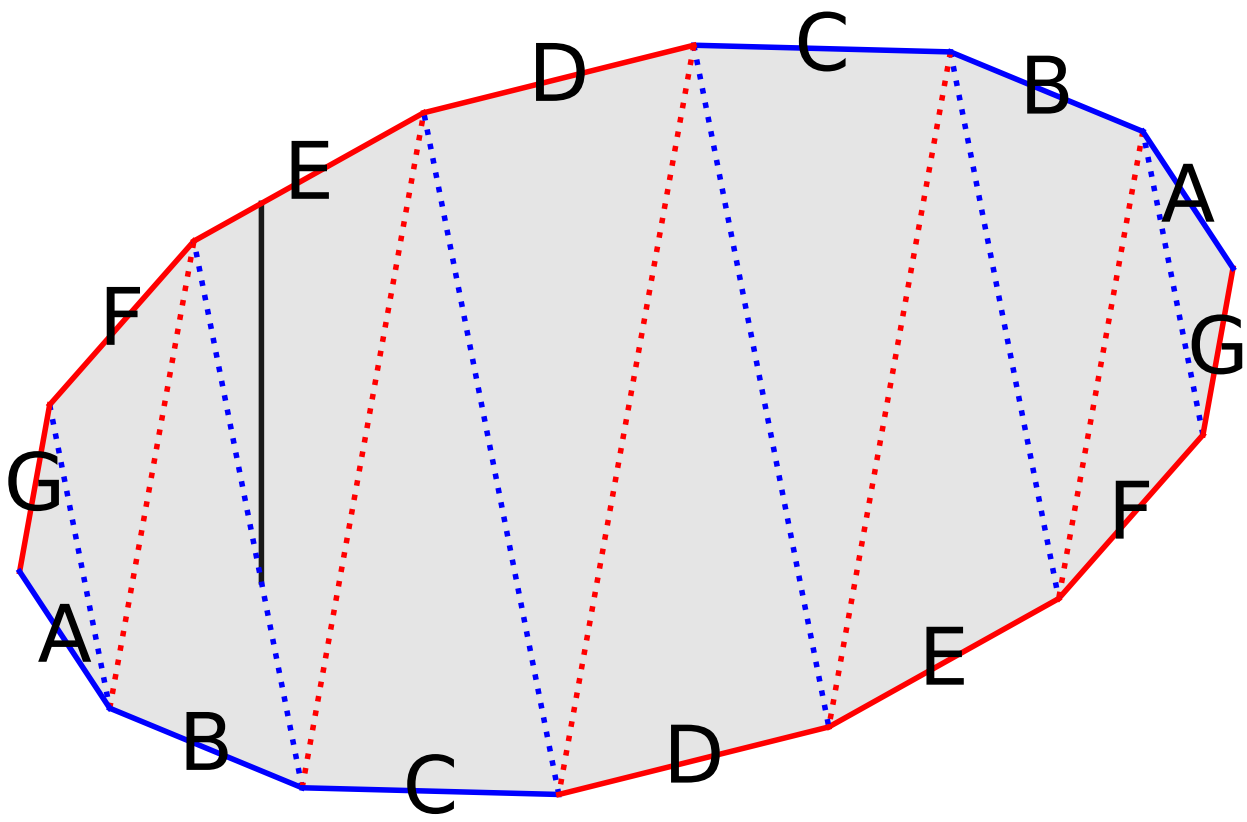
Computing the multiscale boundary of linear subvarieties

V. Delecroix (j.w. K. Fu and B. Zykoski)

JHH80, Toulouse, June 2025

Moduli space of translation flows

Translation surfaces and translation flows



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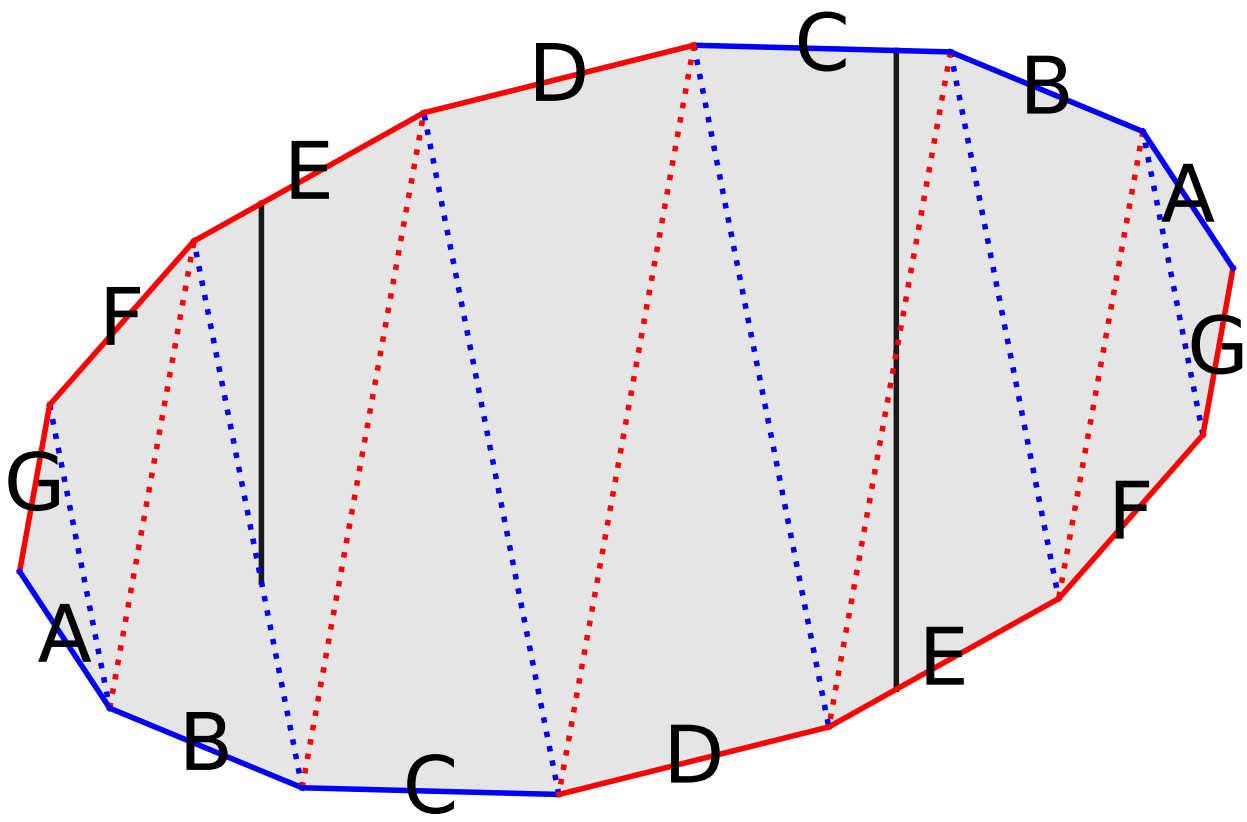

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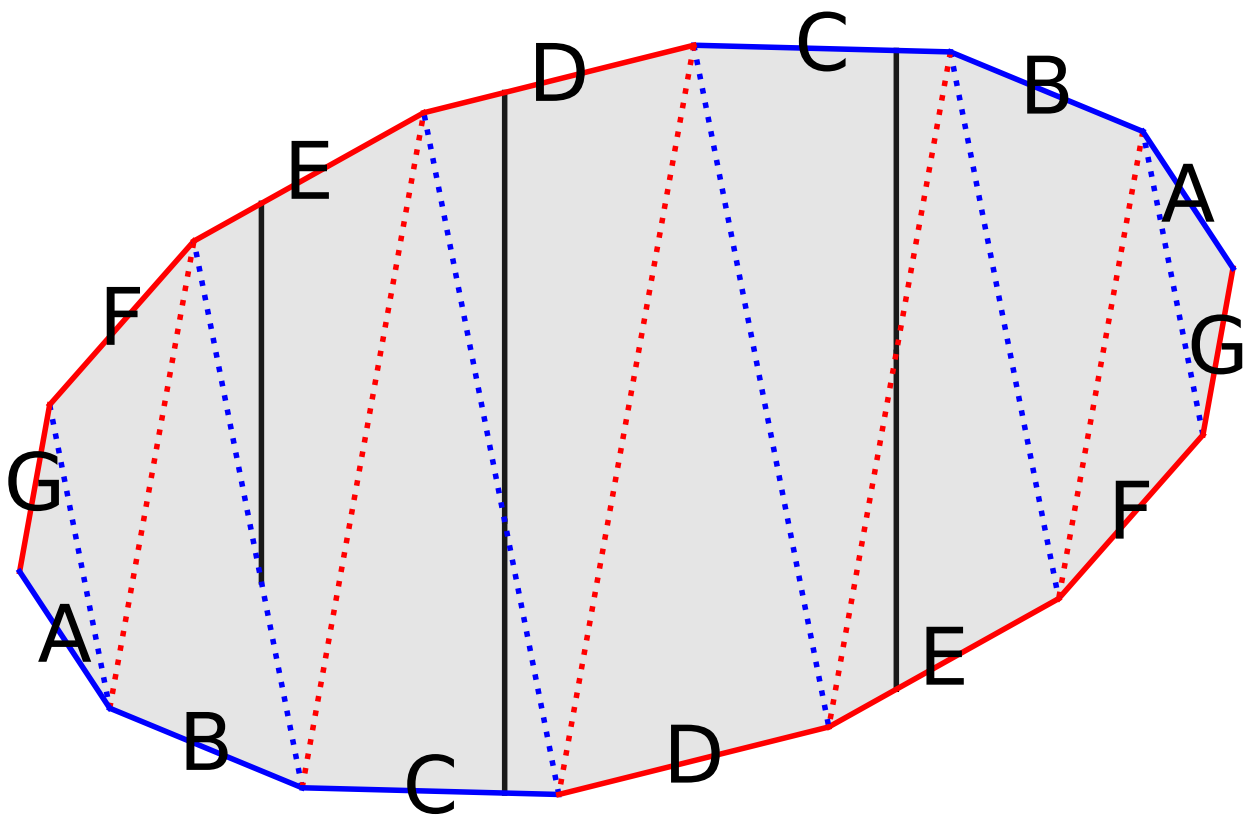

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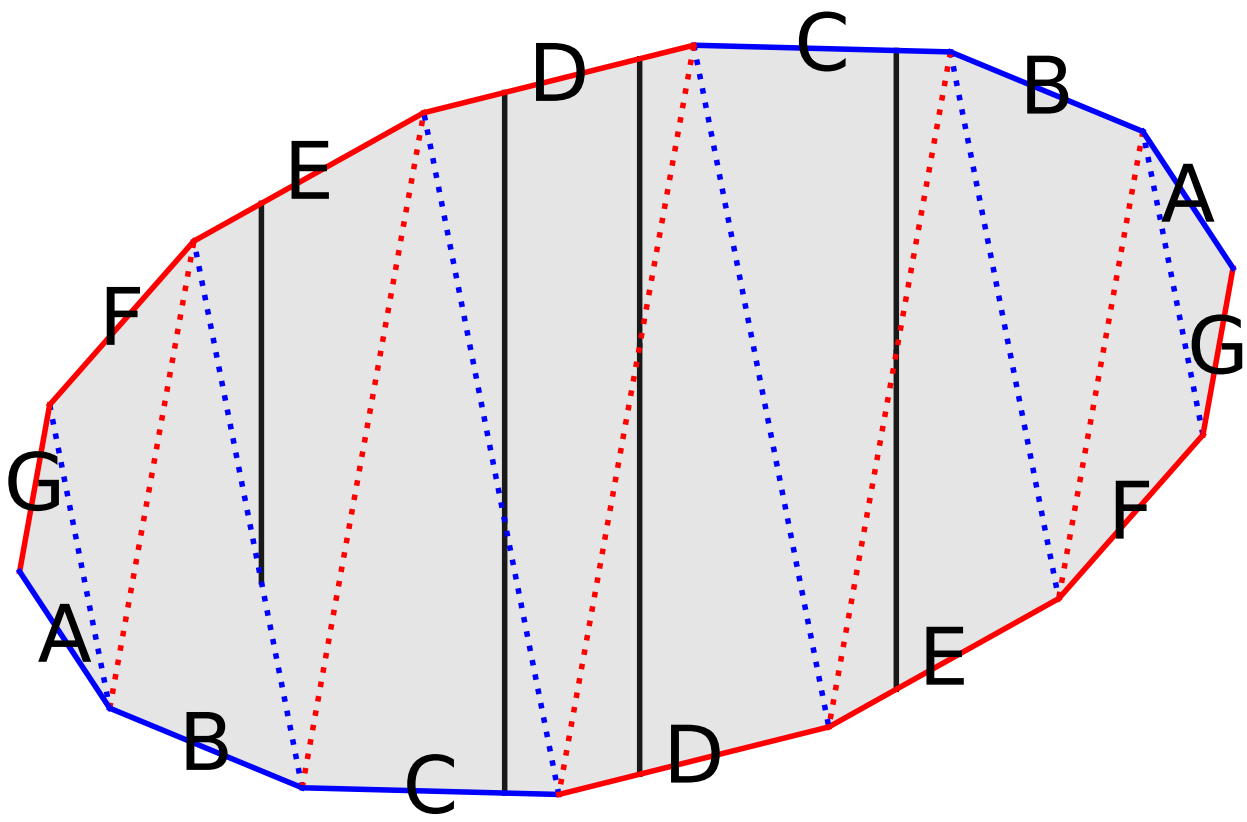

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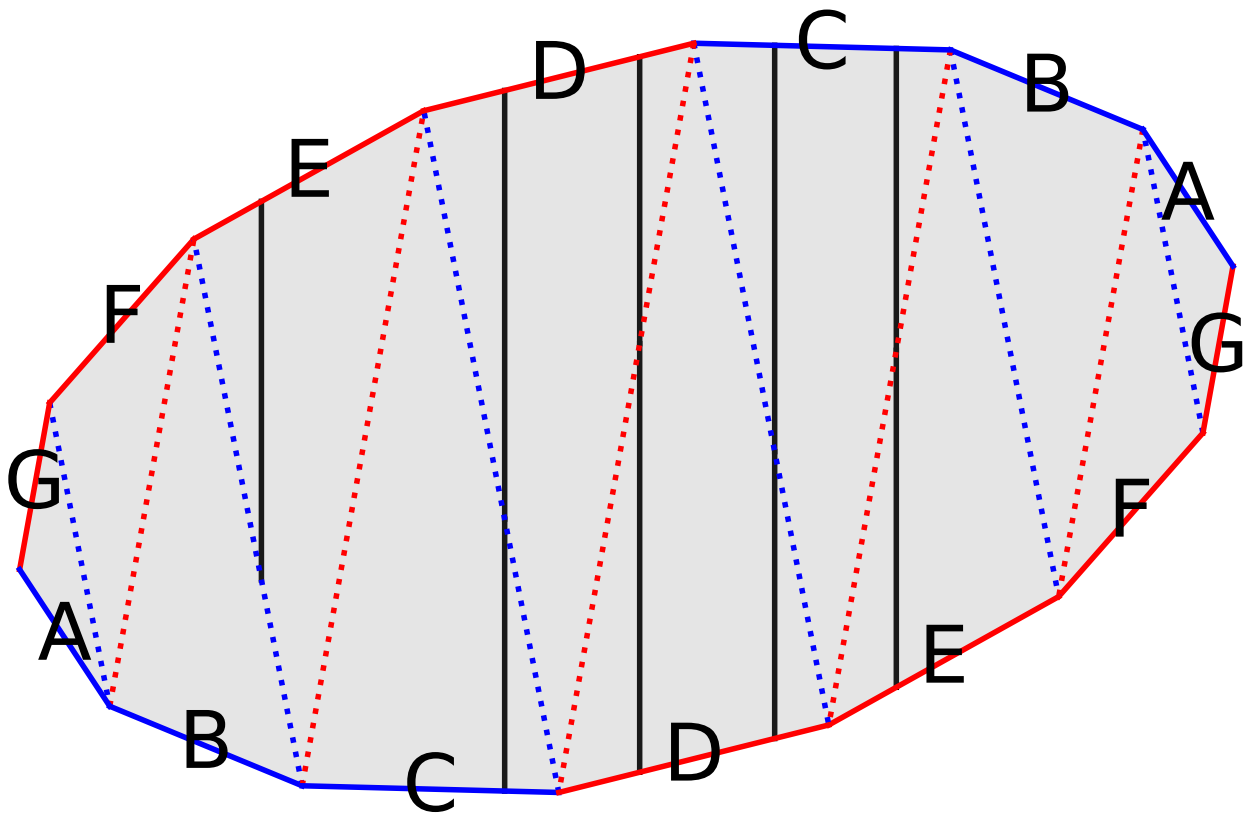

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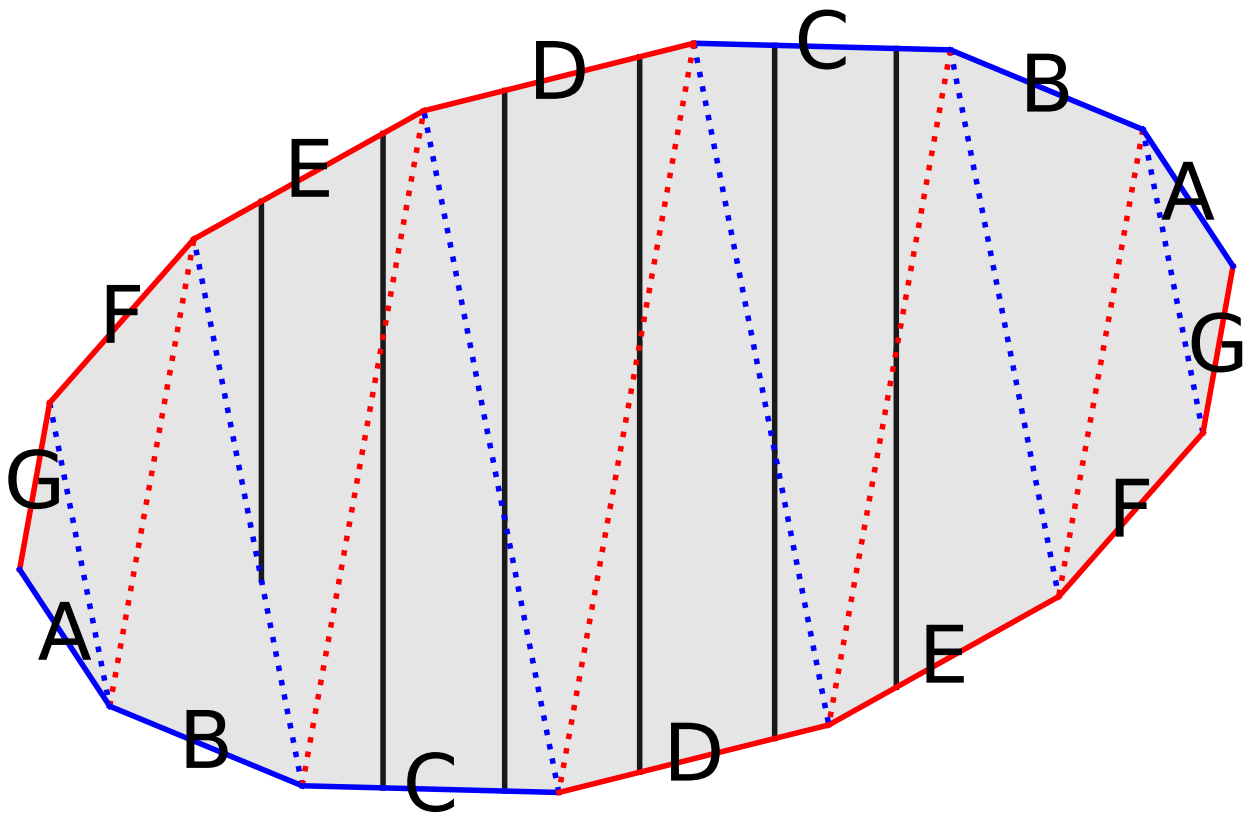

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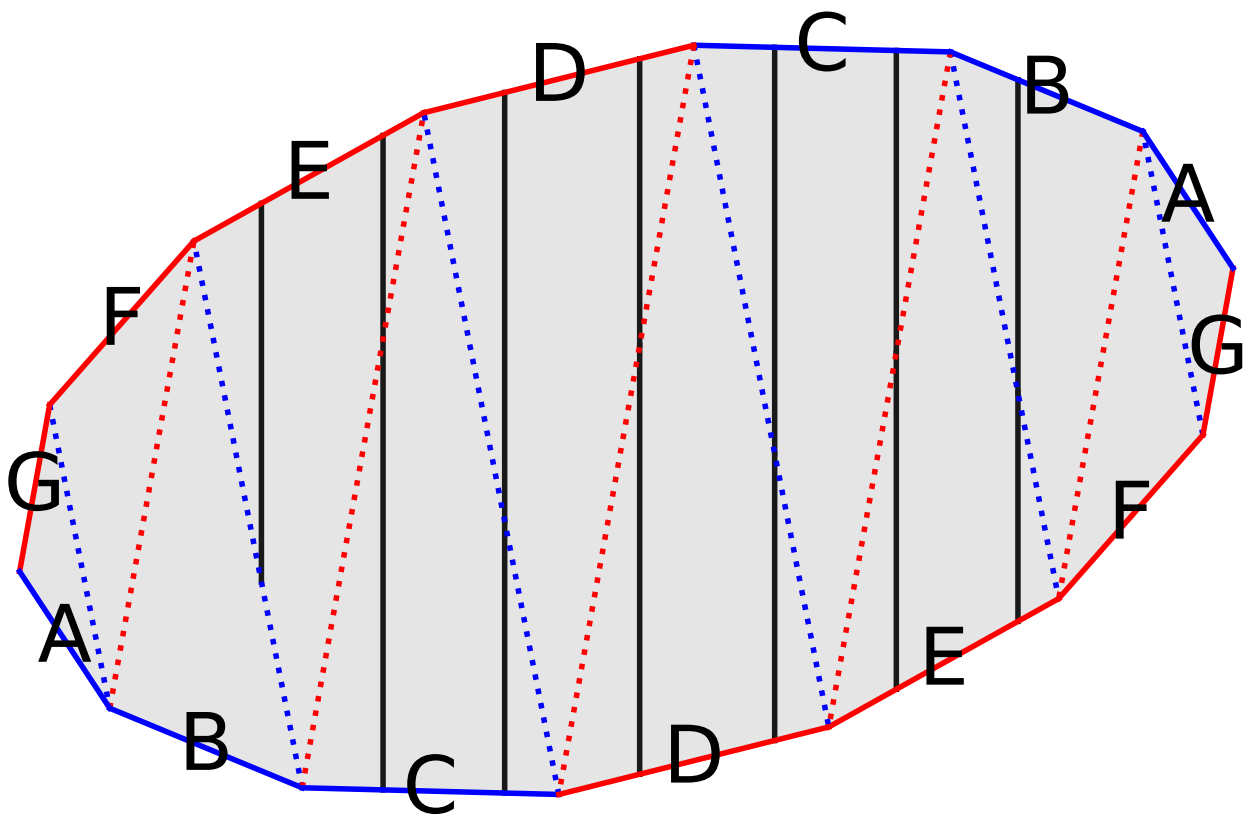

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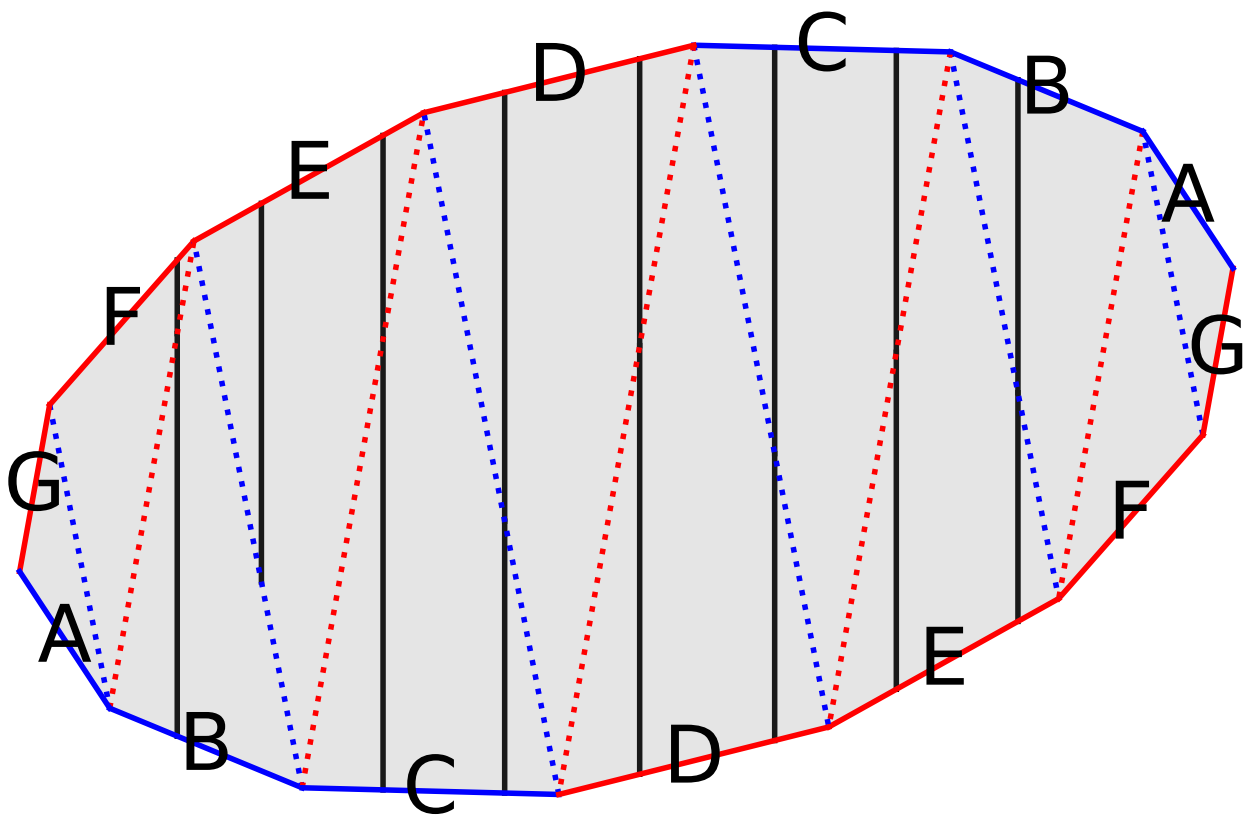

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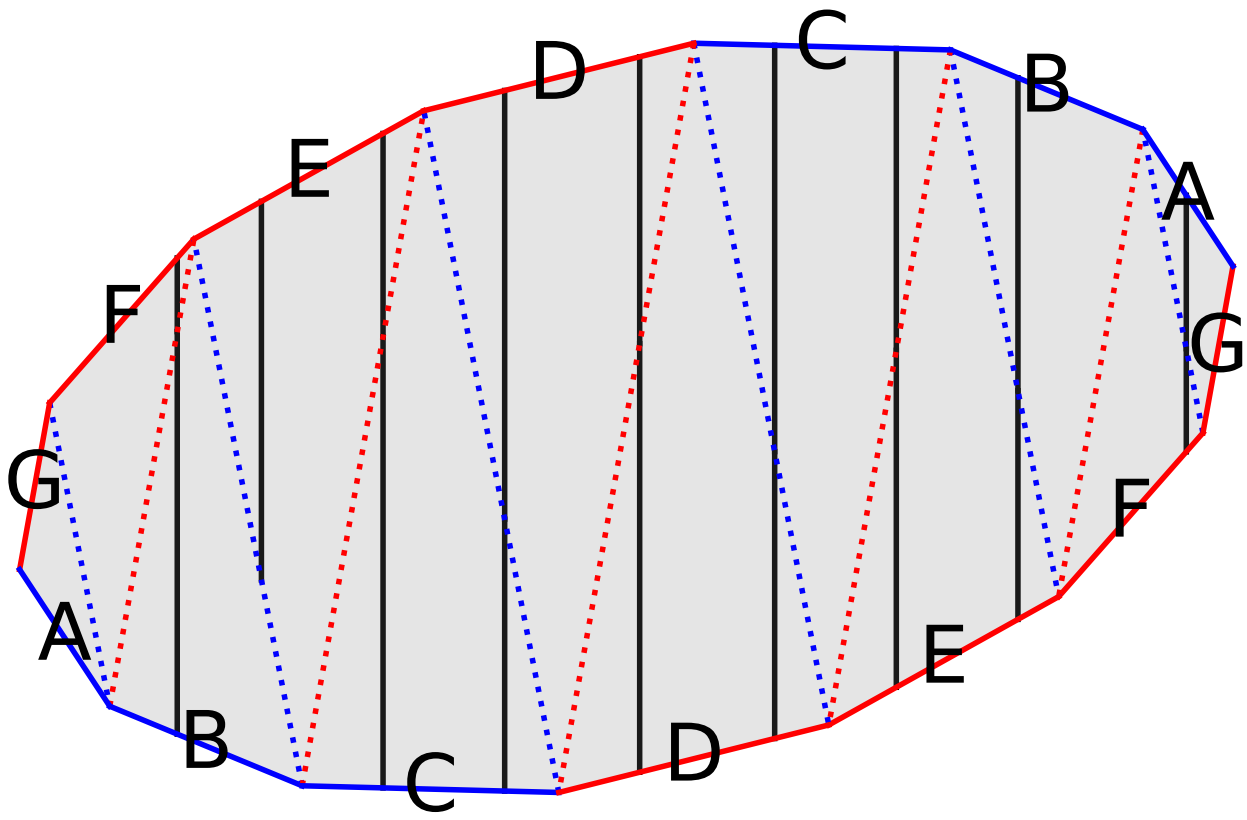

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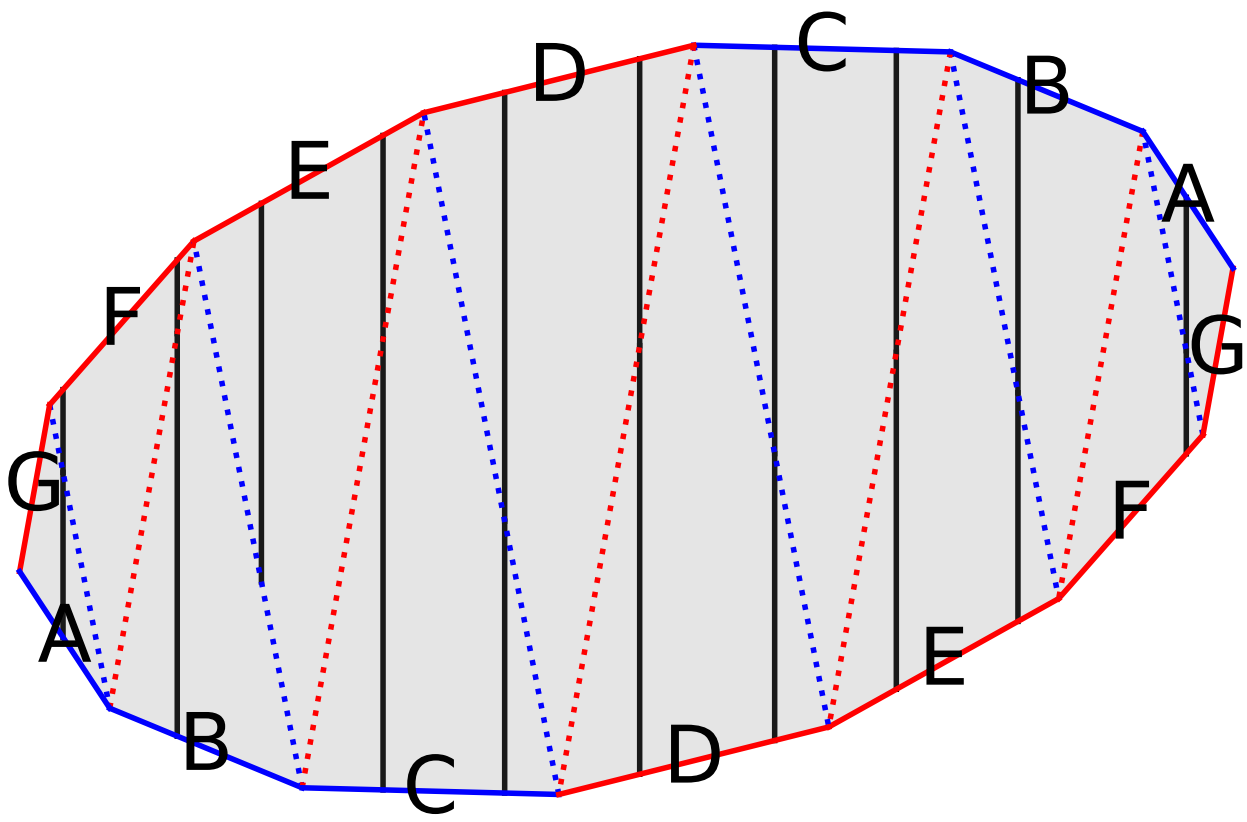

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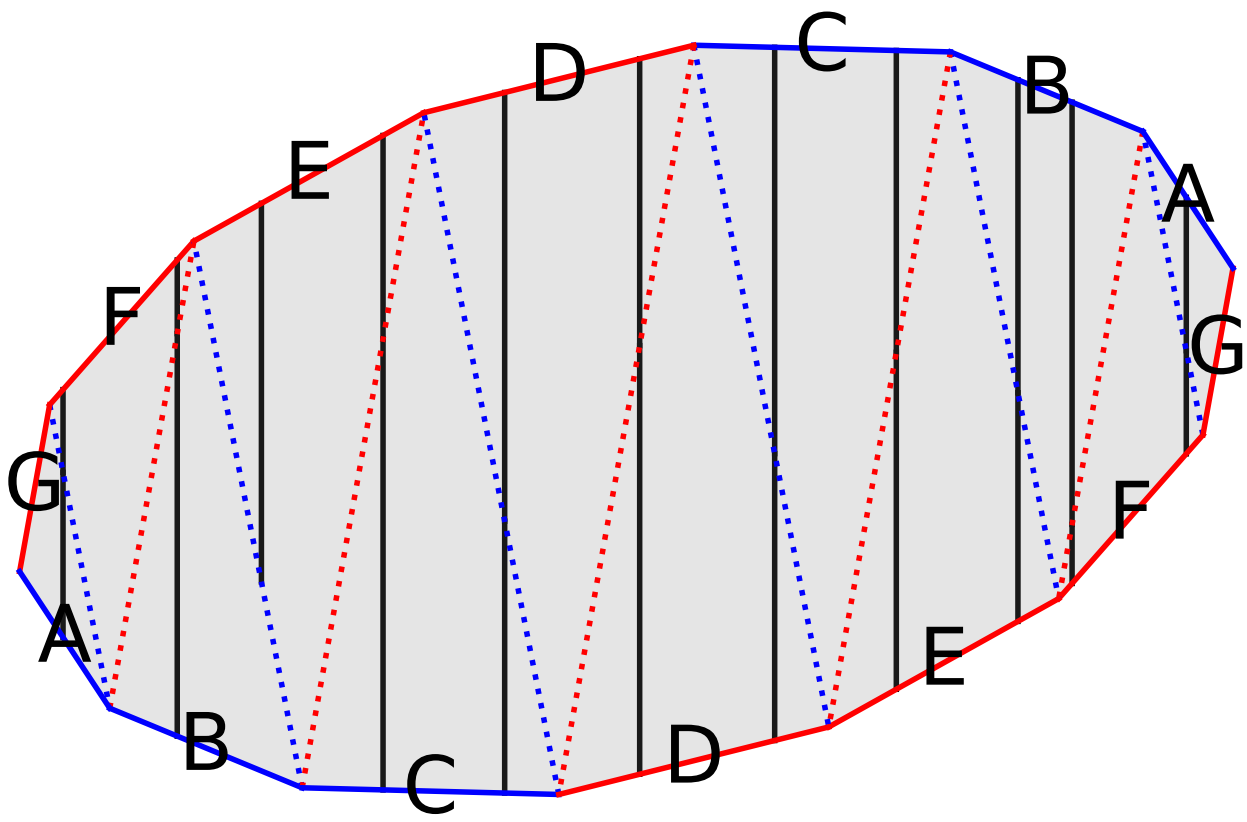

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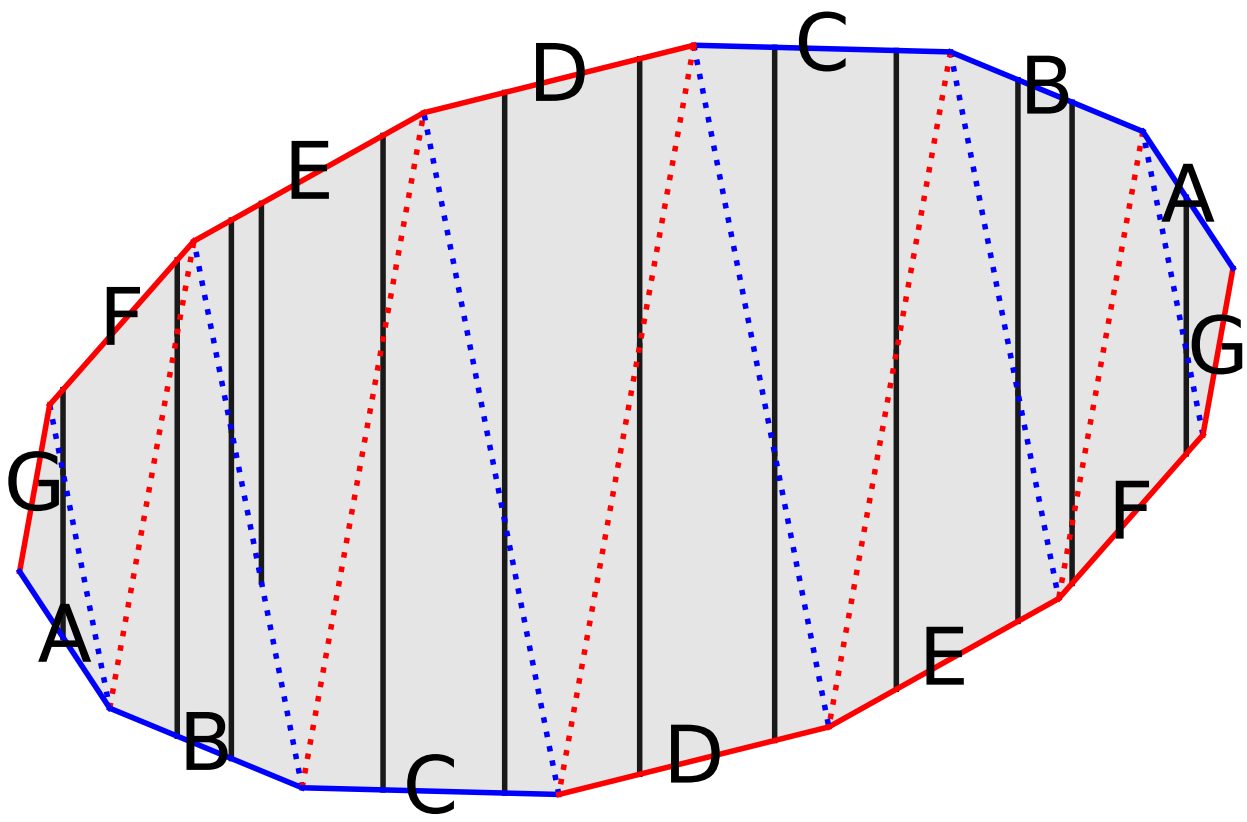

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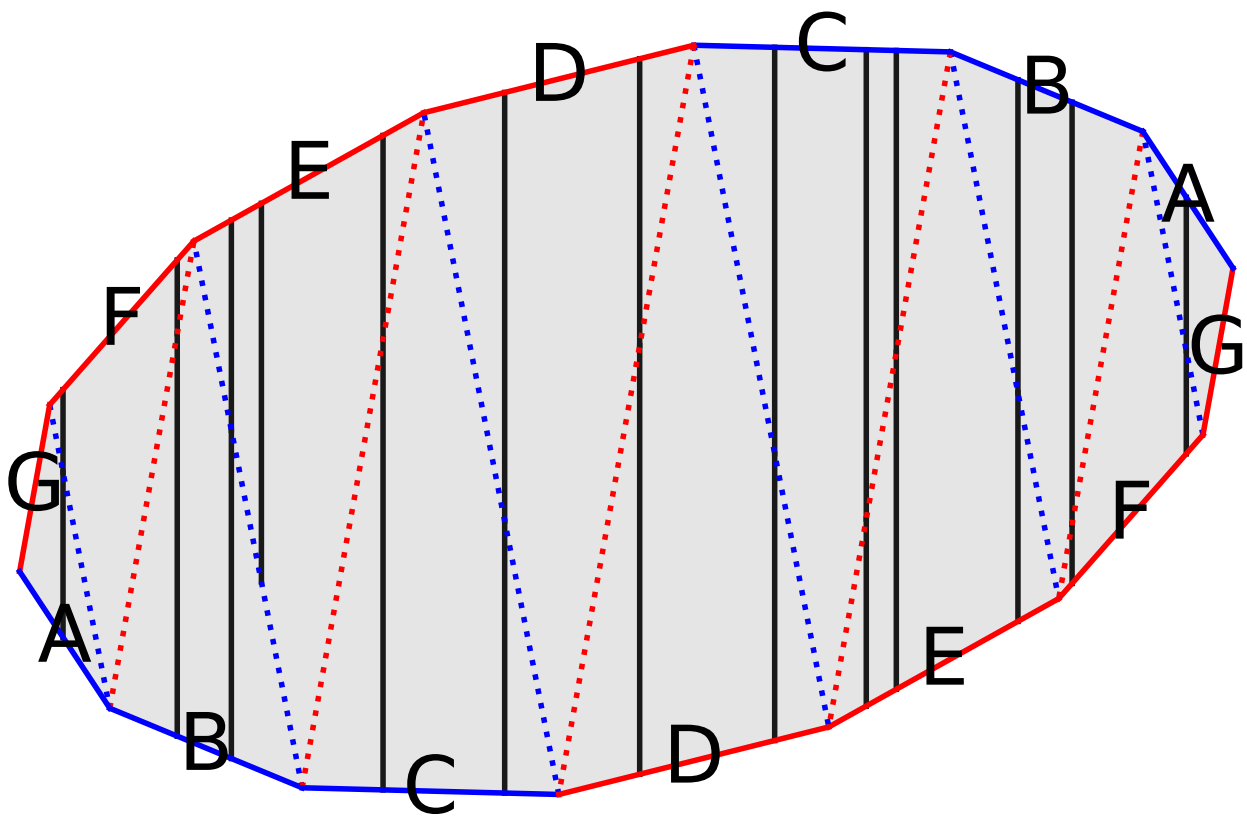

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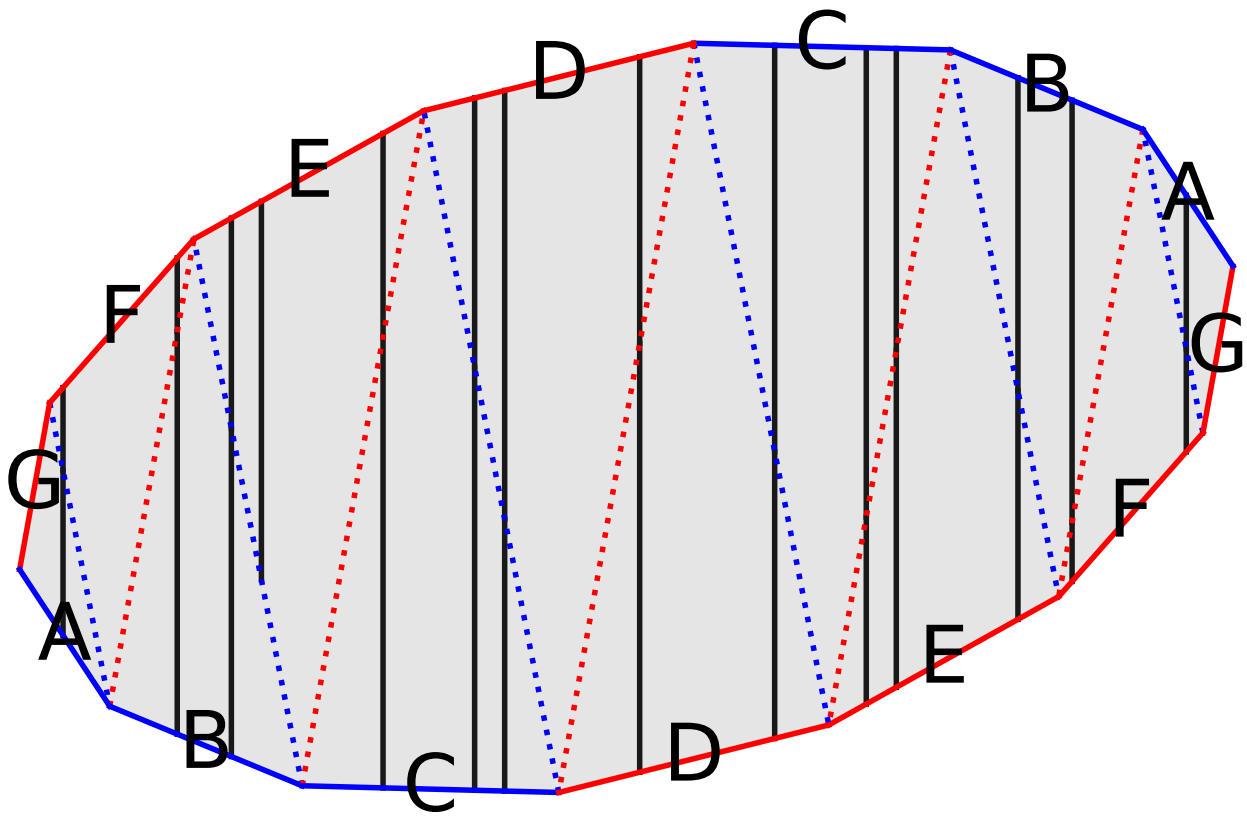

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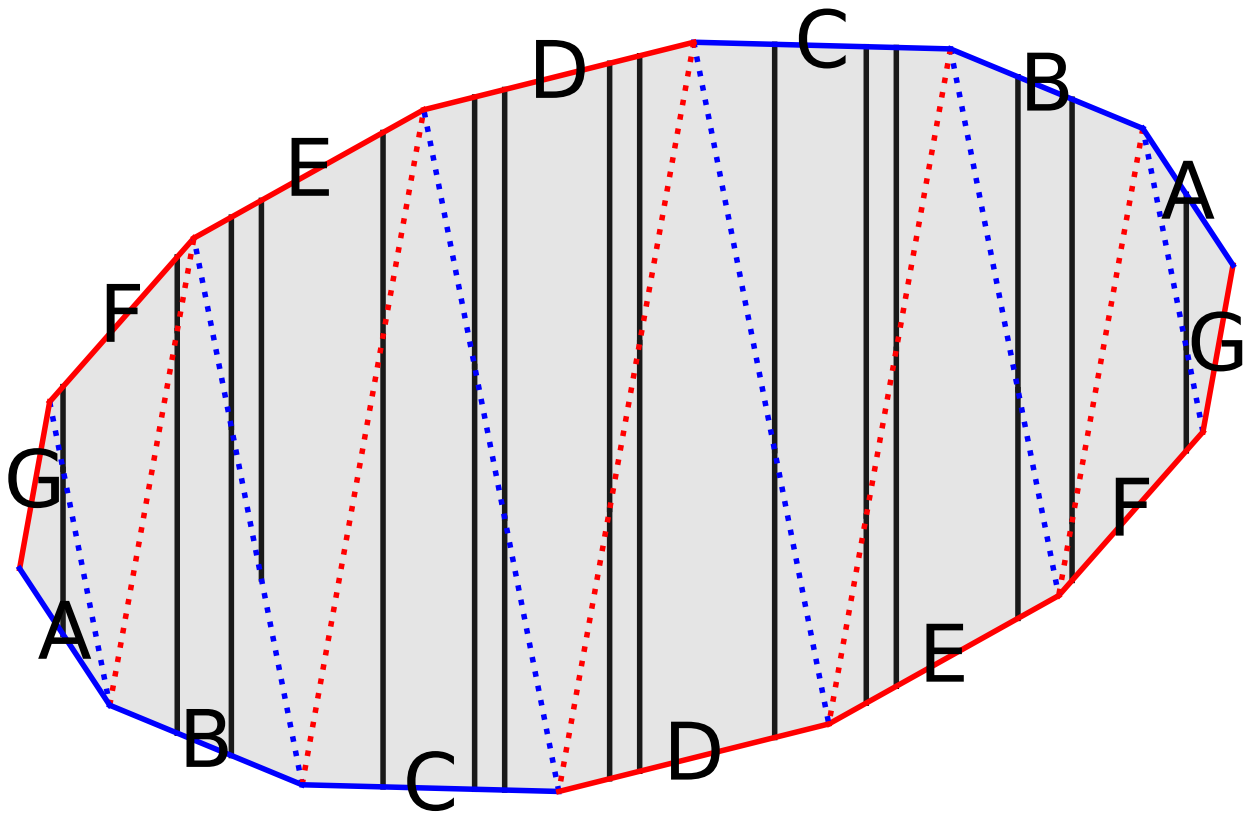

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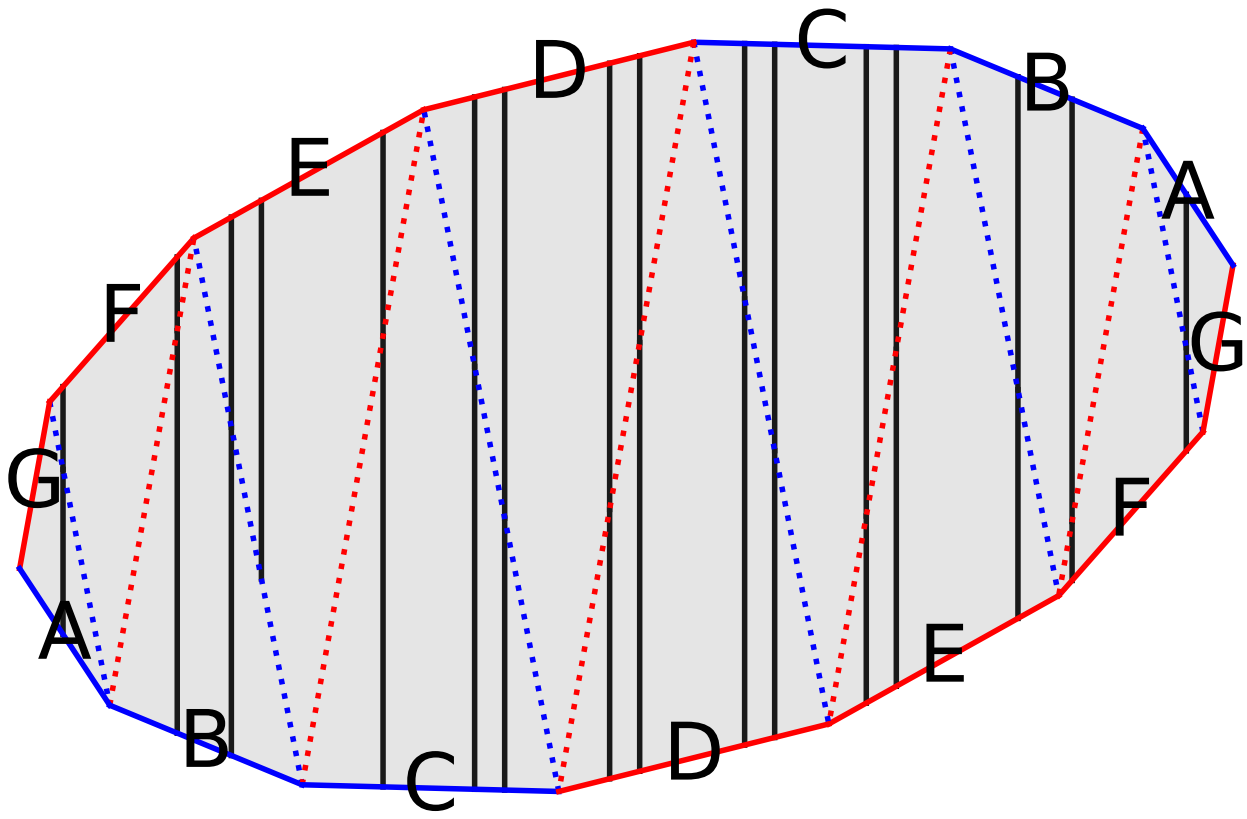

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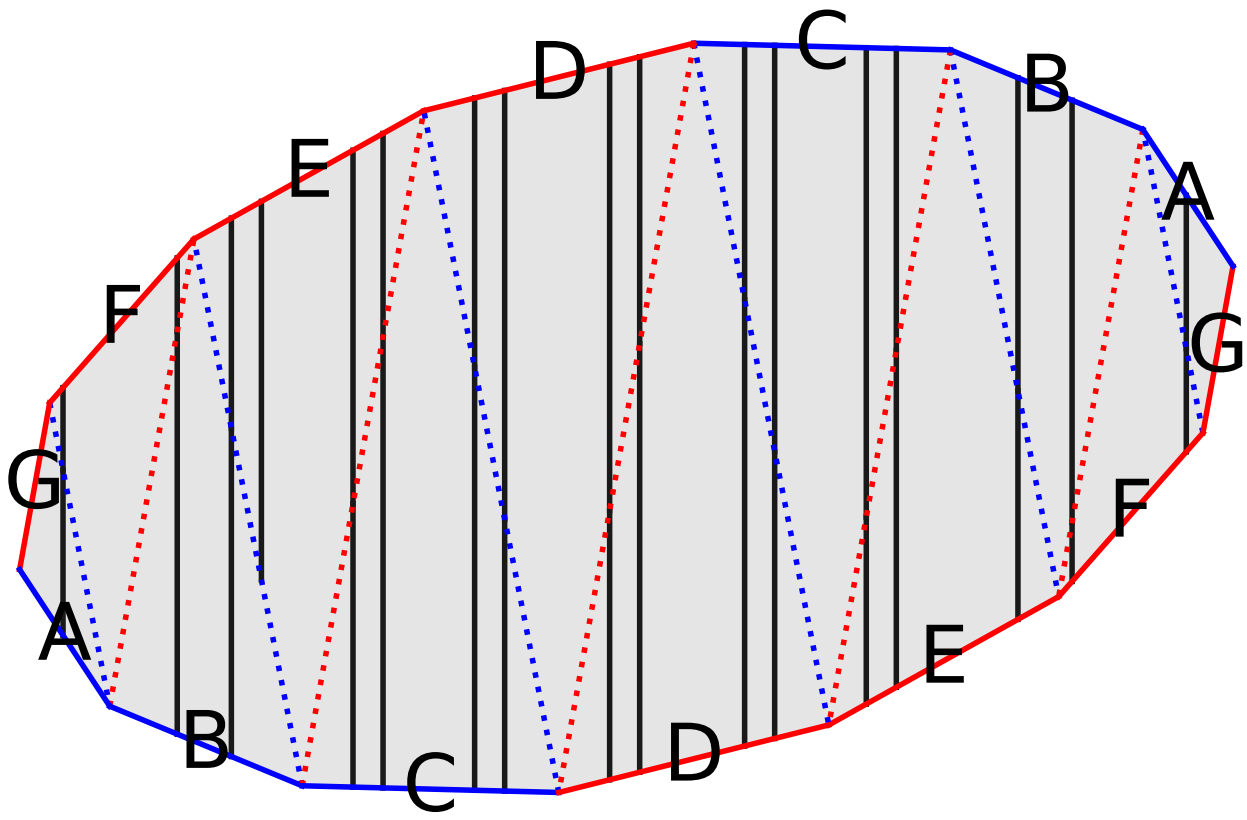

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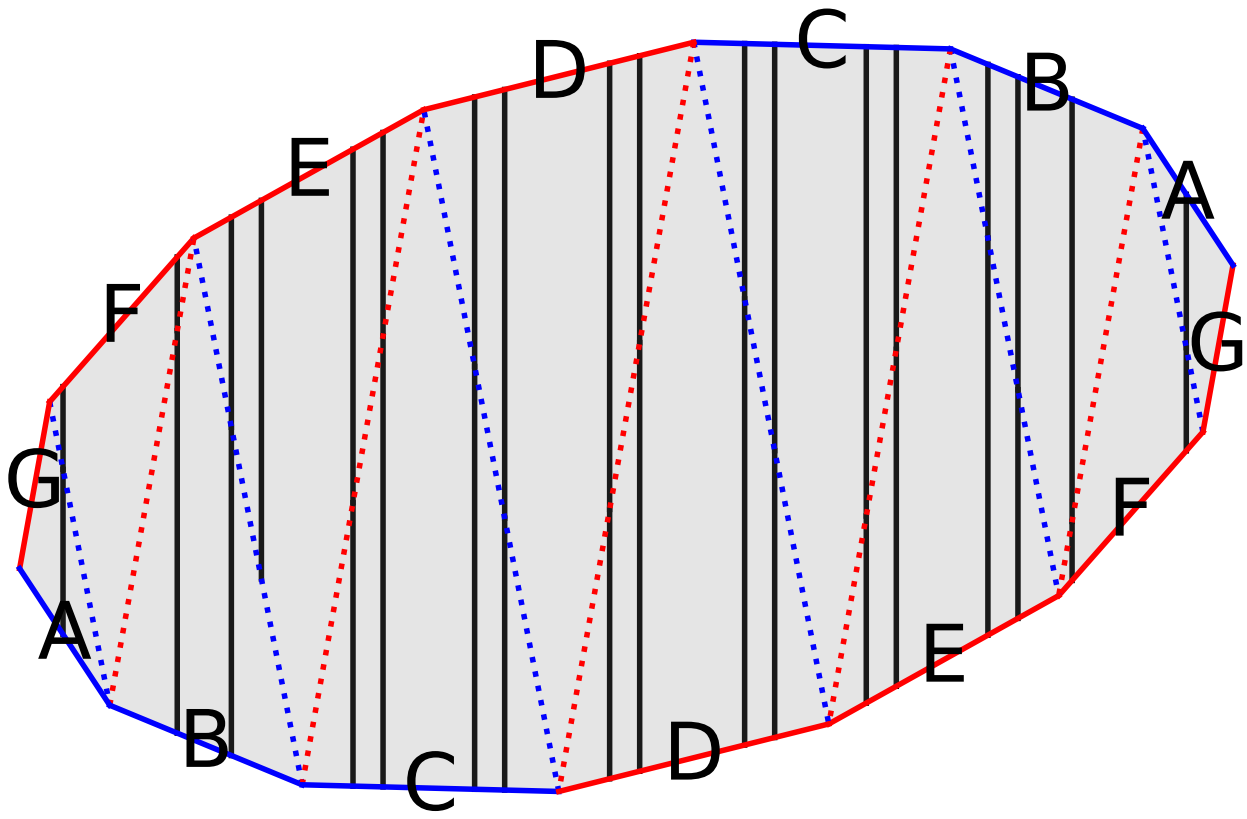

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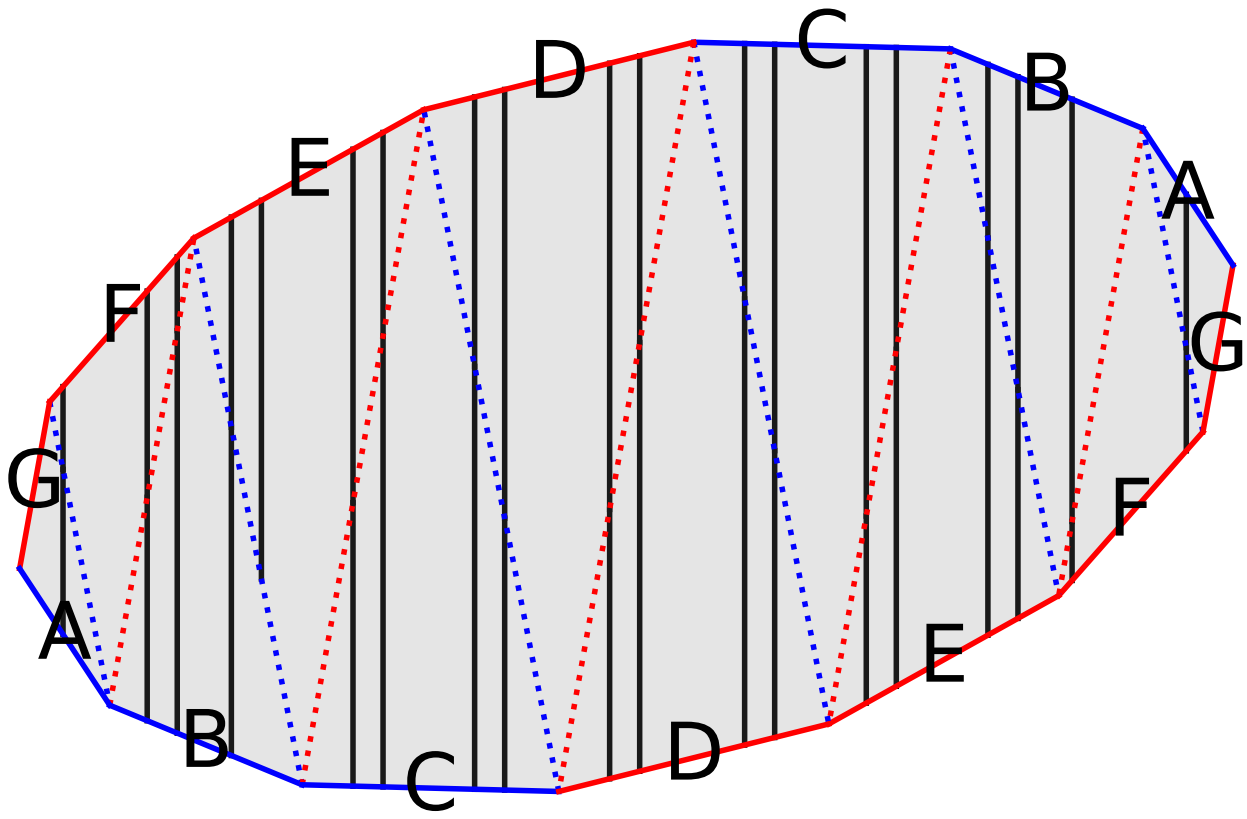

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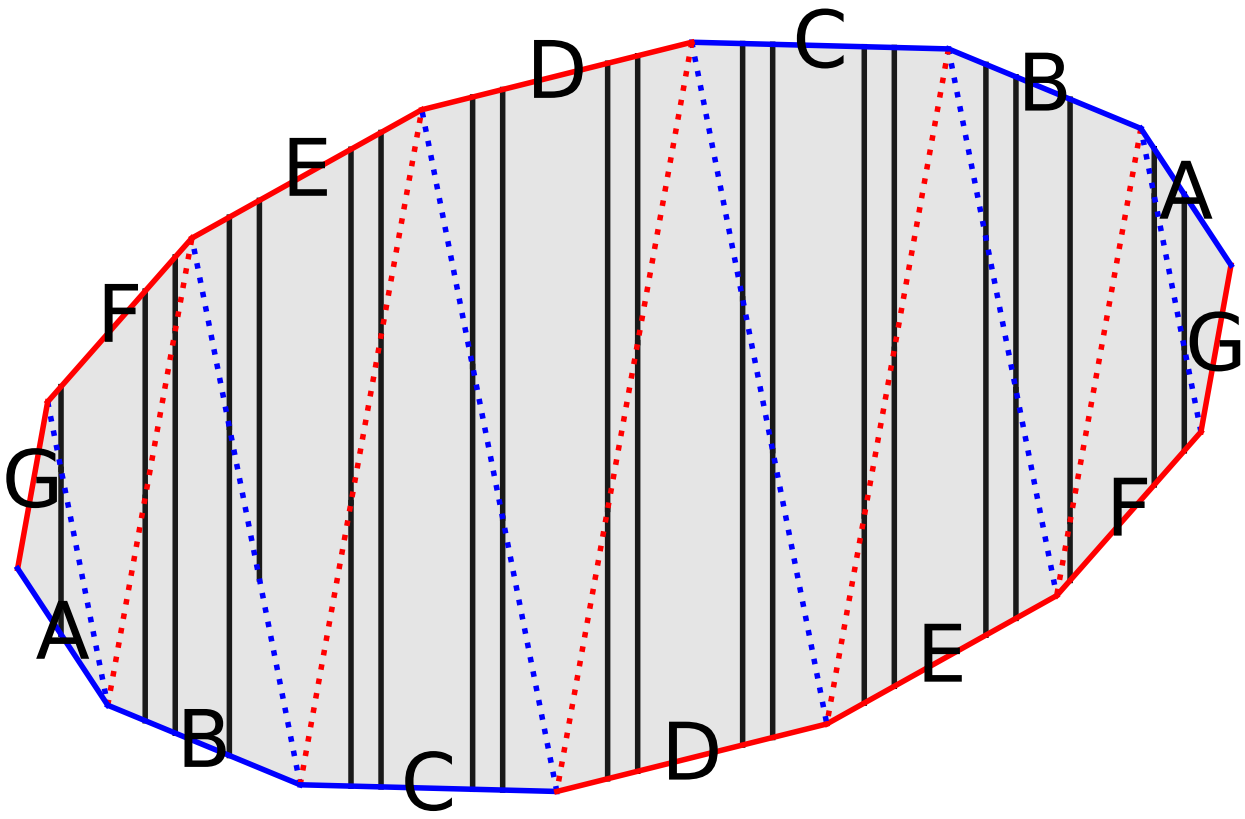

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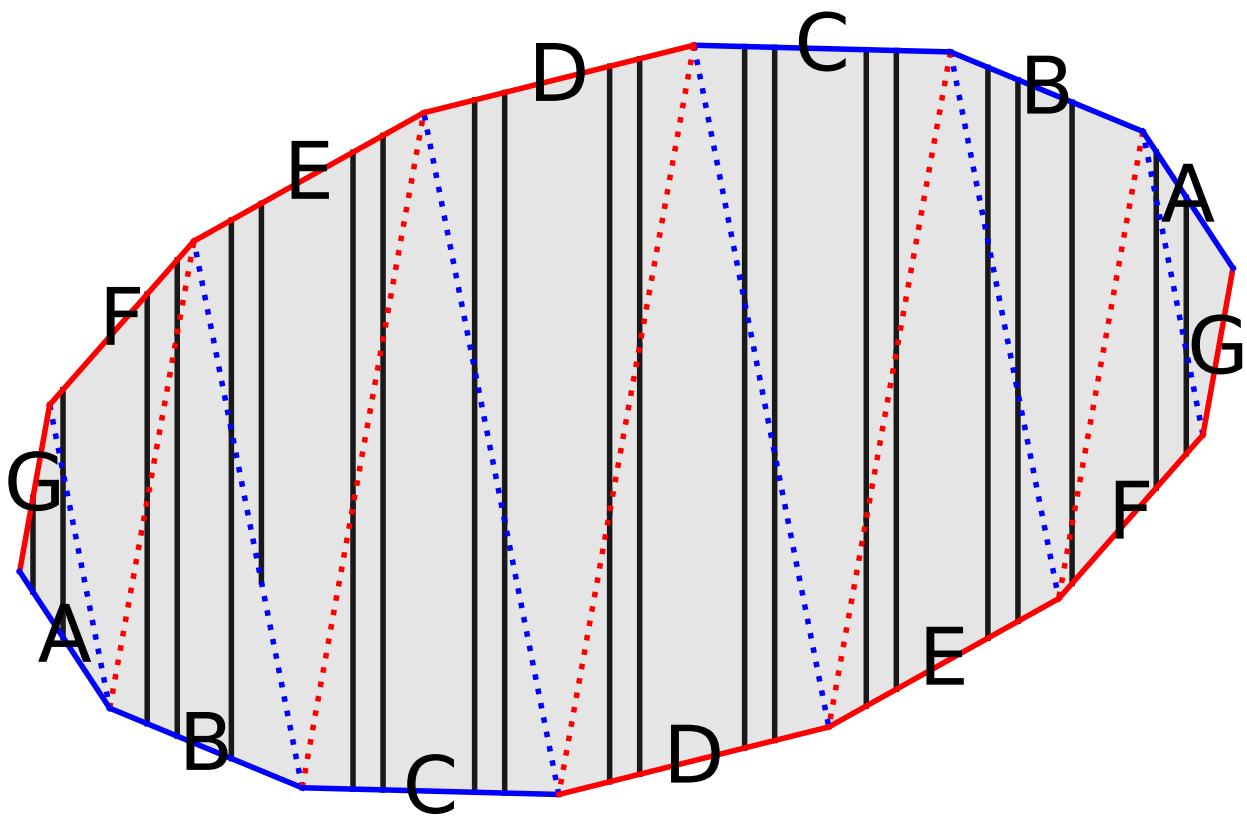

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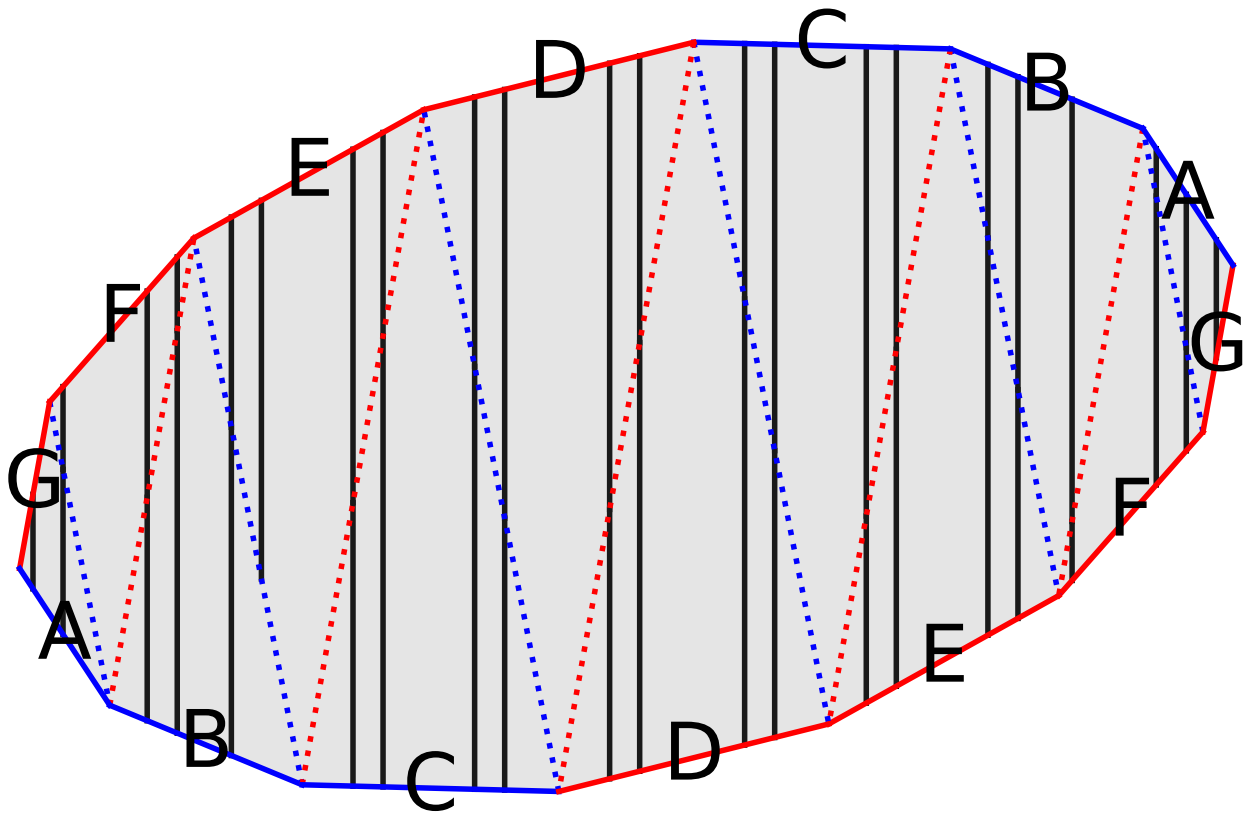

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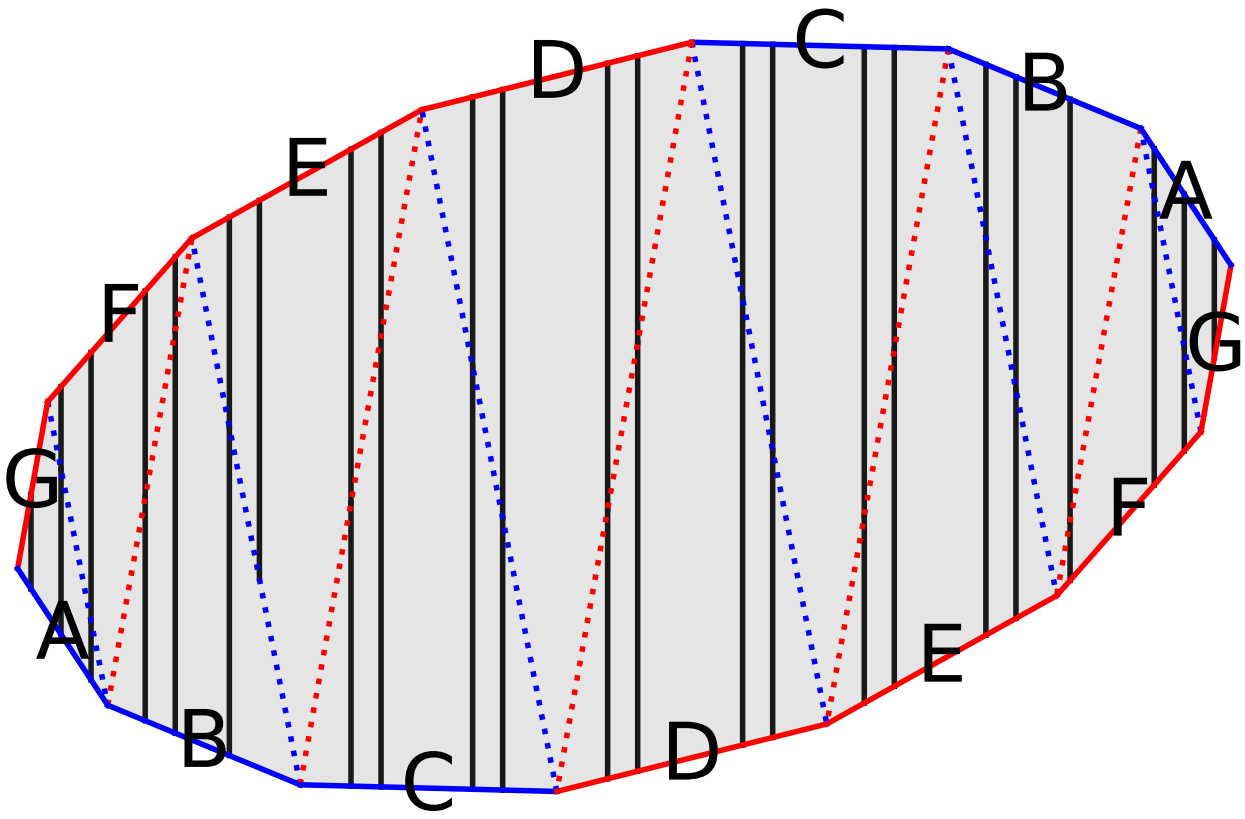

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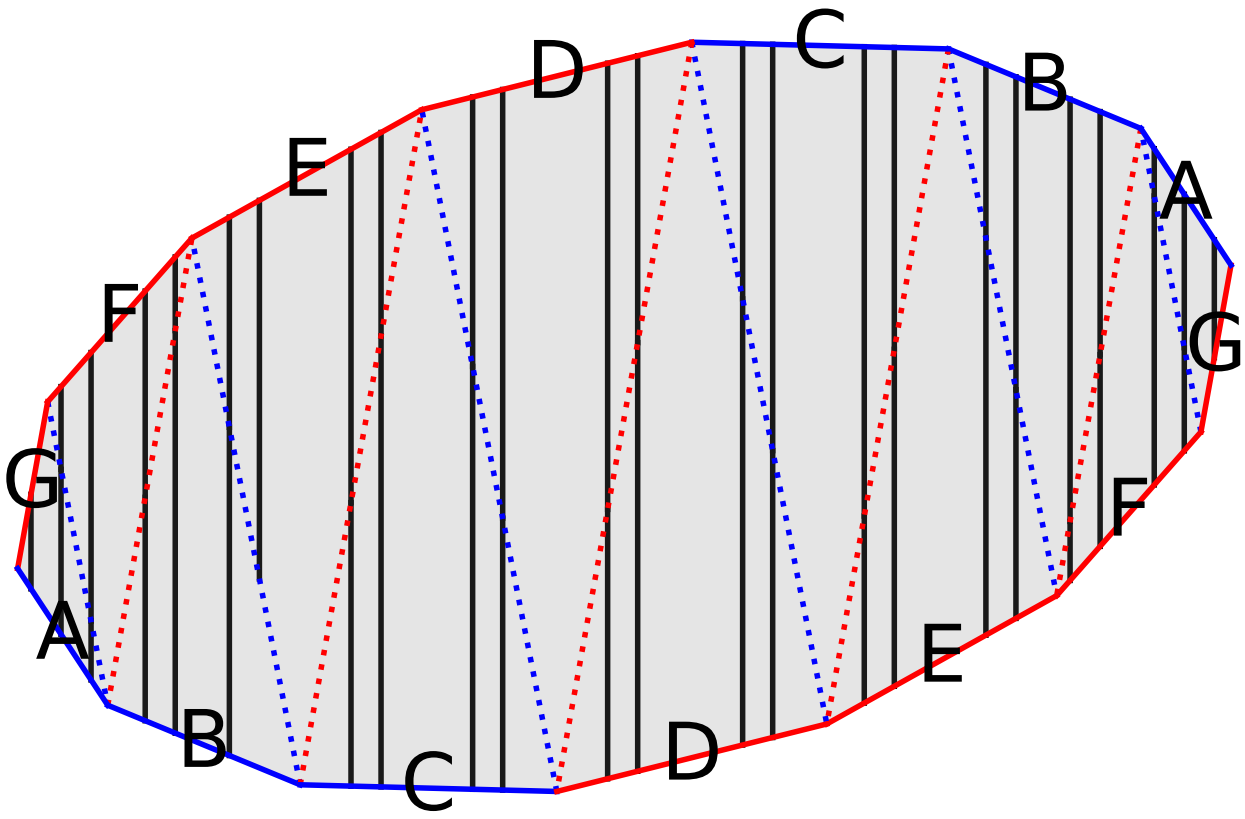

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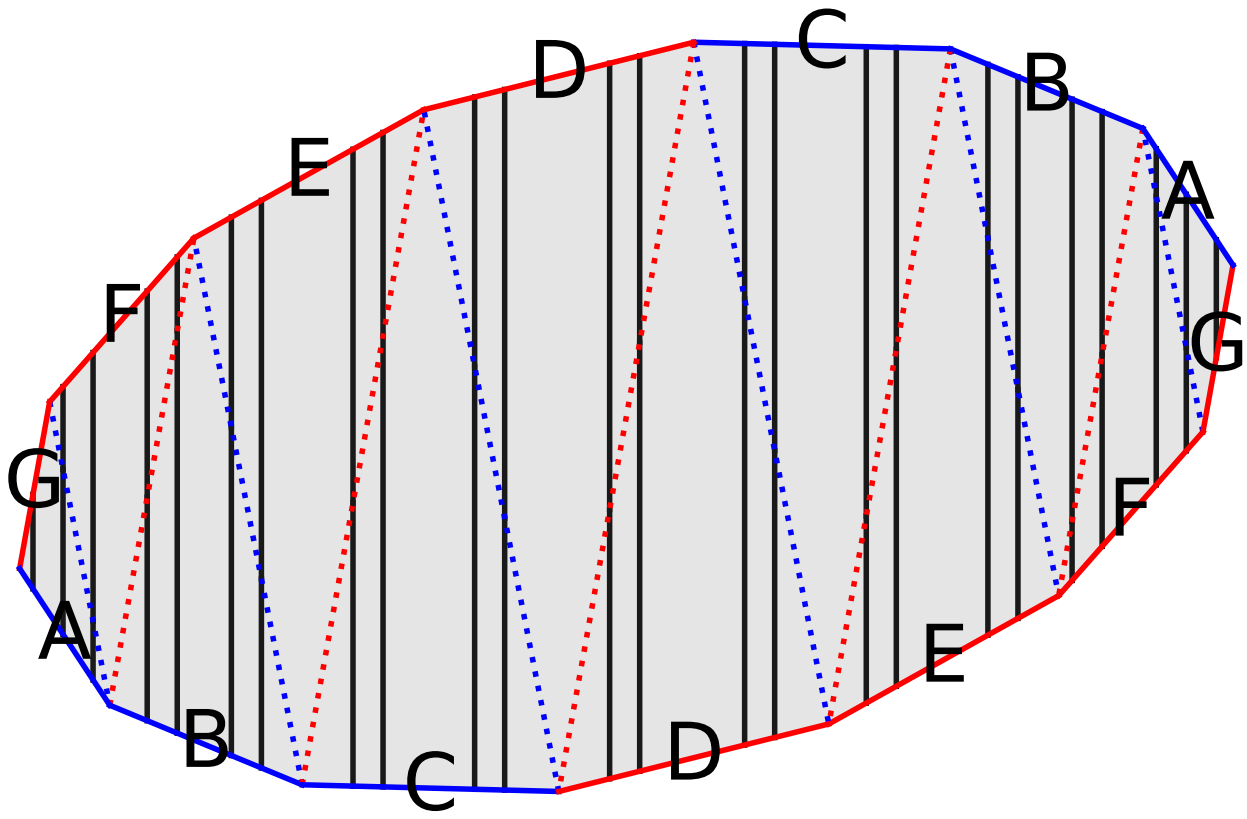

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4. either there exists $\psi : S \to \mathbb{S}^1$, $\theta_0 \in \mathbb{S}^1$ s.t. $\psi \circ F_\theta^t = \psi + \cos(\theta + \theta_0) \cdot t$ or for a.e. θ, F_θ^t is weakly mixing

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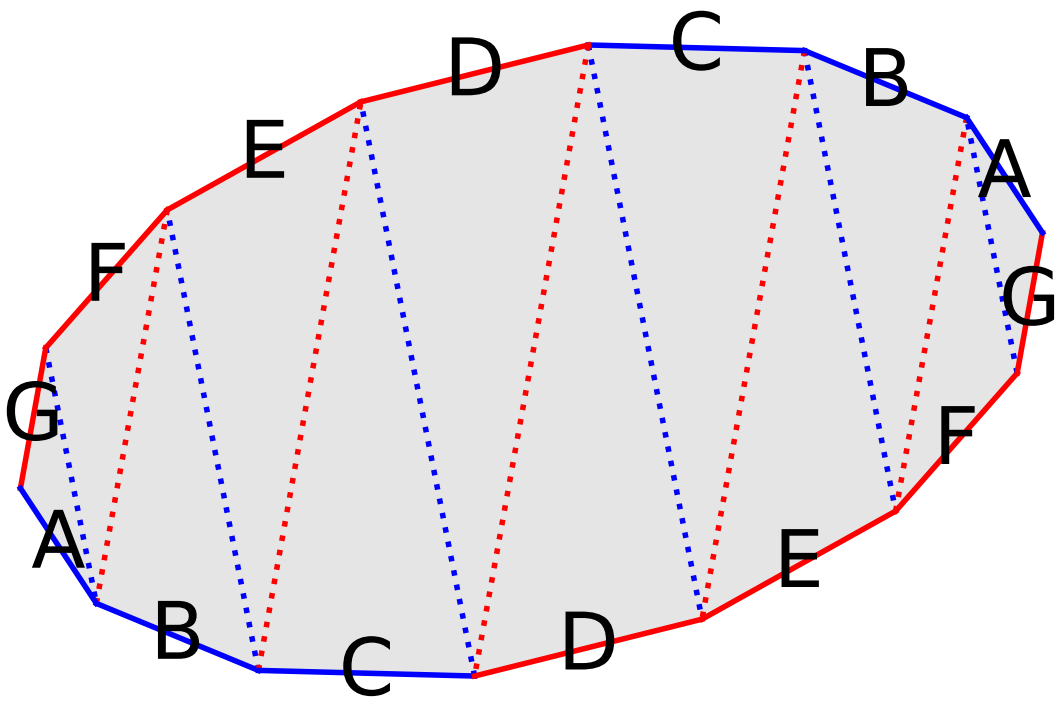
6. *"non-renormalizable" translation flows*

Moduli spaces

$\mathcal{H}_g(\mu) = \mathcal{H}_g(\mu_1, \ldots, \mu_n)$: translation surfaces with singularity pattern μ



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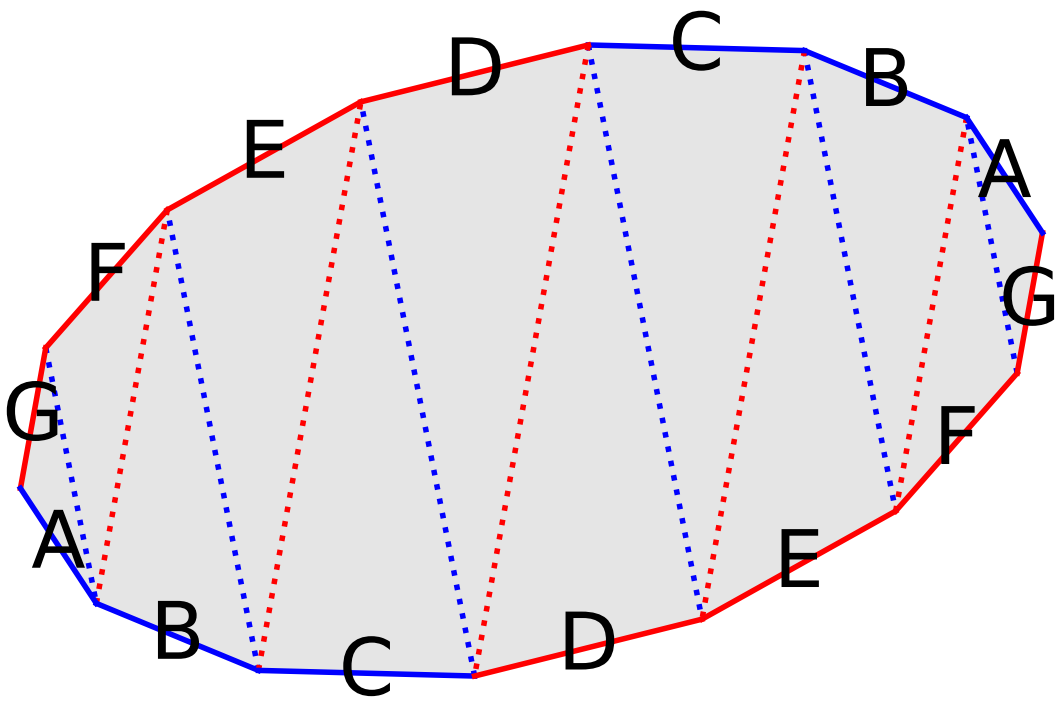

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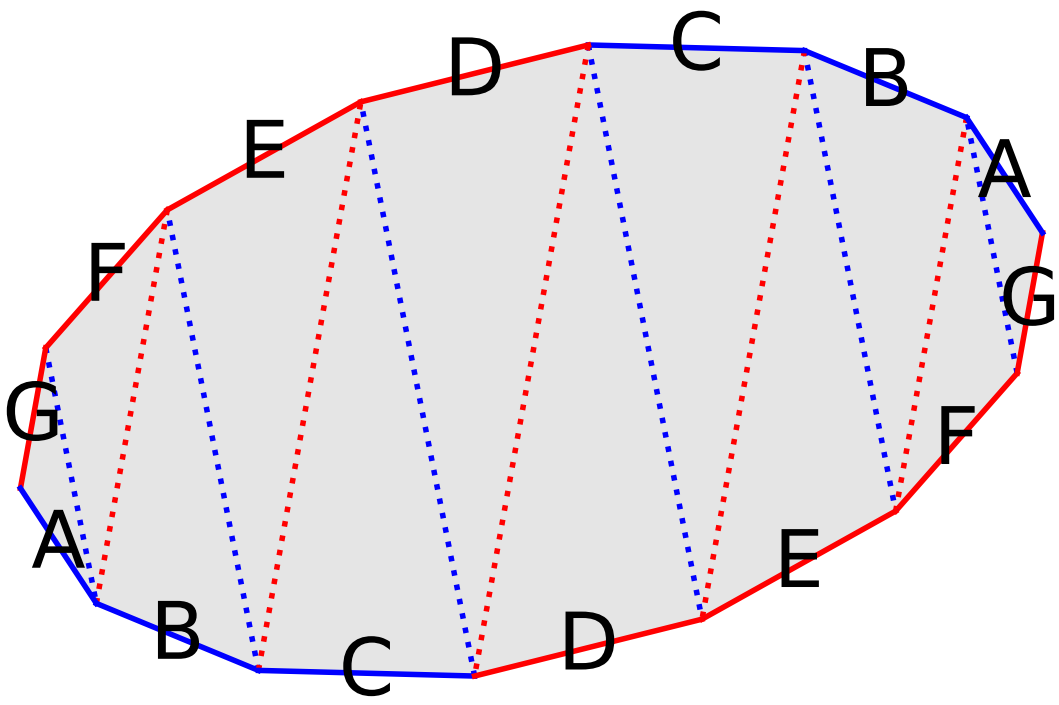

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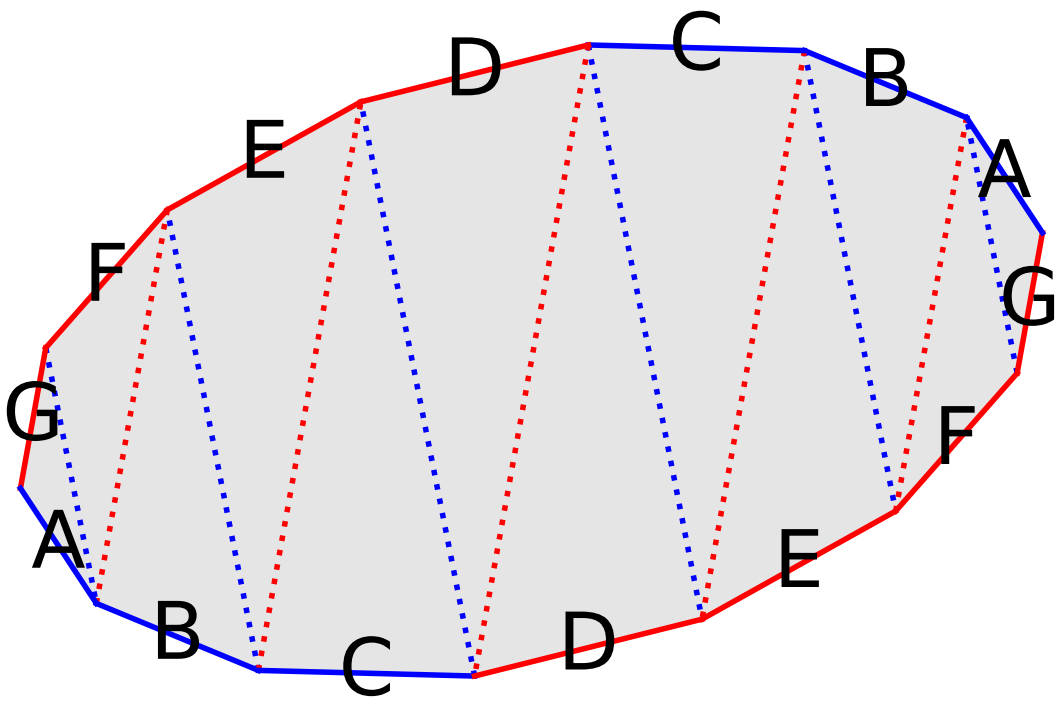

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- Douady-Hubbard (1975): introduction of period coordinates
- Hubbard-Masur (1979): another parametrization of $\mathcal{H}_g(\mu)$

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- admits a (non-algebraic) $\mathrm{GL}_2^+(\mathbb{R})$-action

Unmystifying the package

Plough in the moduli space to harvest in the parameter space.

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- $\lambda_2(S) = \lambda_2(\mathcal{L})$: Lyapunov exponent of $g^t : \mathcal{L} \to \mathcal{L}$
- (*EKZ formula*) $\lambda_1(\mathcal{L}) + \cdots + \lambda_g(\mathcal{L}) = \kappa(\mathcal{H}_g(\mu)) + c_1(\mathcal{L})$

Linear subvarieties

A *linear subvariety* in $\mathcal{H}_g(\mu)$ is an algebraic subvariety which is linear with real coefficients.

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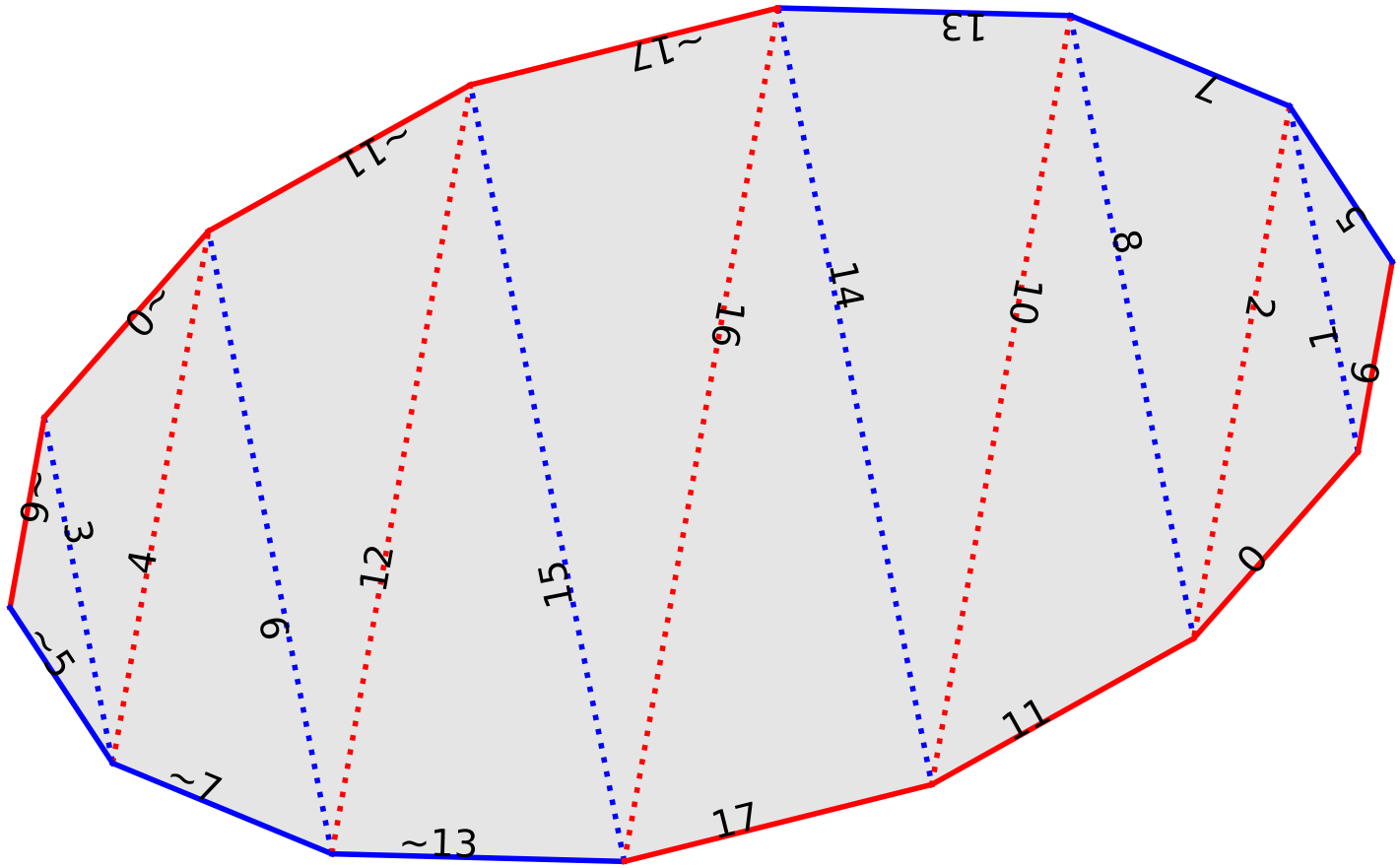
(conversely: linear subvarieties are closed and $\mathrm{GL}_2^+(\mathbb{R})$-invariant)

Examples of linear subvarieties

Regular $2n$-gons (Veech)

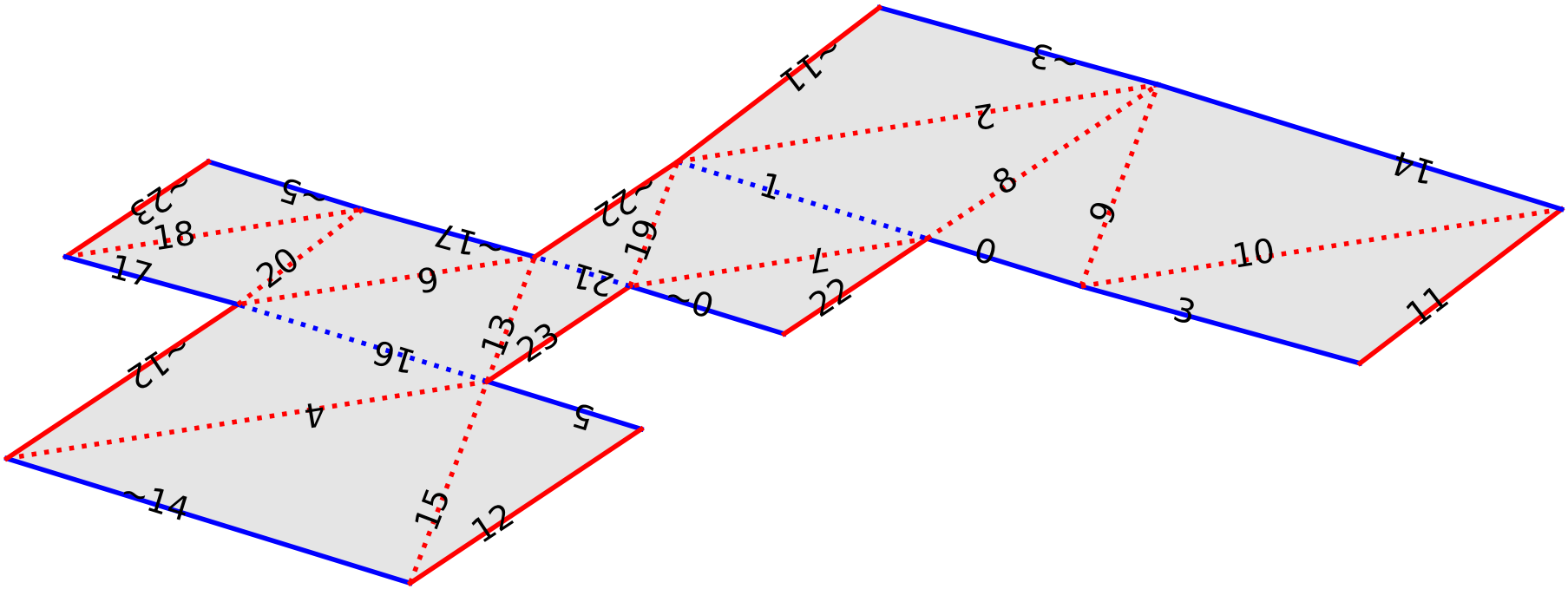
$\dim \mathbb{C}\mathcal{L} = 2,\ \mathrm{rank}(\mathcal{L}) = 1,\ k(\mathcal{L}) = \mathbb{Q}[2\sin(\pi/n)]$



The $(3, 4, 13)$-triangle unfolding (Delecroix-Rüth + DFZ)

$\dim_{\mathbb{C}} \mathcal{L} = 4$, $\mathrm{rank}(\mathcal{L}) = 2$, $k(\mathcal{L}) = \mathbb{Q}[\varphi]$ where $\varphi = \frac{1+\sqrt{5}}{2}$

$z_1 = \varphi \cdot z_0 \qquad z_3 = \varphi \cdot z_{17} \qquad z_2 = \varphi \cdot z_6 \qquad z_4 = \varphi \cdot z_k$



Classification of linear subvarieties (Apisa, Aulicino, Bouw, Delecroix, Eskin, Hooper, Lanneau, McMullen, Mirzakhani, Möller, Mukamel, Nguyen, Rüth, Ward, Wright, . . .)

Exhaustive list of known examples

1. *covering constructions* (including double cover of quadratic differentials)
2. *Bouw-Möller Teichmüller curves*
3. *discriminant loci in arithmetic rank 2 linear subvarieties*: infinitely many rank 1 linear subvarieties defined over $\mathbb{Q}$ or quadratic fields $\mathbb{Q}[D]$
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Open question: any other?

Certifying linear subvarieties

The finiteness result

$\mathcal{H}_g(\mu)$: moduli space of (possibly meromorphic) translation surfaces

Theorem (DFZ)

There exists a finite collection of linear maps $s^\tau : P^\tau \to \mathcal{H}_g(\mu)$ where P^τ are polytopes such that

- $s^\tau(P^\tau) \cap s^{\tau'}(P^{\tau'}) \neq \emptyset$ *if and only if* $\tau = \tau'$,
- $\mathcal{H}_g(\mu) \setminus \bigcup s^\tau(P^\tau)$ *has codimension one (the walls)*

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It satisfies that

1. *A linear subvariety $\mathcal{L} \subset \mathcal{H}_g(\mu)$ intersects each $s^\tau(P^\tau)$ finitely many times and is transverse to the walls.*
2. *If $f : \mathcal{L} \to \mathcal{H}_g(\mu)$ is linear with $\mathcal{L}$ complete linear orbifold and $f(\mathcal{L}) \cap D^\tau$ is finite for each τ, then $f(\mathcal{L})$ is a subvariety.*

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The $s^\tau : P^\tau \to \mathcal{H}_g(\mu)$ are called *Delaunay cells*.

The finiteness result

Corollary

Given a translation surface $M \in \mathcal{H}_g(\mu)$ and a (real) subspace $E \subset T_M\mathcal{H}_g(\mu)$, there is a program that terminates if and only if E is the tangent space of a linear subvariety $\mathcal{L}$.

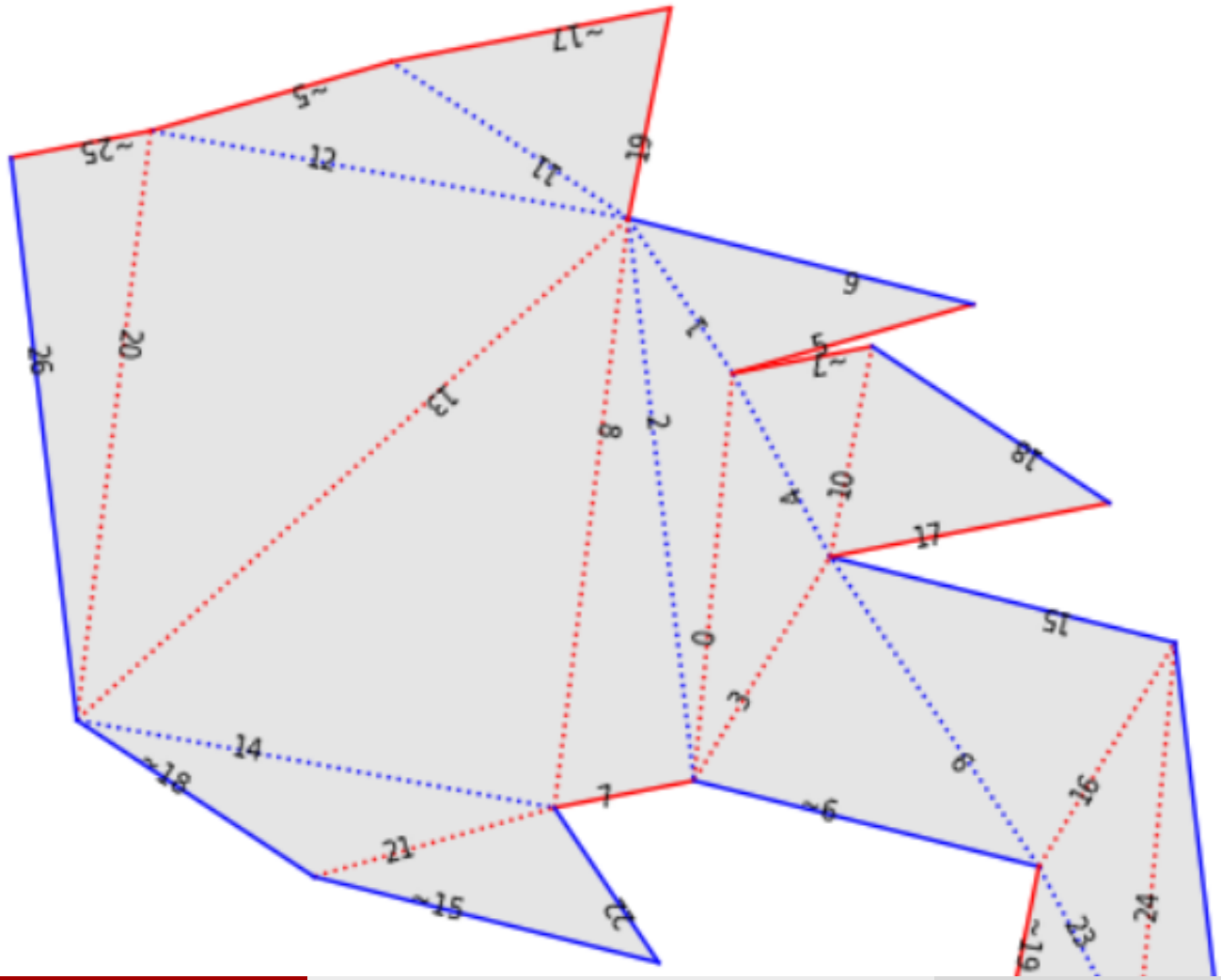
The gothic locus

```
triangles = "(0,1,2)(3,4,~0)(5,6,~1)(7,~2,8)(9,~3,~6)(10,~7,~4)(11,~5,12)(13,14,~8)
            "(17,18,~10)(19,~17,~11)(20,~13,~12)(21,~14,~18)(22,~21,~15)(23,24,~16)
            "(26,~20,~25)(~26,~24,~22)"
colouring = "RBBRBRBRRBRBBRBBRRBRRRBBRRB"
vt = VeeringTriangulation(triangles, colouring)
subspace = [(1, 0, -1, 0, -1, 0, 0, 0, 1, 0, 1, -1, -1, 0, -1, 0, 0, 0, -1, 1, 1, 0
            (0, 1, 1, 0, 0, 1, 2, 0, -1, 2, 0, 0, 1, 0, 1, 2, 0, 0, 0, 0, -1, 1, 1,
            (0, 0, 0, 1, 1, 1, 1, 0, 0, 0, -1, 1, 2, 2, 2, 1, 1, 0, 1, -1, 0, 1, 0,
            (0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 1, 2, 1, 0, 0, 2, 1, 1, 1, 0, 0, 0
F = VeeringTriangulationLinearFamily(vt, subspace)
print(F.stratum())
```

```
H_4(2^3)
```

The gothic locus

```
F.flat_structure_geometric_middle().plot()
```



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```
%%time
L = F.linear_subvariety(multiscale_structure=False)
```

```
CPU times: user 6min 15s, sys: 3.97 s, total: 6min 19s
Wall time: 6min 20s
```

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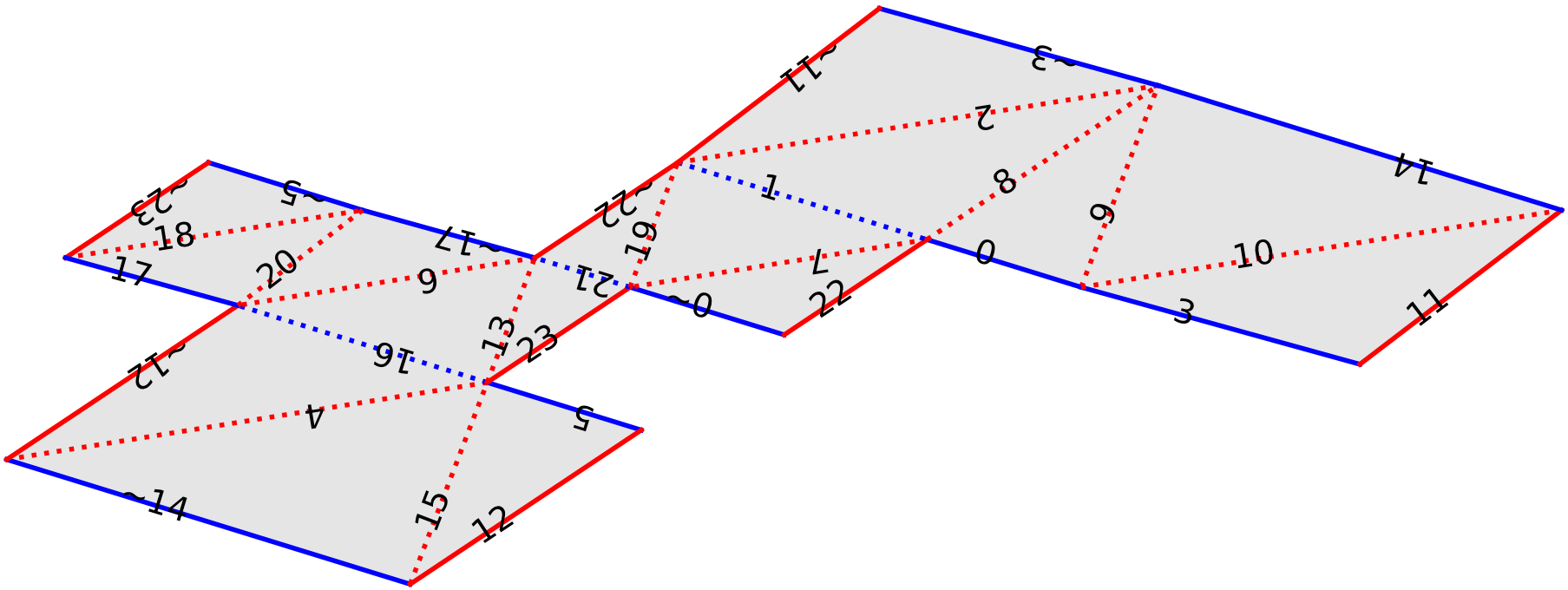
Required the computation of the 47448 components $s^\tau(P^\tau) \cap \mathcal{L}$.

The Delaunay cells $s^\tau : P^\tau \to \mathcal{H}_g(\mu)$

Definition

A *veering triangulation* τ is a triangulation together with a red-blue coloring of its edges so that there are neither monochromatic face nor monochromatic vertex.



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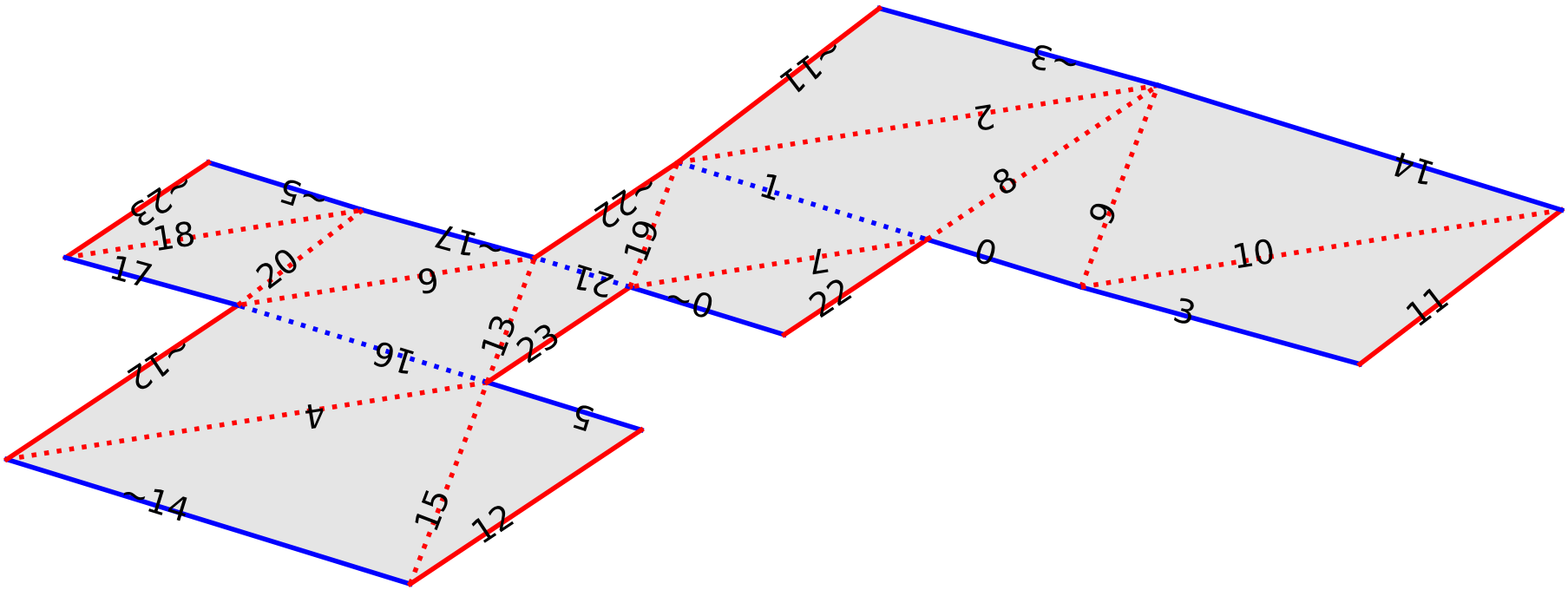

$s^\tau : V^\tau \times H^\tau \to \mathcal{H}_g(\mu)$ where $(x, y) \in V^\tau \times H^\tau \subset (\mathbb{R}^{E(\tau)})^2$ i.f.f.

- non-negativity: for each $e \in E(\tau)$, $x_e \geq 0$ and $y_e \geq 0$
- triangular equalities: e.g. $x_{10} = x_3 + x_{11}$ or $y_{15} = y_{12} + y_5$

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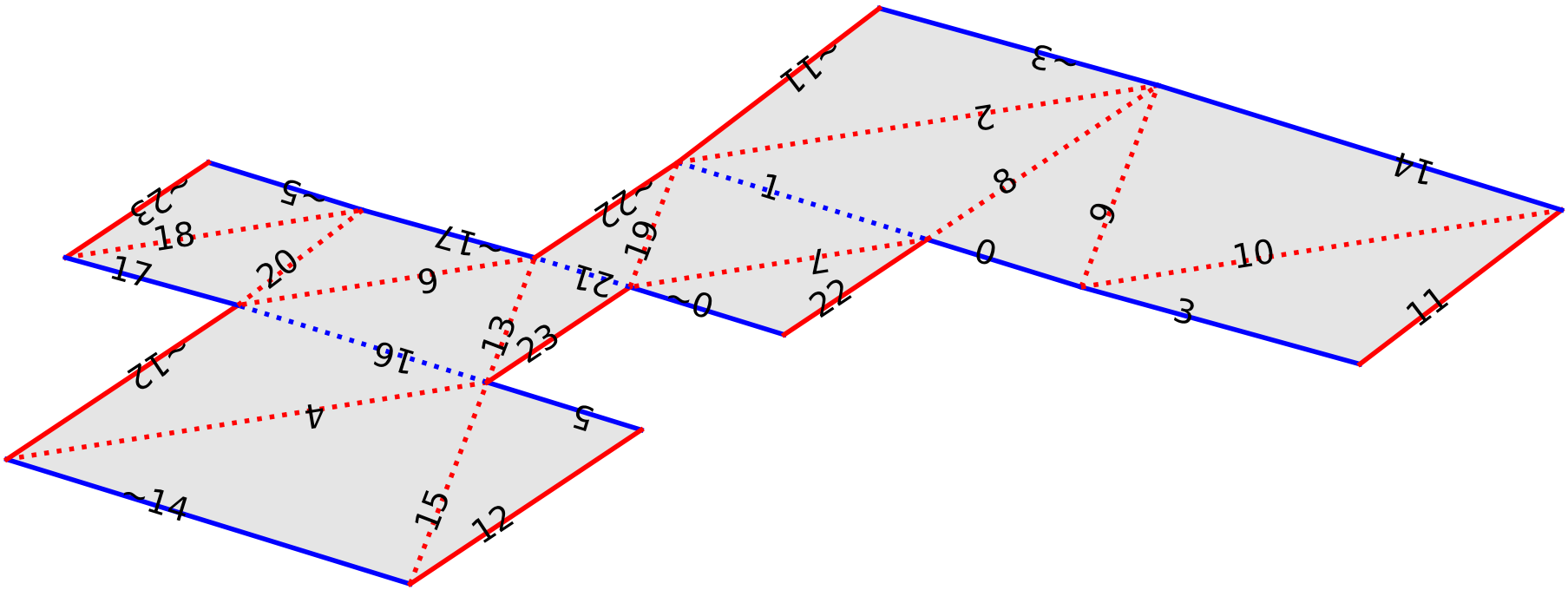

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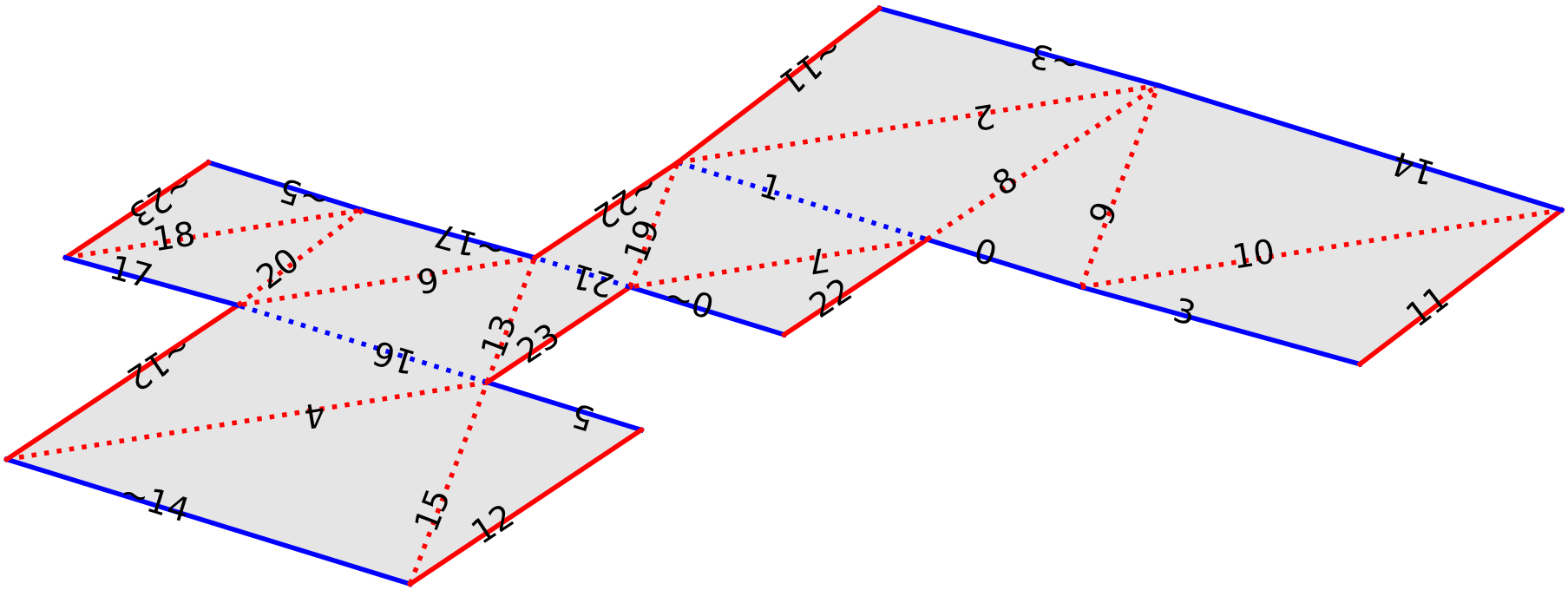
minor problem: local parametrization

major problem: injectivity

The Delaunay cells $s^\tau : P^\tau \to \mathcal{H}_g(\mu)$



$s^\tau : V^\tau \times H^\tau \to \mathcal{H}_g(\mu)$

$P^\tau \subset V^\tau \times H^\tau$ is defined by adding one set of inequations

- non-negativity: for each $e \in E(\tau)$, $x_e \geq 0$ and $y_e \geq 0$
- triangular equalities such as $x_{10} = x_3 + x_1 1$ or $y_{15} = y_{12} + y_5$
- Delaunay condition: e.g. $x_{14} + x_{15} \leq y_{15} + y_{16}$

The Delaunay cells $s^\tau : P^\tau \to \mathcal{H}_g(\mu)$

The obtained Delaunay cells $s^\tau : P^\tau \to \mathcal{H}_g(\mu)$ satisfy

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The Delaunay cells and linear subvarieties

Now to show

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one uses the following ingredients

- understand the geometry of $\mathcal{L}$ near the boundary and use compactness and transversality away from the boundary
- o-minimality (only required for certifying algebraicity in the meromorphic case)

The multiscale compactification

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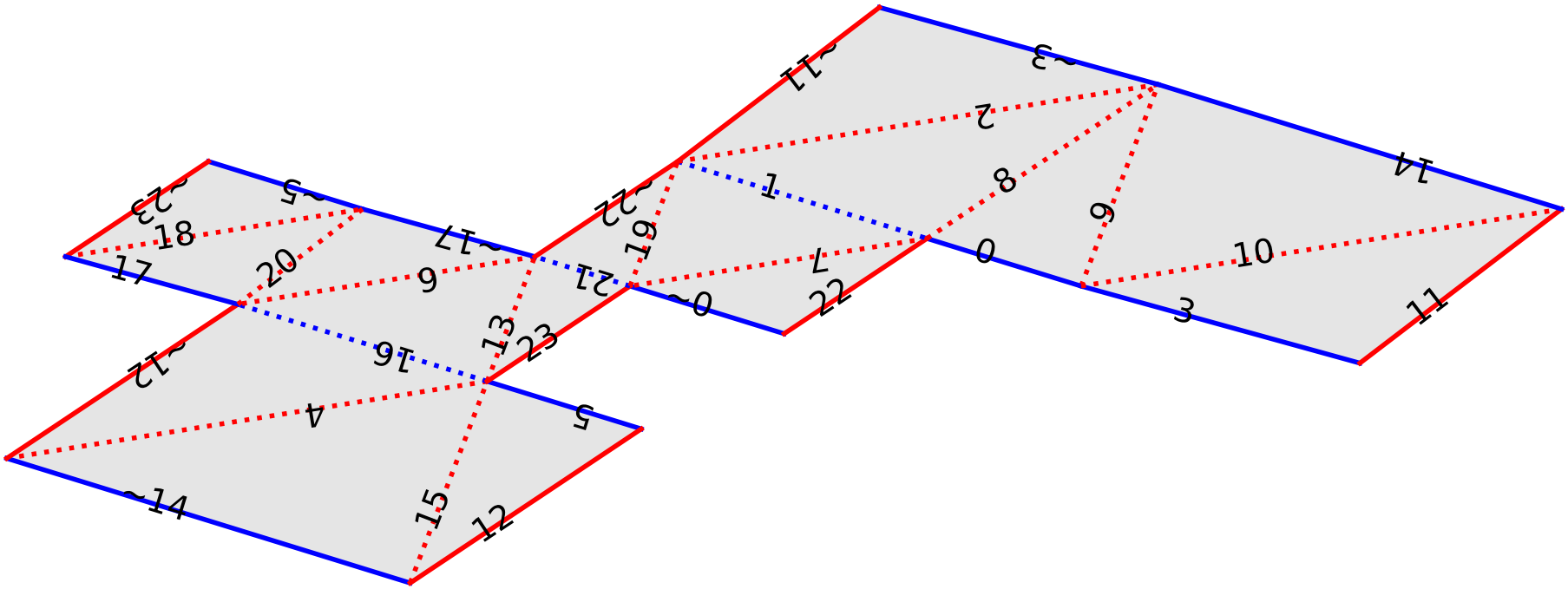
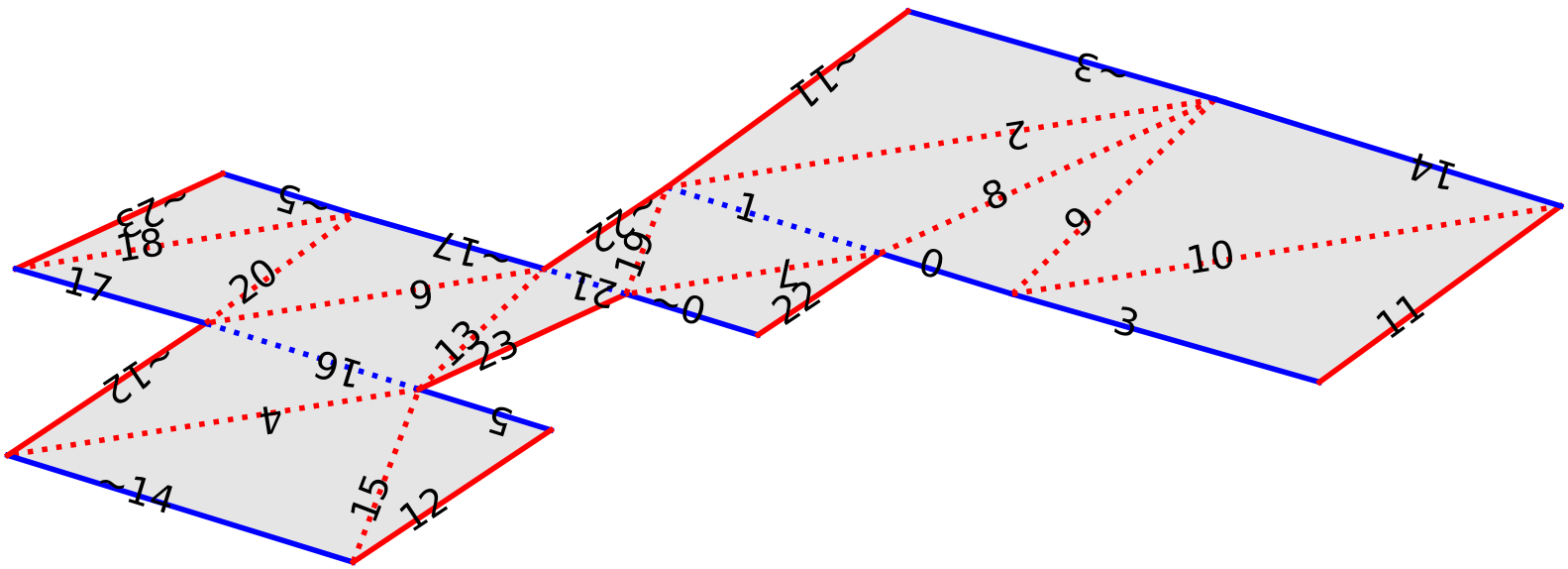


$\mathcal{Q}(4, 11)$

The multiscale compactification



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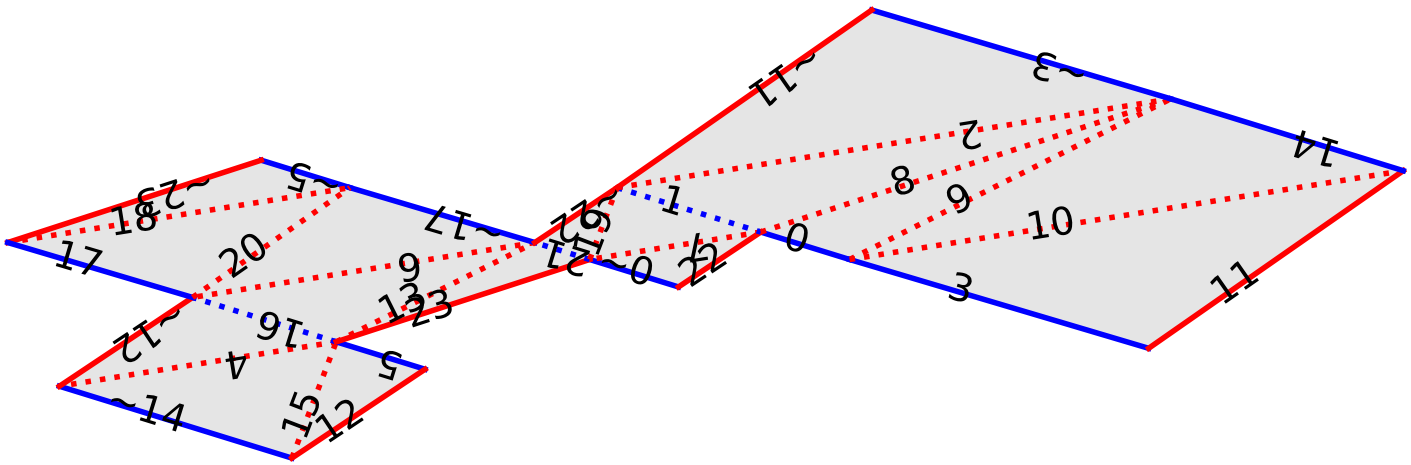

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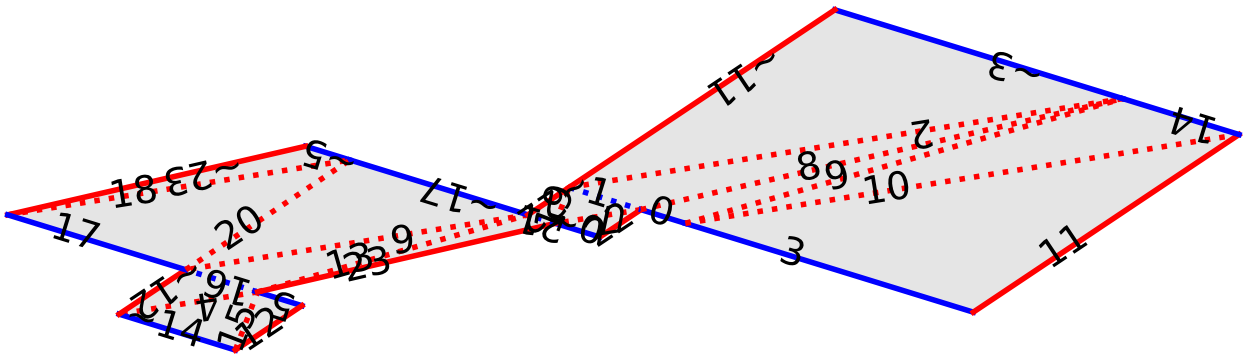

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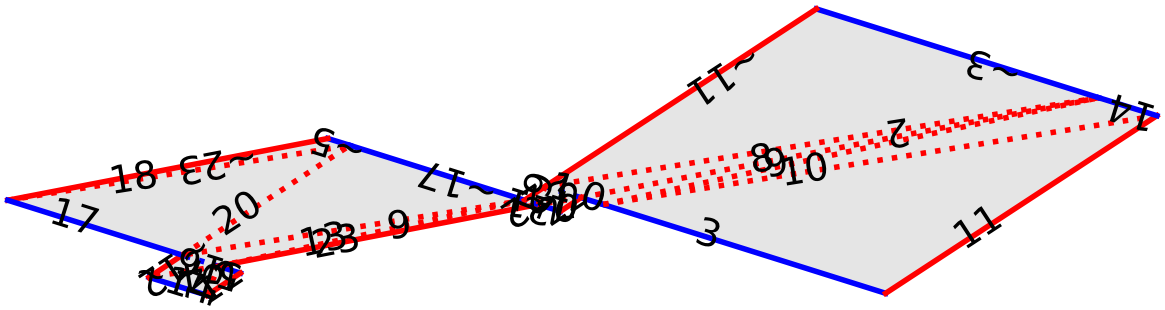

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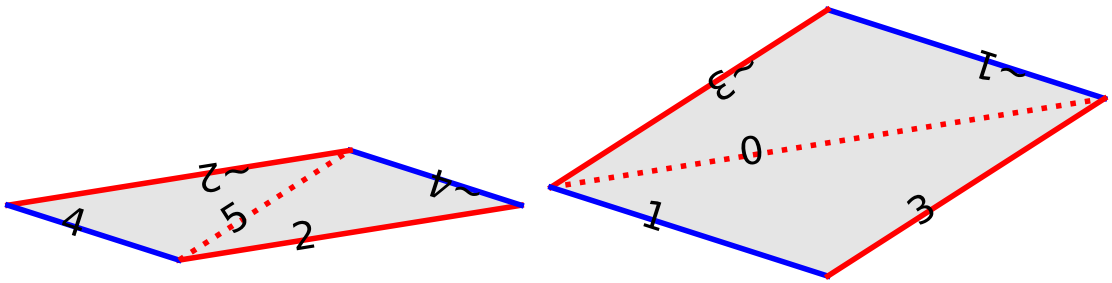
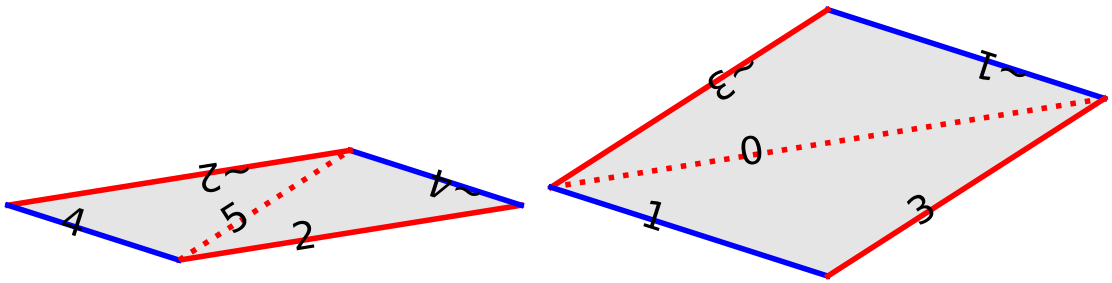


$\mathcal{H}(0) \times \mathcal{H}(0)$

The multiscale compactification



$\mathcal{H}(0) \times \mathcal{H}(0)$ . . . picture misses the lower level in $\mathcal{Q}_2(11, 1, -4^2)$

Blowups of $\overline{P^\tau}$ and multiscale compactification

$\mathcal{H}_g(\mu)$: (non-projectivized) moduli space of (possibly meromorphic) translation surfaces
$\overline{\mathcal{H}_g(\mu)}$: (non-projectivized) multiscale bordification
$s^\tau : P^\tau \to \mathcal{H}_g(\mu)$: Delaunay cell

Theorem (V. Delecroix, K. Fu, B. Zykoski)

There exists a blow-up $\widehat{s^\tau} : \widehat{P^\tau} \to \overline{\mathcal{H}_g(\mu)}$ such that $\widehat{s^\tau}(\partial P^\tau)$ is a union of (refined) Delaunay cells in $\partial\mathcal{H}_g(\mu)$. Moreover, for a point $M \in s^{\tau'}(P^{\tau'})$ in some Delaunay cell $s^{\tau'} : P^{\tau'} \to \overline{\mathcal{H}_g(\mu)}$ and a path $(z_t)_{t\in(0,\epsilon)} \subset P^\tau$ we have equivalence between

- *$s^\tau(z_t)$ converges to M in $\overline{\mathcal{H}_g(\mu)}$ (as $t \to 0$)*
- *z_t converges in $\widehat{P^\tau}$ and $M = \lim \widehat{s^\tau}(z_t)$ (as $t \to 0$)*