On the Hubbard program for Hénon mappings

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Polynomial automorphism of $\mathbb{C}^2$

Theorem (Friedland, Milnor 1989)

Every polynomial automorphism of $\mathbb{C}^2$ is conjugate by a polynomial automorphism to one of the following maps:

(a) *affine maps* $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} k \\ k' \end{pmatrix}$, $ad - bc \neq 0$

(b) *elementary maps* $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} ax + p(y) \\ by + c \end{pmatrix}$, $ab \neq 0$

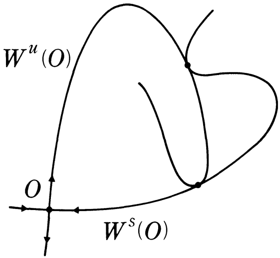
(c) *compositions of Hénon maps* $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} p(x) - ay \\ x \end{pmatrix}$, $a \neq 0$

Palis, Takens: Hénon-like maps are universal:

$$f_{c,a}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x^2+c-ay\\x\end{pmatrix} + g(x,y), \text{ with } g \text{ of small norm}$$

appear in unfoldings of quadratic homoclinic tangencies between the stable and unstable manifold of a saddle periodic point in dissipative systems with one unstable Lyapunov exponent.

The study of part of the local dynamics in these unfoldings is reduced to the study of the dynamics of Hénon-like maps.



Hénon maps

A complex Hénon map $H_{c,a} : \mathbb{C}^2 \to \mathbb{C}^2$ is a diffeomorphism of $\mathbb{C}^2$

$$H_{c,a}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + c - ay \\ x \end{pmatrix}, \quad a \neq 0.$$

Dynamical objects/The cast of players:

$$K^{\pm} = \text{points in } \mathbb{C}^2 \text{ with bounded forward/backward orbit}$$

$$U^{\pm} = \mathbb{C}^2 - K^{\pm} \ \ \text{(escaping sets)}$$

$$J^{\pm} = \partial K^{\pm} = \partial U^{\pm} \ \ \text{and} \ \ J = J^- \cap J^+ \ \ \text{(Julia sets)}$$

$$J^* = \text{closure of saddle periodic points} \subseteq J \ \ (\text{“small” Julia set})$$

Some milestones (general, for all Hénon maps)

- In 1982 Calabi proposed to investigate the basin of attraction of the Hénon map as a candidate for a Fatou-Bieberbach domain.
- Hubbard set the ground for the study of the complex Hénon maps in 1986. Hubbard-Oberste-Vorth 1990-93 classified the analytic structure of the escaping set U^+.
- Fornæss-Sibony 1991 developed pluripotential theory for Hénon maps.
- Bedford-Smillie 1991-2002 have many papers on the subject. Among their results is the general criterion for the connectivity of the Julia set: J is connected if and only if $W^u(p) \cap K^+$ is connected for any saddle periodic point p.
- Hubbard and Papadantonakis developed several programs for exploring the dynamical and parameter space of Hénon maps (FractalAsm, SaddleDrop, Tangencies - work on PowerPC). Later Hubbard and Matt Noonan wrote FractalStream which works on Mac OS X up to 10.14, but not so much on 64bits.

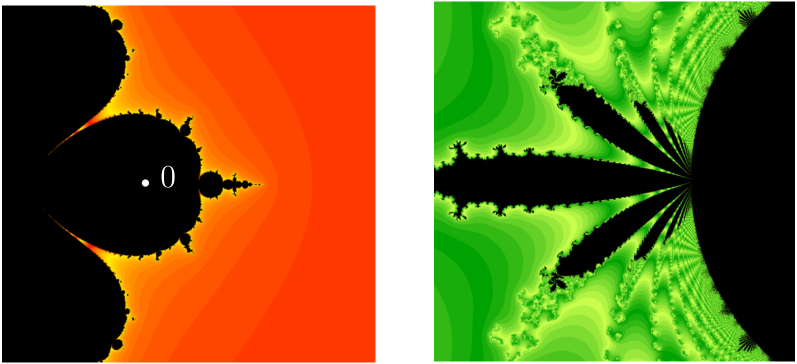


Figure: Parameter pictures for complex Hénon maps. LEFT: a fixed point with an eigenvalue -1 RIGHT: a transverse over the picture to the left - not accurate!

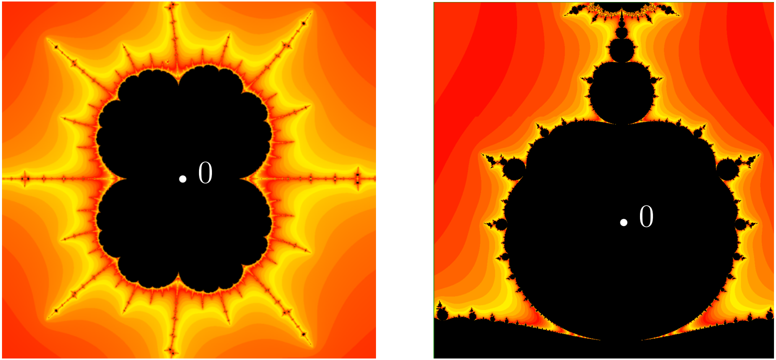


Figure: Parameter pictures for complex Hénon maps with a 3-cycle with an eigenvalue 1 (LEFT), respectively a fixed point with an eigenvalue -1 (RIGHT).

More milestones (specific dissipative families)

A lot of progress has been made in the study of Hénon maps with small Jacobian:

- Benedicks-Carleson (existence of strange attractors),
- Hubbard-Oberste-Vorth (perturbations of hyperbolic polynomials with connected Julia sets),
- Fornæss-Sibony (perturbations of hyperbolic polynomials with disconnected Julia sets),
- Lyubich-Martens-de Carvalho and Lyubich-Crovisier-Yang-Pujals (renormalization, existence of Cantor attractors, study of infinitely renormalizable maps),
- Bedford-Smillie-Ueda (parabolic implosion),
- Radu-Tanase (structure of perturbations of parabolic polynomials and continuity of Julia sets),
- Gaidashev-Radu-Yampolsky, Yampolsky-Yang (perturbations of Siegel polynomials),
- Berger (perturbations of a composition of two quadratic poly),
- Avila-Lyubich (existence of two coexisting attractors), etc.

Hubbard filtration

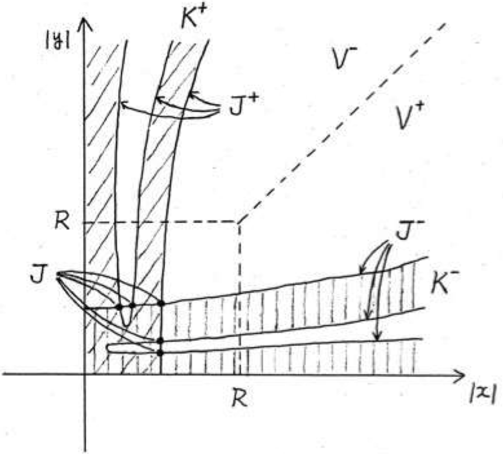


Figure: A schematic interpretation by Yutaka Ishii.

The forward escaping set is $U^+ = \bigcup_{k\geq 0} H^{-\circ k}(V^+)$.

The backward escaping set is $U^- = \bigcup_{k\geq 0} H^{\circ k}(V^-)$.

Theorem (Hubbard, Oberste-Vorth)

The structure of U^+ is well-understood:

$$U^+ = (\mathbb{C} - \overline{\mathbb{D}}) \times \mathbb{C}/\Gamma_{p,a}\ , \quad \text{where}$$

$\Gamma_{p,a} \subset \mathrm{Aut}((\mathbb{C} - \overline{\mathbb{D}}) \times \mathbb{C})$ is a discrete group isomorphic to $\mathbb{Z}[\frac{1}{2}]/\mathbb{Z}$.

Remark: $\mathrm{Aut}((\mathbb{C} - \overline{\mathbb{D}}) \times \mathbb{C})$ is the set of all

$$(\xi, z) \to (\lambda\xi, \alpha(\xi)z + \beta(\xi))\,, \quad \text{where}$$

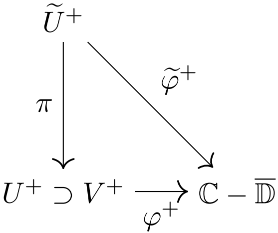
$\alpha : \mathbb{C} - \overline{\mathbb{D}} \to \mathbb{C}^*$, $\beta : \mathbb{C} - \overline{\mathbb{D}} \to \mathbb{C}$ are arbitrary analytic functions and $|\lambda| = 1$.

Remark: The analytic structure of U^+ uniquely determines the Hénon map in degree 2 (proved by Bonnot, Radu, T). In higher degrees, it determines it up to pre and post composition by affine maps (proved by Ratna Pal).

There exists an unique holomorphic function $\varphi^+ : V^+ \to \mathbb{C} - \overline{\mathbb{D}}$,

- $\varphi^+ \circ H = (\varphi^+)^2$
- $\varphi^+(x,y) \sim x$, when $(x,y) \to \infty$ in V^+.

Remark: φ^+ does not extend holomorphically to U^+, but it does extend along curves contained in U^+ starting in V^+ so it is well defined on a covering manifold $\widetilde{U}^+$ of U^+.



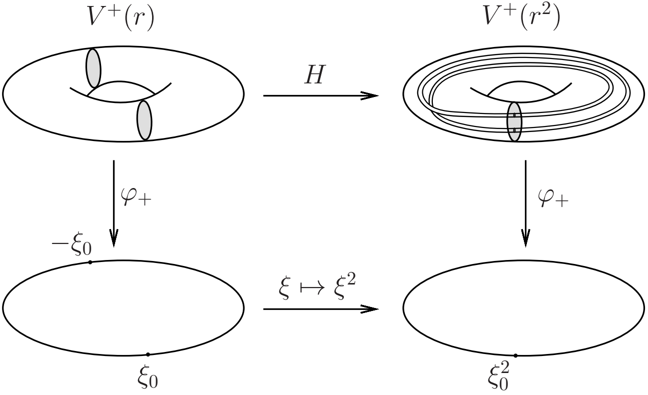
Theorem (Hubbard, Oberste-Vorth)

The covering manifold $\widetilde{U}^+$ is biholomorphic to $(\mathbb{C} - \overline{\mathbb{D}}) \times \mathbb{C}$.

For large r, the set $V^+(r) = \{(x,y) \in V^+, |\varphi^+| = r\}$ is homeomorphic to a solid torus, and

$$\varphi^+ : V^+(r) \to \{z, |z| = r\}$$

is a fibration with fibers homeomorphic to closed disks.



Topology of U^+

The set U^+ is an increasing union of sets biholomorphic to V^+

$$V^+ \subset H^{-1}(V^+) \subset H^{-2}(V^+) \subset \ldots \subset H^{-n}(V^+) \subset \ldots$$

This induces a sequence of homomorphisms on fundamental groups

$$\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 2} \ldots \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 2} \ldots$$

So $\pi_1(U^+) = \varinjlim(\mathbb{Z}, \cdot 2) = \mathbb{Z}\left[\frac{1}{2}\right]$.

$$\begin{array}{ccc} \widetilde{U}^+ & \xrightarrow{\simeq} & (\mathbb{C} - \overline{\mathbb{D}}) \times \mathbb{C} \\ \pi \downarrow & & \downarrow \pi \\ U^+ & \xrightarrow{\simeq} & (\mathbb{C} - \overline{\mathbb{D}}) \times \mathbb{C} / \Gamma_{p,a} \end{array}$$

The group $\Gamma_{p,a} \simeq \pi_1(U^+)/\pi_1(\widetilde{U}^+)$, hence isomorphic to $\mathbb{Z}\left[\frac{1}{2}\right]/\mathbb{Z}$.

Hope: The group exists for all Hénon maps, but only operates discontinuously on $S^1 \times \mathbb{C}$ [$(\mathbb{C} - \overline{\mathbb{D}}) \times \mathbb{C}$] if K is connected.

Otherwise it has a limit set (which could perhaps be drawn).

THE HENON MAPPING IN THE COMPLEX DOMAIN

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6. A PROGRAM FOR DESCRIBING MAPPINGS IN THE HENON FAMILY

We propose, as a strategy for studying a mapping F in the Henon family, to try to understand how the fibers $U_+(s)$ collapse onto ∂K_+ as $s \to 0$.

Grand View

U^+ is naturally foliated by copies of $\mathbb{C}$, each dense in a 3-sphere with a solenoid removed $U^+(s) = \{G^+ = s\}$. Extend this to J^+ whenever possible.

- Give a combinatorial and topological description of the Julia set J^+ as a (quotient of a) 3-sphere with a solenoid removed.
- Describe the Julia set J^+ in terms of group actions on $\mathbb{S}^1 \times \mathbb{C}$.
- Describe the Julia set J as a (quotient of a) solenoid.
- Build 4-dimensional topological models for the filled-in Julia set set K^+ of Hénon maps (*pinched ball models*), analogue to the one-dimensional *pinched disk model* for polynomials. The underlying model where the nontrivial pinching takes place is the unit 3-sphere with a solenoid removed. The pinching is done inside the unit 4-ball.
- Many more directions of research.

Hubbard–Oberste-Vorth: consider Hénon maps which are singular perturbations of polynomials with connected Julia set.
The Hénon map is hyperbolic and its dynamics inside V fibers over the dynamics of the polynomial p. The description of the Julia sets J, J^- and J^+ involves projective and inductive limits of objects defined in terms of p alone.

The Julia set J^+ has a complicated topology and is not globally a fiber bundle over J_p!

Radu-T.: we have extended the HOV characterization of J and J^+ to semi-parabolic Hénon maps which are small perturbations of parabolic polynomials, but we focus on hyperbolic maps in this talk.

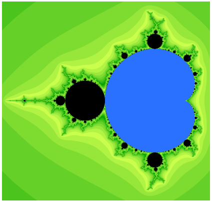
The Bonnot model

Theorem (Bonnot)

Let $p(x) = x^2 + c$, with c from the main cardioid of the Mandelbrot set. Then there is $a_0 > 0$ such that if $0 < |a| < a_0$ there exist a homeomorphism Φ for which the diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\Phi} & \mathbb{C}^2 \\ {\scriptstyle (r,\theta)\mapsto(r^2,\sigma(\theta))}\downarrow & & \downarrow {\scriptstyle H_{p,a}} \\ X & \xrightarrow{\Phi} & \mathbb{C}^2 \end{array}$$

In this model, K^+ is the 4-ball without Σ^- and J^+ is $\mathbb{S}^3 - \Sigma^-$.



The unit sphere $\mathbb{S}^3$ can be written as a union of two solid tori $\mathbb{T}_0$ and $\mathbb{T}_1$, glued along their boundaries. After rescaling, $\mathbb{T}_0 = \mathbb{S}^1 \times D$, and we consider σ a homeomorphism of $\mathbb{S}^3$,

$$\sigma(\xi, z) = (\xi^2, \xi + \epsilon \frac{z}{\xi}) \quad \text{on } \mathbb{T}_0,$$

It comes with an attracting, respectively a repelling solenoid:

$$\Sigma^+ = \bigcap_{n \geq 0} \sigma^n(\mathbb{T}_0) \quad \text{and} \quad \Sigma^- = \bigcap_{n \geq 0} \sigma^{-n}(\mathbb{T}_1)$$

Write $\mathbb{R}^4$ with its polar coordinates as $(r, \theta) \in \mathbb{R}^+ \times \mathbb{S}^3$ and consider the model space

$$X = \mathbb{R}^4 - \{(r, \theta) \ : r \geq 1 \text{ and } \theta \text{ from the repelling solenoid } \Sigma^-\}$$

Theorem (T.)

Let p be a hyperbolic quadratic polynomial with connected Julia set. There exists a_0 such that for all a with $0 < |a| < a_0$ the group $\Gamma_{p,a}$ can be continuously extended to $\mathbb{S}^1 \times \mathbb{C}$. Moreover, the group $\Gamma_{p,a}$ acts freely and properly discontinuously on $\mathbb{S}^1 \times \mathbb{C}$.

The Julia set J^+ is homeomorphic to a quotient of $\mathbb{S}^1 \times \mathbb{C}/\Gamma_{c,a}$ by an equivalence relation (which depends only on the polynomial p).

Corollary

If p has an attractive fixed point, then the Julia set J^+ of the Hénon map $H_{p,a}$ is a topological manifold and $J^+ \simeq \mathbb{S}^1 \times \mathbb{C}/\Gamma_{p,a}$ for $|a|$ small.

Theorem (Radu's thesis)

Globally, J^+ is a quotient of $\mathbb{S}^3 - \Sigma^-$ by an equivalence relation $\sim_p$.

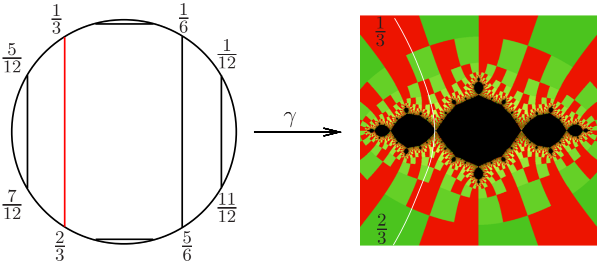


Figure: The lamination for $z \mapsto z^2 - 1$.

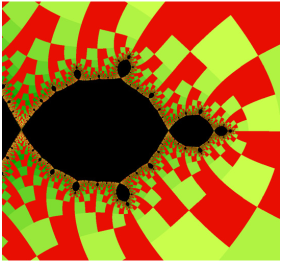


Figure: $W^u(q)$ for a Hénon map small perturbation of $z \mapsto z^2 - 1$.

Conjecture A (Hubbard)

$J = J^*$, where J^* is the closure of the saddle periodic points.

Proved by

- Bedford-Smillie in the hyperbolic case
- Fornæss-Guerini-Peters when the Hénon map is (partially) hyperbolic on $J^* \subset J$
- Radu-Tanase for small perturbations of parabolic polynomials (the only explicit examples which are not hyperbolic)
- Lyubich-Peters if partial hyperbolicity on J and $|a| < 1/4$

If this conjecture does not hold, then it would probably be in the conservative case.

Conjecture B

Let p be a fixed point in J then p is in J^*.

Theorem (Firsova, Lyubich, Radu, T.)

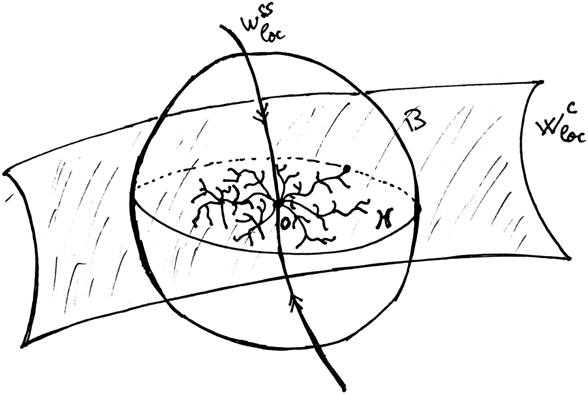
Let f be a germ of holomorphic diffeomorphisms of $(\mathbb{C}^n, 0)$ with a semi-neutral fixed point at 0 with exactly one neutral eigenvalue $|\lambda| = 1$. Consider an open ball $B \subset \mathbb{C}^n$ centered at 0 such that f is partially hyperbolic on a neighborhood of $\overline{B}$. There exists a set $\mathcal{H} \subset \overline{B}$ such that:

a) $\mathcal{H} \Subset W^c_{\mathrm{loc}}(0)$, where $W^c_{\mathrm{loc}}(0)$ is any local center manifold of the origin corresponding to the neutral eigenvalue, and constructed relative to $\overline{B}$.

b) $\mathcal{H}$ is compact, connected, full and f-invariant. Moreover $0 \in \mathcal{H}$, $\mathcal{H} \cap \partial B \neq \emptyset$.

Theorem (Lyubich, Radu, T.)

If the Hénon map $H_{\lambda,\mu}$, with $\lambda = e^{i\theta}$, $\theta \notin \mathbb{Q}$ and $|\mu| < 1$ is not linearizable at the origin, then $\mathcal{H} \subset J^*$.



The hedgehog $\mathcal{H}$ inside a center manifold. In fact, all center manifolds of 0 contain the hedgehog.

Question: This solves Conjecture B in the dissipative case. What about the conservative case $|\lambda| = |\mu| = 1$?

Écalle-Hakim theory

Let $f \neq Id$ be a germ of diffeomorphisms of $(\mathbb{C}^n, 0)$. Then f is tangent to the identity of order k at 0 if its Jacobian at 0 is the identity, and the first non-zero homogeneous polynomial in the Taylor series expansion of $f - Id$ about 0 is of degree $k+1$, i.e. for $z \in \mathbb{C}^n$ sufficiently small one has

$$f(z) = z + p_{k+1}(z) + p_{k+2}(z) + h.o.t,\ p_{k+1} \neq 0.$$

A characteristic direction of f is an element $[v] \in \mathbb{P}^{n-1}_{\mathbb{C}}$ such that $p_{k+1}(v) = \alpha v$ for some $\alpha \in \mathbb{C}$; it is called degenerate if $\alpha = 0$, and non-degenerate otherwise.

Theorem (Écalle, Hakim)

Let f be a germ of holomorphic diffeomorphisms of $(\mathbb{C}^n, 0)$ tangent to the identity at 0 of order k. If v is a non-degenerate characteristic direction for f then there exist (at least) $k-1$ parabolic curves tangent to v.

A **parabolic-attracting curve** $\mathcal{P}_{att}$ is a one-dimensional f-invariant holomorphic disk which contains 0 in its boundary and where the forward orbits converge to 0 along $[v]$, at a subexponential rate.

A **parabolic-repelling curve** $\mathcal{P}_{rep}$ for f is a parabolic-attracting curve for f^{-1}.

Hakim and Vivas also give sufficient conditions under which petals can be fat (i.e. open subsets of $\mathbb{C}^n$ or complex manifolds of lower dimension $1 \leq d < n$) according to some finer invariants associated to the non-degenerate characteristic directions.

We extend Écalle-Hakim theory to complex Hénon maps

$$H_{\lambda,\mu}(x,y) = (\lambda x, \mu y) + h.o.t.$$

with a double parabolic fixed point ($\lambda \neq \mu$ are distinct roots of unity $\lambda = e^{2\pi i p_1/q_1}$ and $\mu = e^{2\pi i p_2/q_2}$, where p_1, q_1 and p_2, q_2 are coprime). Let $q = lcm(q_1, q_2)$. The q-th iterate $H^q_{\lambda,\mu}$ is "tangent to the identity". However, for Hénon maps most of the characteristic directions are degenerate.

Theorem (Firsova, Radu, Raissy, T., Vivas – work in progress)

Generically there are q_i parabolic-attracting and q_i parabolic-repelling petals tangent to a characteristic direction, which are locally graphs of holomorphic functions (hence one-dimensional disks) containing the origin in their boundary.

Many results on parabolic dynamics in $\mathbb{C}^n$ are based on performing a sequence of blow-ups to resolve the singularities of an associated formal vector field whose time one flow is the germ f. The resolution of singularities for vector fields is well-understood for $n = 2$.

Abate: if the fixed point is isolated then parabolic curves always exist in dimension two for germs tangent to the identity, even if all characteristic directions are degenerate

López-Hernanz and Rosas: for each characteristic direction $[v]$ of f there exists either a curve $\mathcal{P}$ tangent to $[v]$ and pointwise fixed by the map f, or a **parabolic-attracting curve** $\mathcal{P}_{att}$ and **a parabolic-repelling curve** $\mathcal{P}_{rep}$ tangent to $[v]$.

Sauzin: for the Hénon map $H_{1,1}$ the attracting and repelling parabolic curves do not intersect in a neighborhood of the origin, although their Taylor series expansions agree up to some high order.

FRRTV: in general, the attracting and repelling parabolic curves as constructed by Hakim do not intersect in a neighborhood of the origin, unless the initial map is conjugate to a skew product.

Question: Do the global parabolic-attracting and parabolic-repelling petals intersect?

For each local parabolic-attracting petal $\mathcal{P}_{att}$, we define global parabolic-attracting petals

$$W^s(\mathcal{P}_{att}) := \bigcup_{n\geq 0} f^{-nq}(\mathcal{P}_{att})$$

as an increasing union of sets $\mathcal{P}_{att} \subset f^{-q}(\mathcal{P}_{att}) \subset \ldots$, each biholomorphic to $\mathcal{P}_{att}$.

Similarly we define global parabolic-repelling petals

$$W^u(\mathcal{P}_{rep}) := \bigcup_{n\geq 0} f^{nq}(\mathcal{P}_{rep})$$

A parabolic homoclinic web

Consider complex Hénon maps with a double parabolic fixed point.

Theorem (Firsova, Radu, Raissy, T., Vivas – work in progress)

1) There exist local 1D attracting and repelling petals $\mathcal{P}_{att}$, $\mathcal{P}_{rep}$.
2) The global attracting/repelling petal is an immersed, but not embedded complex submanifold of $\mathbb{C}^2$ biholomorphic to $\mathbb{C}$, dense in the Julia sets J^+, respectively in J^-.
3) The $W^s(\mathcal{P}_{att}) \cap W^u(\mathcal{P}_{rep}) \neq \emptyset$ is dense in J^*.

Corollary

The double-parabolic fixed point is in J^*.

Our philosophy: in the Hénon family treat the global parabolic-attracting/repelling petals of a parabolic point as stable/unstable manifolds of a saddle. However, the double parabolic fixed point is not a regular point of J^*.

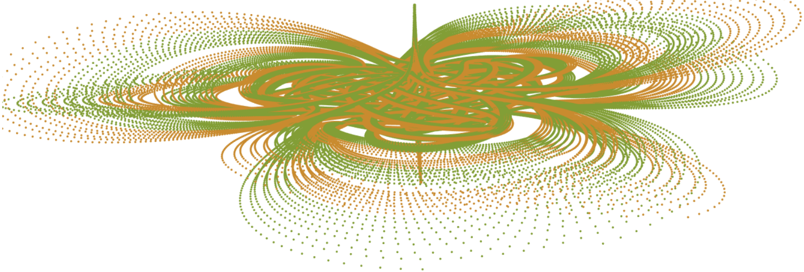


Figure: $H_{\lambda,\mu}$ with $\lambda = e^{2\pi i1/3}$ and $\mu = 1$. Alternation of petals near the origin. Notice that they are tangent to the corresponding characteristic directions near the origin, but do not lie in the same planes in $\mathbb{C}^2$.

Happy birthday!!