S} G(w)\,dm_{\mathcal{C}}^{g_2}(w) = \int_{T^1_{g_1}S} G\circ\Psi^{g_1\to g_2}(v)\times\mathcal{E}^{g_1\to g_2}(v)\,dm_{\mathcal{C}}^{g_1}(v)\,.$$

If you succeeded to prove that, you forgot the normalization. Indeed, $m_{\mathcal{C}}^{g_i}$ *are the normalized probability measures on* $T^1_{g_i}S$ *associated through quotient by* Γ *and Hopf coordinates* H^{g_i} *to the measure* $\mathcal{C}\otimes dt$*. Show that the correct formula is*

$$\int_{T^1_{g_2}S} G(w)\,dm_{\mathcal{C}}^{g_2}(w) = \frac{\int_{T^1_{g_1}S} G\circ\Psi^{g_1\to g_2}(v)\times\mathcal{E}^{g_1\to g_2}(v)\,dm_{\mathcal{C}}^{g_1}(v)}{\int_{T^1_{g_1}S}\mathcal{E}^{g_1\to g_2}(v)\,dm_{\mathcal{C}}^{g_1}(v)}$$

2.5 Shadows are shadows

Exercise 2.6 *Show that*

$$\mathcal{O}_x^{g_1}(B^{g_1}(y,r))\subset\mathcal{O}_x^{g_2}(B^{g_2}(y,r+C))\,.$$

Exercise 2.7 *Prove that* $m_{\mathcal{C}}^{g_2}$ *is a Gibbs measure wrt the potential* G *iff* $m_{\mathcal{C}}^{g_1}$ *is a Gibbs measure wrt the potential* $G\circ\mathcal{E}^{g_1\to g_2}$*.*

Exercise 2.8 *Prove that*

$$h(m_{\mathcal{C}}^{g_2}) = \frac{1}{\int_{T^1_{g_1}S}\mathcal{E}^{g_1\to g_2}dm_{\mathcal{C}}^{g_1}}\times h(m_{\mathcal{C}}^{g_1})$$