Topology of Hénon maps.

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Polynomial Automorphisms of $\mathbb{C}^2$.

- Complex Hénon map

$$H\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p(x) - ay \\ x \end{pmatrix}$$

where p is a polynomial, is an example of an automorphism.

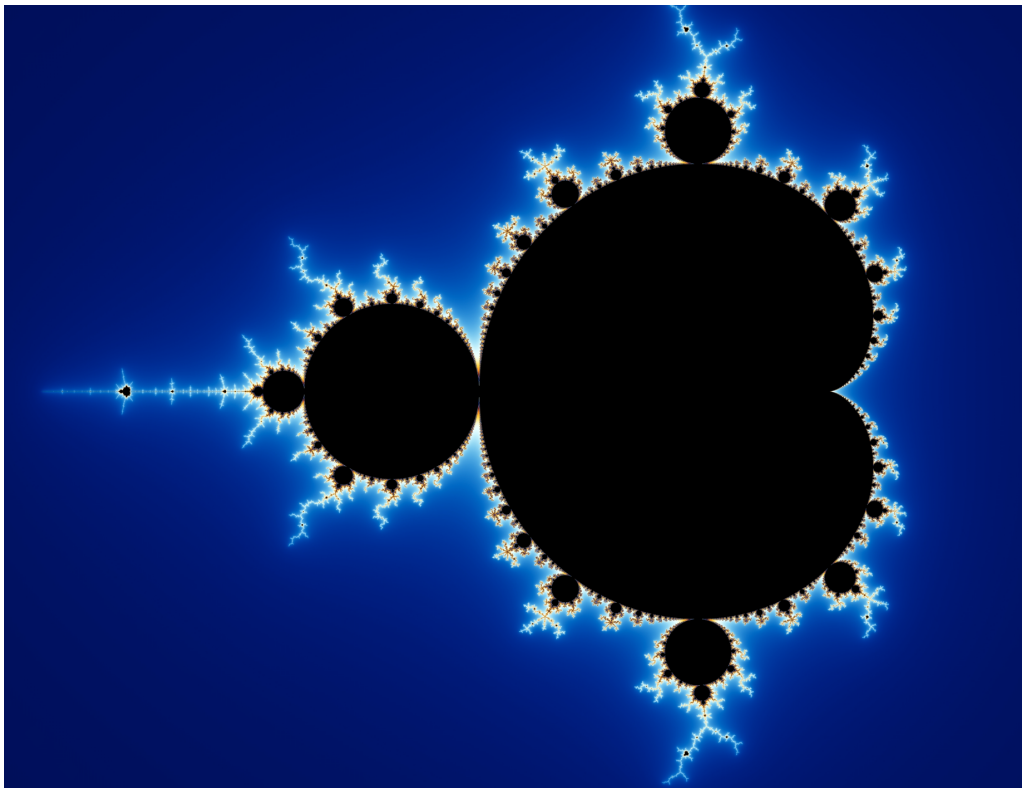
$$H^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ \frac{p(y)-x}{a} \end{pmatrix}$$

- The real version was introduced by Hénon as an approximation of the first return map of the Lorenz flow.
- Each polynomial automorphism of $\mathbb{C}^2$ with positive entropy is conjugate to a composition of Hénon maps (Friedland-Milnor).
- In this talk, we consider quadratic Hénon maps, $p(z) = z^2 + c$.

Family $p_c(z) = z^2 + c$

Mandelbrot set

$$M = \{c \in \mathbb{C} : \ p_c^{\circ n}(0) \not\to \infty, n \to \infty\}$$



Böttcher coordinates and Green's functions for Quadratic Polynomials

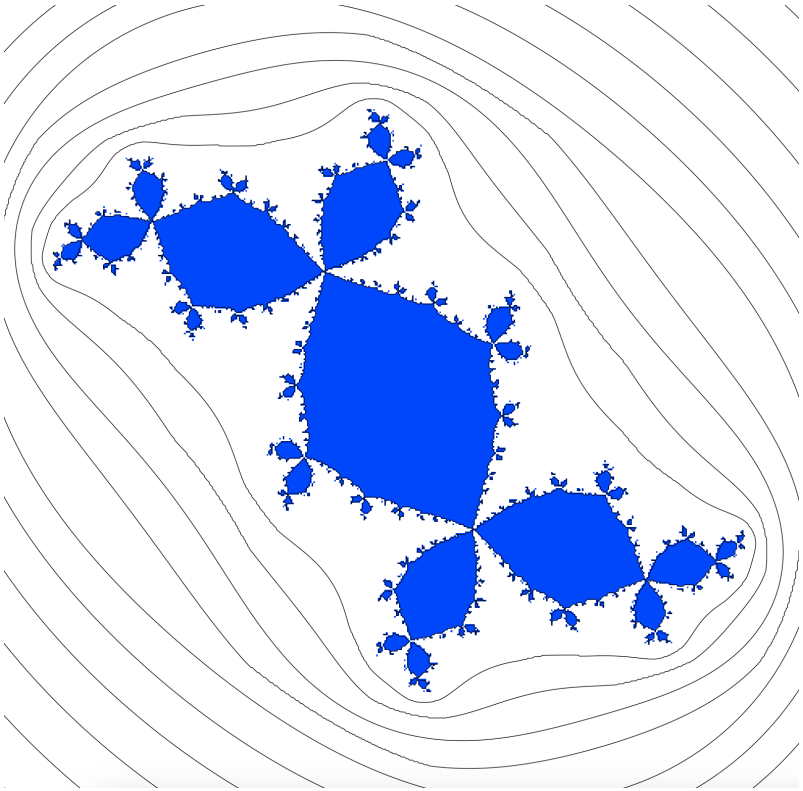


Figure: Rabbit Julia set
$p_c(z) = z^2 + c$,
$c \simeq -0.1226 + 0.7449i$

Böttcher coordinate

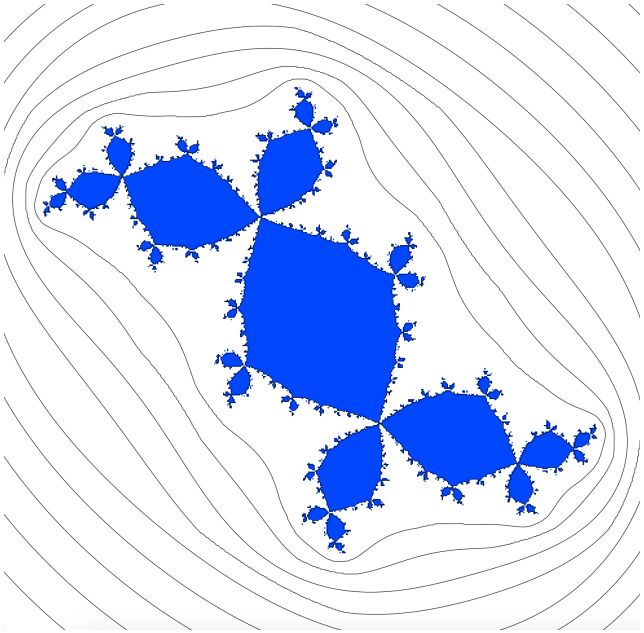
$$\phi_c(p_c) = \phi_c^2$$

$\phi_c \sim x$ as $x \to \infty$

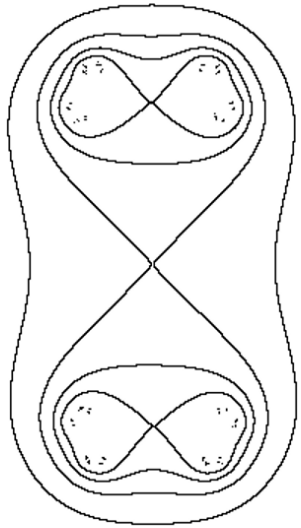
Green's function

$$G_c(x) = \lim_{n\to\infty} \frac{\log^+ |p_c^n(x)|}{2^n}$$

Dichotomy for $z \to z^2 + c$



For $c \in M$,
Julia set J_c is connected,
escape locus U_c is a punctured disk, ϕ_c is well-defined in U_c.



For $c \notin M$,
Julia set J_c is a Cantor set.
ϕ_c is not well-defined in U_c, but for every point in U_c, there is n such that $\phi_c^{2^n}$ is well-defined.

Dynamical Filtration for Hénon Maps

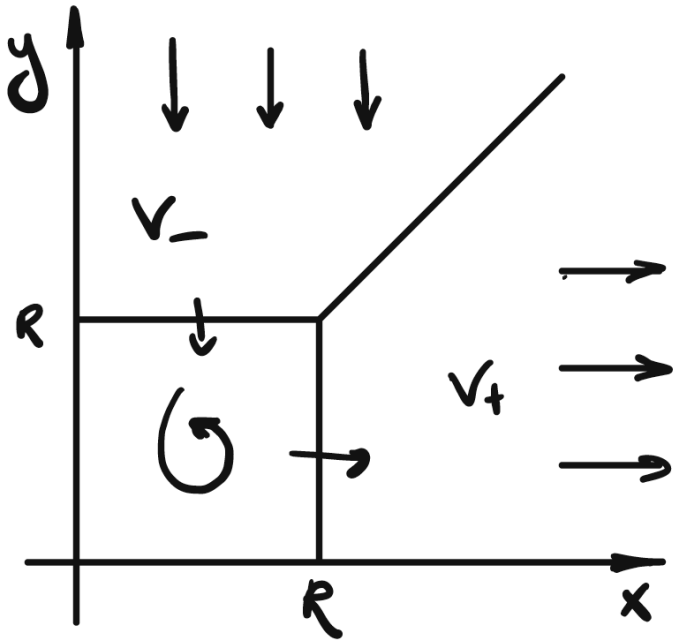


Figure: Dyn. Filtration

$$V^+ = \{|x| > |y|, |x| > R\}$$
$$V^- = \{|y| > |x|, |y| > R\}$$

Escape Sets U^+, U^-

$$U^+ = \{(x,y) : \lim_{n\to\infty} |H^n(x,y)| = \infty\}$$
$$U^+ = \cup_{n=0}^{\infty} H^{-n}(V^+)$$
$$U^- = \{(x,y) : \lim_{n\to\infty} |H^{-n}(x,y)| = \infty\}$$
$$U^- = \cup_{n=0}^{\infty} H^{n}(V^-)$$

$$K^+ = \mathbb{C}^2 \backslash U^+, \; J^+ = \partial K^+$$
$$K^- = \mathbb{C}^2 \backslash U^-, \; J^- = \partial K^-$$
$$K = K^- \cap K^+, \; J = J^- \cap J^+$$

Sets J^+, J^-, J are Julia sets of the Hénon map H

Böttcher coordinates, Green's Functions.

- In V^+ : $\phi_+(H) = \phi_+^2$, $\phi_+ \sim x$ as $x \to \infty$
- Böttcher coordinate ϕ_+ was introduced by Hubbard and Oberste-Vorth.
- ϕ_+ does not extend to all of U^+, but for every point in U^+ there is $n \geq 0$ s.t. $\phi_+^{2^n}$ is well-defined.
- The stable foliation $\mathcal{F}^+$ in V^+ consists of level sets of ϕ_+. It can be propagated by dynamics to all of U^+.
- Green's function measures the exponential rate of escape to infinity under forward iterates of the map.

$$G_+(x,y) = \lim_{n\to\infty} \frac{\log^+ |H^n(x,y)|}{2^n}$$

$$G_+ = \log |\phi_+| \quad (\text{in } V^+)$$

- G_+ is a plurisubharmonic function in $\mathbb{C}^2$, pluriharmonic on U^+.
- Each leaf of $\mathcal{F}^+$ is dense in $\{G_+ = r\}$.

Structure of the escape set

Theorem (Hubbard, Oberste-Vorth)

For a Hénon map, U^+ is foliated by copies of $\mathbb{C}$ given by the extended fibers of φ^+.

$$U^+ = (\mathbb{C} - \overline{\mathbb{D}}) \times \mathbb{C}/\Gamma, \quad \textit{where}$$

$\Gamma \subset \textit{Aut}((\mathbb{C} - \overline{\mathbb{D}}) \times \mathbb{C})$ *is a discrete group isomorphic to* $\mathbb{Z}[\frac{1}{2}]/\mathbb{Z}$.

The same description holds for U^-.

Question

Describe the structure of $U^+ \cap U^-$.

Theorem

For any Hénon map, escape set $U^+ \cap U^-$ is connected.

Shift Locus

- A Hénon map is called a shift map if dynamics on K is conjugate to the 2-shift on bi-infinite duadic sequences.
- For any pair of shift Hénon maps H_1, H_2 there is a homeo $\phi : \mathbb{C}^2 \to \mathbb{C}^2$ such that $H_1 \circ \phi = \phi \circ H_2$. (Buzzard, Verma)
- For any pair of shift Hénon maps, $U^+ \cap U^-$ are homeomorphic.

Theorem

(work in progress) For a shift Hénon map, $H^1(U^+ \cap U^-) = \mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}[\frac{1}{2}]$

Question

Describe $\pi_1(U^+ \cap U^-)$ for shift Hénon maps.

Question

Describe $H^1(U^+ \cap U^-)$, $\pi_1(U^+ \cap U^-)$ for a general Hénon map. Does the description depend on the properties of J?

Mayer-Vietroris sequence for shift Hénon maps

For a shift Hénon map,

$$U^+ \cup U^- = \mathbb{C}^2 \backslash J$$

Take H to be a small perturbation of a quadratic polynomial with Julia set that is a Cantor set on the real line, then

$$H_1(\mathbb{C}^2 \backslash J) = 0,$$

$$H_2(\mathbb{C}^2 \backslash J) = 0$$

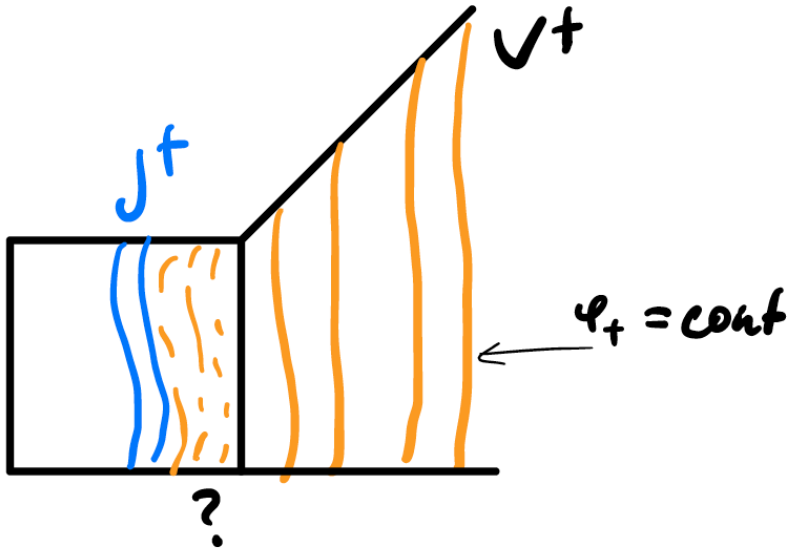
Exact sequence

$$H_2(U^+ \cap U^-) \to H_1(U^+ \cap U^-) \to H_1(U^+) \oplus H_1(U^-) \to H_1(U^+ \cap U^-)$$

$$H_2(\mathbb{C}^2 \backslash J) \to H_1(U^+ \cap U^-) \to H_1(U^+) \oplus H_1(U^-) \to H_1(\mathbb{C}^2 \backslash J)$$

$$H_1(U^+ \cap U^-) = \mathbb{Z}[\tfrac{1}{2}] \oplus \mathbb{Z}[\tfrac{1}{2}]$$

Extension of the Böttcher coordinate



Can ϕ_+ "be extended all the way to J^+"?

Theorem (Tanase)

For a Hénon map that is a perturbation of a hyperbolic polynomial with connected Julia set, the Julia set J^+ is homeomorphic to the quotient of $\mathbb{S}^1 \times \mathbb{C}$ by a discrete group of automorphisms isomorphic to $\mathbb{Z}[\frac{1}{2}]/\mathbb{Z}$ and an equivalence relation.

Critical points and Connectivity of the Julia set

1D: Critical points play a fundamental role for the dynamics of polynomials.

- The Julia set of a polynomial is connected if and only if all critical points have bounded forward orbits.
- A polynomial is hyperbolic if and only if the closure of the orbits of the critical points are disjoint from the Julia set.
- Easy to plot in the parameter space.
- A polynomial of degree d has at most $d-1$ non-repelling cycles.

2D: The Hénon map is a biholomorphism of $\mathbb{C}^2$, hence it has no critical points in the usual sense.

Critical locus

The critical locus $\mathcal{C}$ is the set of tangencies between the foliations of the escaping sets U^+ and U^-. $\mathcal{C}$ can also be described as the set of zeroes of the holomorphic function w, where

$$d\log\varphi^+ \wedge d\log\varphi^- = w(x,y)dx \wedge dy.$$

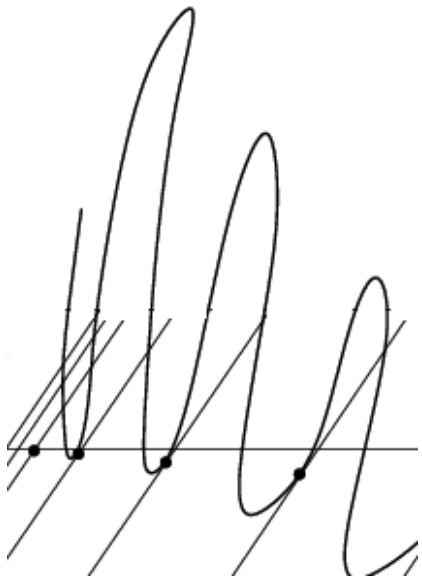


Figure: Tangencies between leaves of two foliations

Bedford-Smillie: The set $\mathcal{C}$ is a nonempty, closed analytic subvariety of $U^+ \cap U^-$ and is invariant under the Hénon map.

Stable/unstable critical loci

Unstable Critical Locus $\mathcal{C}^u$: the set of tangencies between the foliation of U^+ and the "lamination" of J^-.

Stable Critical Locus $\mathcal{C}^s$: set of tangencies between the foliation of the escaping set U^- and the "lamination" of J^+.

J^+ and J^- are not always laminar.
Bedford-Smillie: If J^+ and J^- are both laminated and if the laminations are transversal along J then f is hyperbolic.

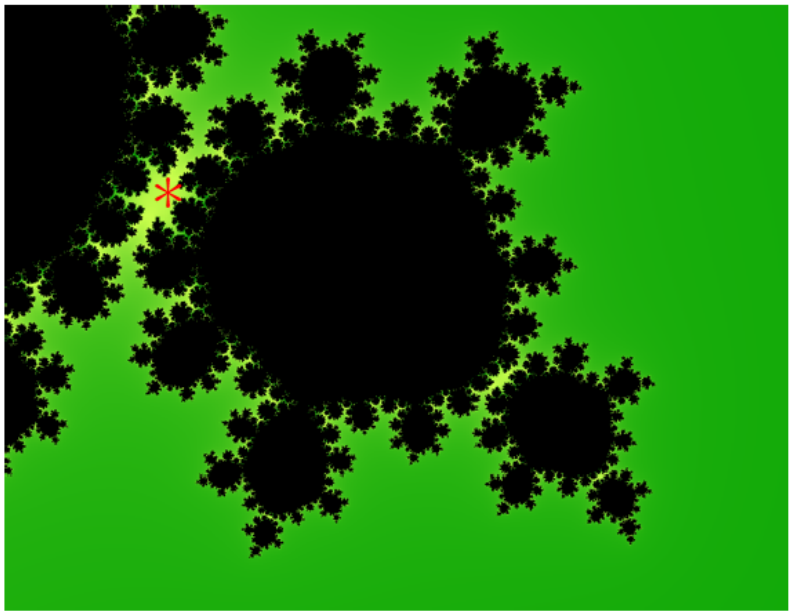
Bedford and Smilie showed that $\mathcal{C}^s$ and $\mathcal{C}^u$ can be empy, $\mathcal{C}$ is never empty. They remark that the relation between $\mathcal{C}$, $\mathcal{C}^s$ and $\mathcal{C}^u$ is "rather mysterious is general".

Unstable connectivity

Theorem (Bedford, Smillie)

For a dissipative complex Hénon map the following are equivalent:

- *J is connected.*
- *$W^u(p) \cap K^+$ is connected for some saddle periodic point p.*
- *$W^u(p) \cap K^+$ is connected for any saddle periodic point p.*
- *$\mathcal{C}^u = \emptyset$.*



'Primary component' of a critical locus,

$H(x,y) = (x^2 + c - ay, x)$, $\delta = \frac{1+|a|}{2}$, $S^h_\delta = \{|y| < \delta\}$.

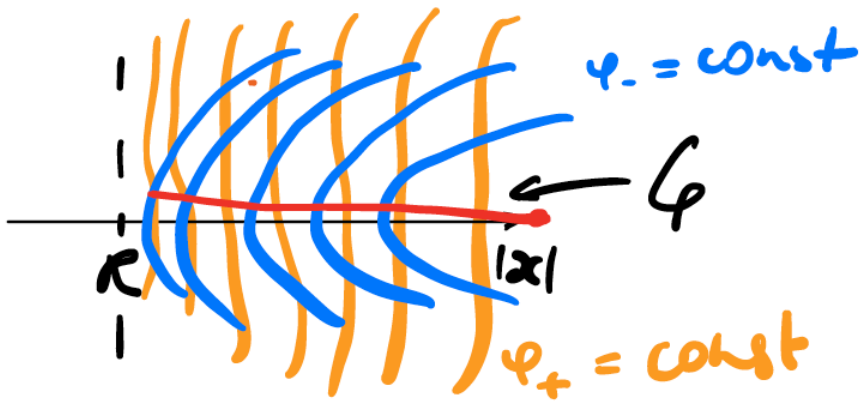
Lemma

(F, Radu, Tanase) If $(x,y) \in \mathbb{C}^2 \setminus \cup_n (H^n)(S^h_\delta)$, *then* $(x,y) \notin \mathcal{C}$.

Theorem

For a Hénon map H there exist R, δ *such that critical locus* $\mathcal{C}^{R,\delta} = \mathcal{C} \cap \{|x| > R\} \cap \{|y| < \delta\}$ *is a punctured disk with a puncture at* $(\infty, 0)$.

Follows from Lyubich, Robertson.



Hubbard's Critical Locus Conjecture for Quadratic Hénon maps

Cor. If K is connected, φ_+ extends to a biholomorphism $C_0 \to \mathbb{C} - \overline{\mathbb{D}}$.

$$C = \bigcup_{n \in \mathbb{Z}} C_n \qquad C_n = H^{0,n}(C_0).$$

Real critical locus

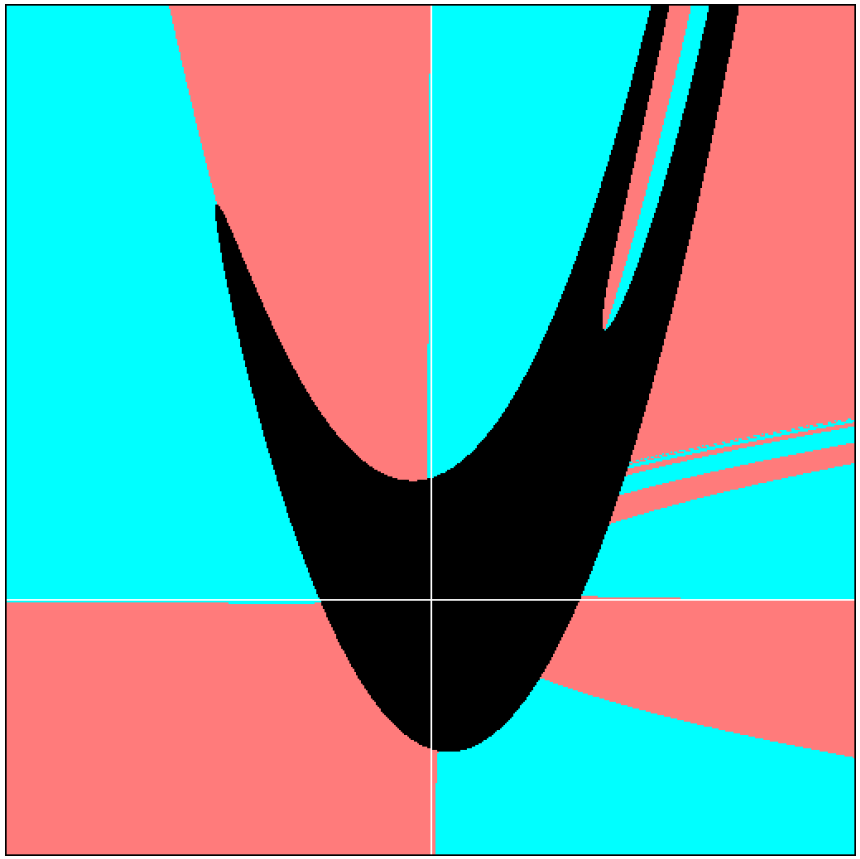


Figure: Critical locus in the real plane for a Henon map with connected K. The critical locus is the boundary of the red/blue regions. Picture drawn with the program Tangencies written by J. Hubbard and implemented by K. Papadantonakis at Cornell.

Models for the critical locus I

When $a \to 0$, the map $H(x, y) = (p(x) - ay, x)$ degenerates to the one-dimensional map $x \mapsto p(x)$ acting on the parabola $x = p(y)$.

Theorem (Lyubich, Robertson)

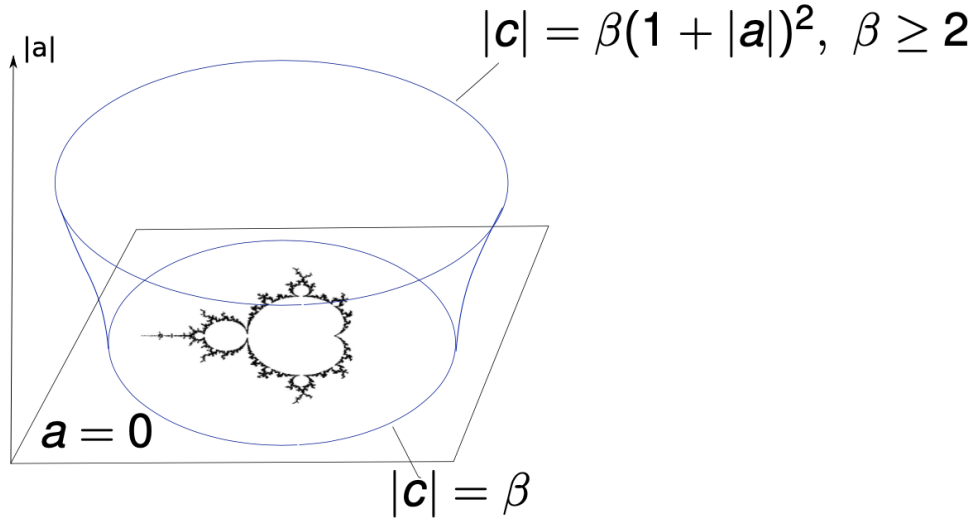
Let H be a Hénon map that is a small perturbation of a hyperbolic quadratic polynomial p with connected Julia set.

- *There exists a unique primary component $\mathcal{C}_0$ of the critical locus, which is asymptotic to the x-axis.*
  $\mathcal{C}_0$ is biholomorphic to $\mathbb{C} - \overline{\mathbb{D}}$ and it is everywhere transverse to the foliations of U^+ and U^-. Its boundary is homeomorphic to the Julia set J_p.
- *All other components of $\mathcal{C}$ are iterates of $\mathcal{C}_0$ under H.*

Shift Locus, HOV-region

$H_{a,c}(x, y) = (x^2 + c, ay)$

- A Hénon map is a shift map if dynamics on K is conjugate to the 2-shift on bi-infinite duadic sequences.
- An HOV region $\{(c, a) : |c| > \beta(1 + |a|)^2\}$, $\beta \geq 2$
- Hubbard, Oberste-Vorth: If (a, c) is in an HOV-region, then $H_{c,a}$ is a complex horseshoe, and hence, it is in the shift locus.



Models for the critical locus II

There exists $\beta > 2$ such that the following results hold throughout the HOV_β region:

Theorem (F., Radu, Tanase.)

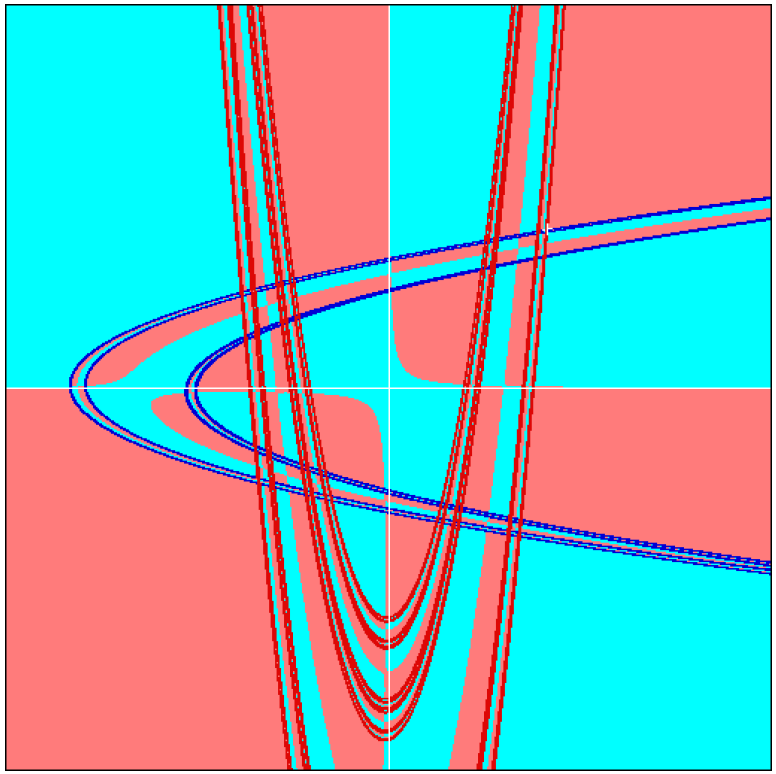
The critical locus of $H_{a,c}$ is a smooth, irreducible complex analytic subvariety of $U^+ \cap U^-$, of pure dimension one.

Theorem (F., Radu, Tanase)

The critical locus of $H_{a,c}$ is homeomorphic to a Riemann surface of infinite genus which is a countable collection of truncated spheres $(\mathcal{S}_n)_{n\in\mathbb{Z}}$ with countably many handles, glued by dynamics.

So the Julia set is disconnected & the critical locus in connected.

Real Critical Locus



Theorem (F., Radu, Tanase)

The accessible boundary of the critical locus $\mathcal{C}$ is $\mathcal{C}^s \cup \mathcal{C}^u$. The boundary of the critical locus $\mathcal{C}$ is $J^+ \cup J^-$.

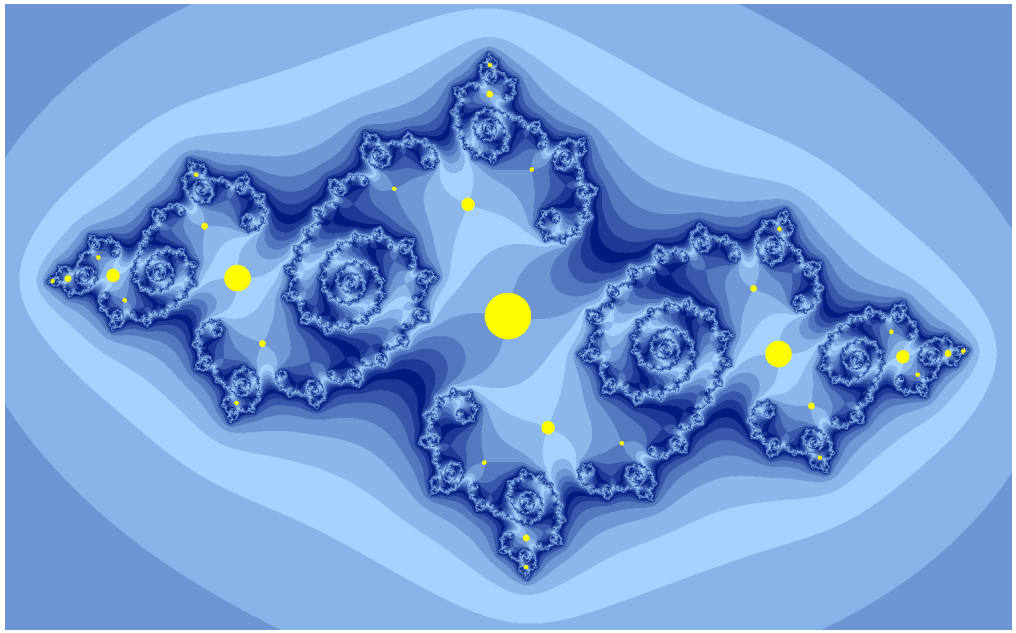
Theorem (F., Radu, Tanase)

For Hénon maps HOV_β region, the critical loci are quasiconformally equivalent.

Remark: The truncated sphere model was conjectured by J. Hubbard and previously proved by F. in perturbative setting (i.e. Hénon maps that are small perturbations of polynomials with disconnected Julia sets). Quasiconformal equivalence for small perturbations was proved by F. and M.Lyubich.

General question (perhaps conjecture): The Julia set J of a quadratic Hénon map is connected if and only if the critical locus $\mathcal{C}$ is disconnected.

Construction of a “main component” the critical locus



- $p(x) = x^2 + c$, c outside of the Mandelbrot set
- D is a disk bounded by an equipotential of p, $D_0, D_1 = p^{-1}(D)$
- J_p is a Cantor set, identified with $\{0,1\}^{\mathbb{N}}$
- $W := W_\emptyset$ removed disk around the critical point 0
- W_{α_n} removed disks around $p^{-n}(0)$

Truncated sphere

Figure: A truncated sphere.

- S^+ denote the disk $\overline{D}$ from which we remove the disks W_{α_n} and the Julia set
- S^- is a copy of S^+, U_{α_n} disks removed from D
- *Truncated sphere* $\mathcal{S}$ is S^+ glued to S^- along ∂D with an extra point removed.

Handles

Figure: The truncated spheres S_{-1} and S_0 connected by 1 handle

Handles

Figure: The truncated spheres S_{-1} and S_1 connected by 2 handles

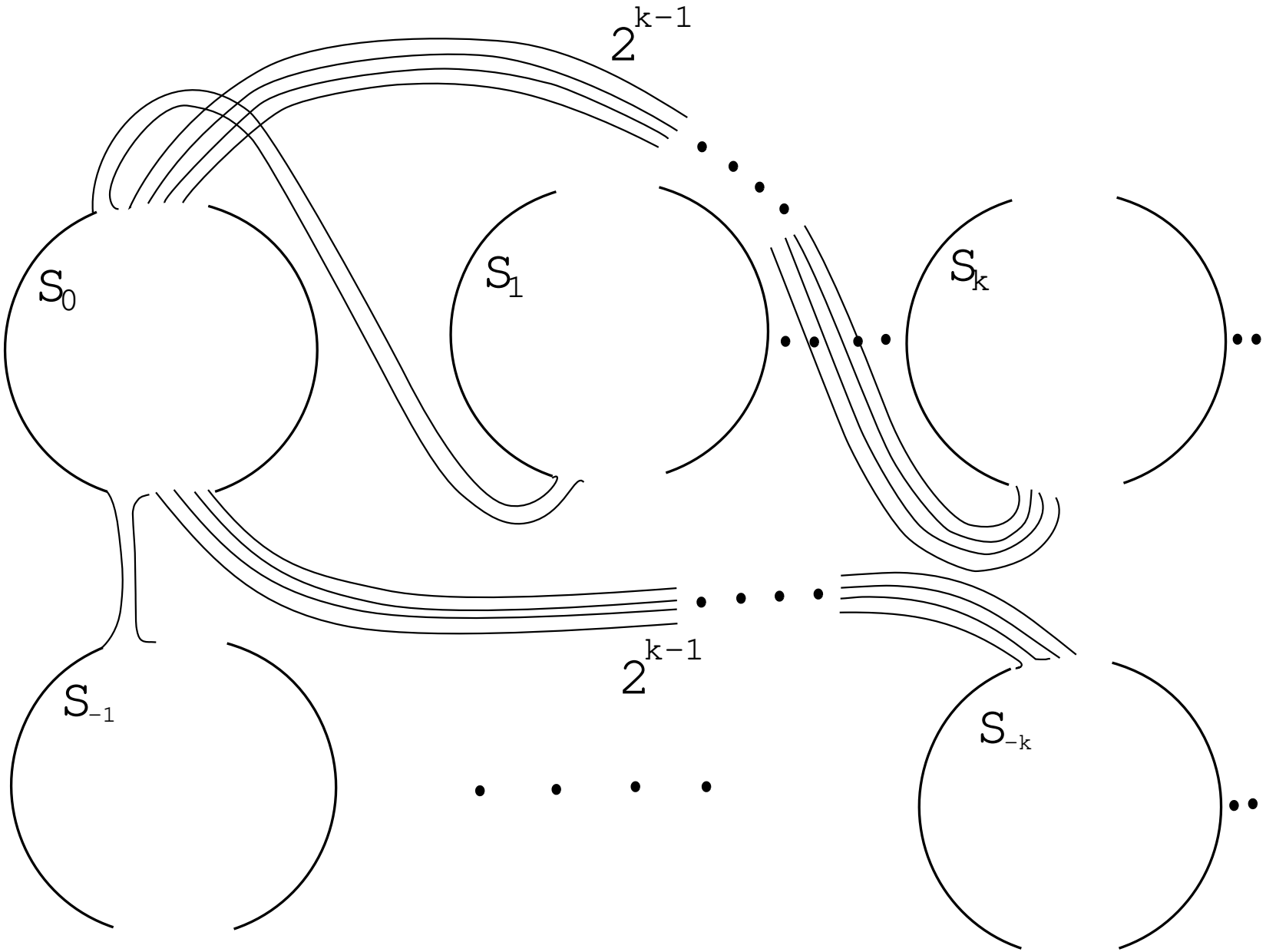
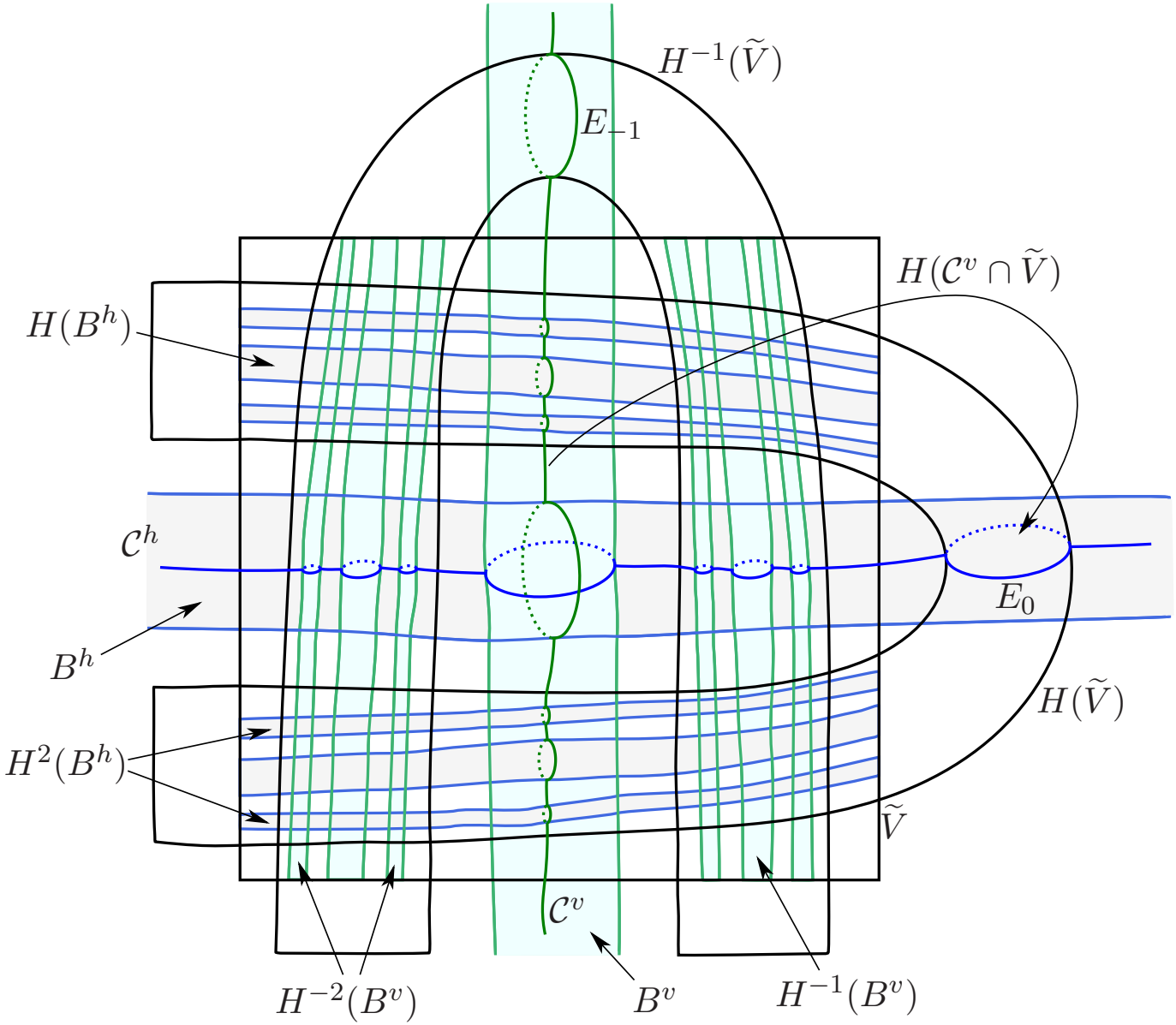


Figure: The topological model.

Handles in $\mathbb{C}^2$



Monodromy

Lemma

A loop in the HOV_β locus induces an isotopy class of maps from the critical locus to itself, which commute with the Hénon map.

Lemma

Let γ be a simple loop around $a = 0$, with base point (c_0, a_0) and image contained in the HOV_β region. γ induces a trivial monodromy action on the Julia set $J_{c_0,a_0} \simeq \Sigma_2$, therefore it induces just the identity map Id from the critical locus to itself.

Monodromy

Lemma

Let γ be a simple loop around the Mandelbrot set, with base point (c_0, a_0) and image contained in the HOV_β region. γ induces the monodromy action δ on the Julia set $J_{c_0,a_0} \simeq \Sigma_2$, where $\delta : \Sigma_2 \to \Sigma_2$ is the homeomorphism which exchanges the symbols 0 and 1, that is $\delta(\omega) = \omega'$, where for every $n \in \mathbb{Z}$, $\omega'_n = 1$ iff $\omega_n = 0$, and $\omega'_n = 0$ iff $\omega_n = 1$. The loop γ induces an involution map η from the critical locus to itself, which commutes with the Hénon map.

Happy Birthday!!!