ON THE DYNAMICS OF TANGENT-LIKE MAPPINGS

JHH80 Dynamical Developements:
Degeneration of Flat Surfaces and Rational Maps

10-13 June, 2025

Núria Fagella
(joint work with Anna M. Benini and Matthieu Astorg)

Facultat de Matemàtiques i Informàtica, UB
Centre de Recerca Matemàtica

I met Hamal ....

... on my very first conference, ... **Hillerod, Summer 1993!**

I met Hamal ....

... on my very first conference, ... **Hillerod, Summer 1993!**

I met Hamal ....

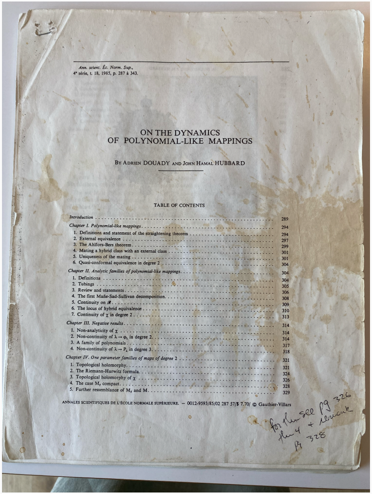
... on my very first conference, ... **Hillerod, Summer 1993!**

I met Hamal ....

... on my very first conference, ... **Hillerod, Summer 1993!**

On the dynamicsl of polynomial-like mappings'85

Undoubtably in the **best-sellers list**, we find the paper from Douady and Hubbard, from 1985:



Ann. scient. Éc. Norm. Sup.,
4e série, t. 18, 1985, p. 287 à 343.

ON THE DYNAMICS
OF POLYNOMIAL-LIKE MAPPINGS

BY ADRIEN DOUADY AND JOHN HAMAL HUBBARD

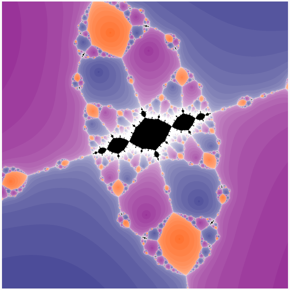
TABLE OF CONTENTS

Introduction 289
Chapter I. Polynomial-like mappings 294
1. Definitions and statement of the straightening theorem 294
2. External equivalence 297
3. The Ahlfors-Bers theorem 299
4. Mating a hybrid class with an external class 301
5. Uniqueness of the mating 301
6. Quasi-conformal equivalence in degree 2 304
Chapter II. Analytic families of polynomial-like mappings 304
1. Definitions 304
2. Tubings 305
3. Review and statements 306
4. The first Mañe-Sad-Sullivan decomposition 308
5. Continuity on $\mathscr{R}$ 309
6. The locus of hybrid equivalence 310
7. Continuity of χ in degree 2 313
Chapter III. Negative results 314
1. Non-analyticity of χ 314
2. Non-continuity of $\lambda \to \varphi_\lambda$ in degree 2 314
3. A family of polynomials 317
4. Non-continuity of $\lambda \to P_\lambda$ in degree 3 318
Chapter IV. One parameter families of maps of degree 2 321
1. Topological holomorphy 321
2. The Riemann-Hurwitz formula 324
3. Topological holomorphy of χ 326
4. The case M_f compact 328
5. Further ressemblance of M_f and M 329

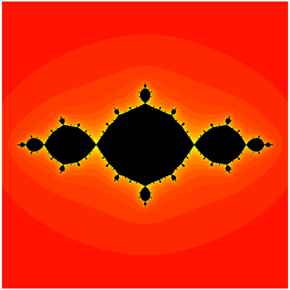
ANNALES SCIENTIFIQUES DE L'ÉCOLE NORMALE SUPÉRIEURE. — 0012-9593/85/02 287 57/$ 7.70/ © Gauthier-Villars

On the dynamicsl of polynomial-like mappings'85

Question: Why do we encounter polynomial Julia sets in the dynamical plane of non-polynomial mappings?

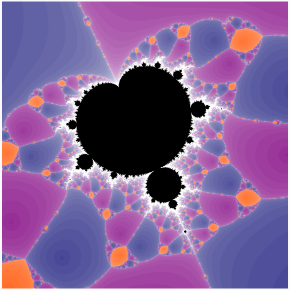


Newton's method of a cubic pol.

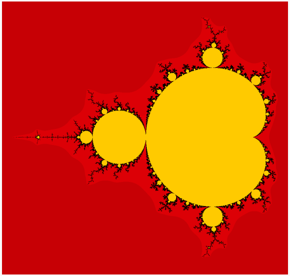


$z \mapsto z^2 - 1$

Question: And Mandelbrot sets in non-polynomial parameter spaces?

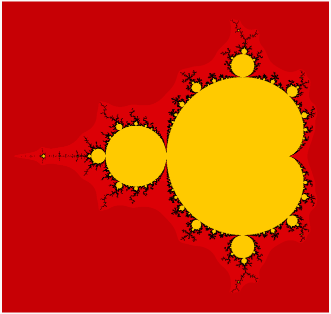


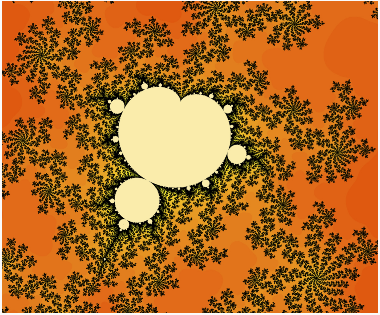
Newton's method of a cubic pol.



The Mandelbrot set.

Question: And copies of the Mandelbrot set within itself, ad infinitum?



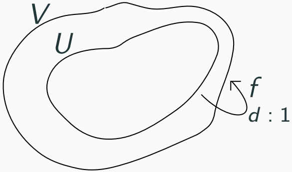


On the dynamicsl of polynomial-like mappings

Local dynamics of f near critical points	compared to	"global" dynamics of a polynomial

Local dynamics of f near critical points	compared to	"global" dynamics of a polynomial

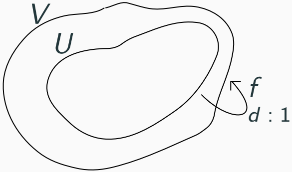
Polynomial-like map of degree $d > 1$:
A triple (f, U, V) such that U, V are Jordan domains, $\overline{U} \subset V$ and $f : U \to V$ holomorphic and proper of degree d.



On the dynamicsl of polynomial-like mappings

Local dynamics of f near critical points	compared to	"global" dynamics of a polynomial

Polynomial-like map of degree $d > 1$:
A triple (f, U, V) such that U, V are Jordan domains, $\overline{U} \subset V$ and $f : U \to V$ holomorphic and proper of degree d.



Filled Julia set:
$K(f) = \{z \in U \mid f^n(z) \in U \text{ for all } n \geq 0\} = \bigcap_n \overline{f^{-n}(U)}$

Julia set:
$J(f) = \partial K(f)$.

Clearly, **Polynomials are polynomial-like**.

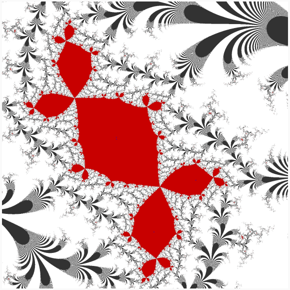
THE STRAIGHTENING THEOREM

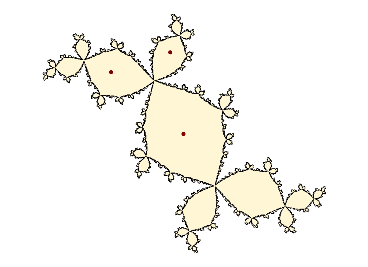
Theorem [DH'85]

Let (f, U, V) be polynomial-like of degree $d > 1$.

Then f is **hybrid equivalent** to a genuine polynomial P, in a neighborhood of $K(f)$.

If $K(f)$ is connected, P is unique up to affine conjugation.





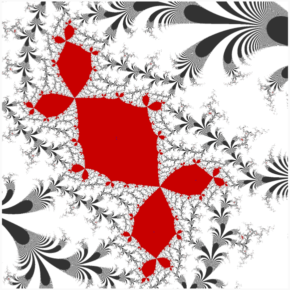
THE STRAIGHTENING THEOREM

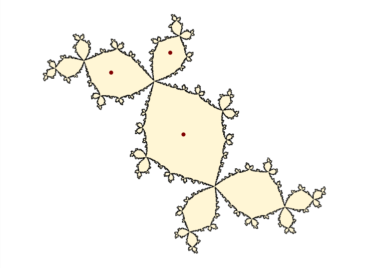
Theorem [DH'85]

Let (f, U, V) be polynomial-like of degree $d > 1$.

Then f is **hybrid equivalent** to a genuine polynomial P, in a neighborhood of $K(f)$.

If $K(f)$ is connected, P is unique up to affine conjugation.





THE PARAMETER VERSION

Theorem [DH'85]

Let $(f_\lambda, U_\lambda, V_\lambda)$ be a "proper" holomorphic family of polynomial-like of degree 2, over $\lambda \in \Lambda$ a topological disk. Let

$$M_f = \{\lambda \in \Lambda \mid K_\lambda \text{ is connected}\}$$

Then, M_f is homeomorphic to the Mandelbrot set M.

THE PARAMETER VERSION

Theorem [DH'85]

Let $(f_\lambda, U_\lambda, V_\lambda)$ be a "proper" holomorphic family of polynomial-like of degree 2, over $\lambda \in \Lambda$ a topological disk. Let

$$M_f = \{\lambda \in \Lambda \mid K_\lambda \text{ is connected}\}$$

Then, M_f is homeomorphic to the Mandelbrot set M.

- $f = P^n \longrightarrow$ **Renormalization**.

THE PARAMETER VERSION

Theorem [DH'85]

Let $(f_\lambda, U_\lambda, V_\lambda)$ be a "proper" holomorphic family of polynomial-like of degree 2, over $\lambda \in \Lambda$ a topological disk.
Let

$$M_f = \{\lambda \in \Lambda \mid K_\lambda \text{ is connected}\}$$

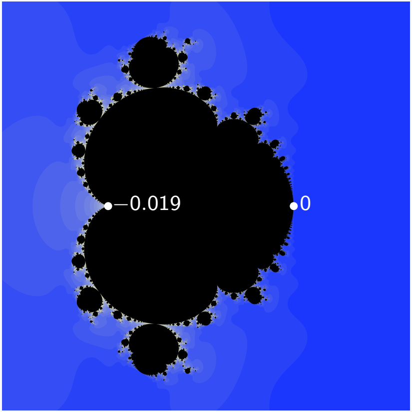
Then, M_f is homeomorphic to the Mandelbrot set M.

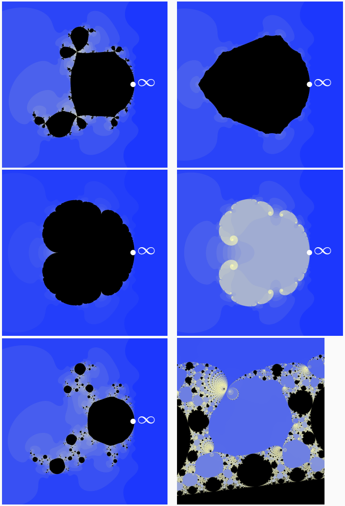
- $f = P^n \longrightarrow$ **Renormalization**.

- [McMullen'00] **Universality** $\longrightarrow$ "Proper" families of polynomial-like maps exist near all Misiurewicz parameters (preperiodic critical points).

THE TANDELBROT SET

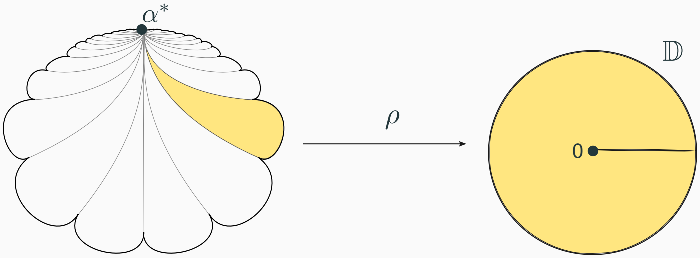
(Mandelshell?? Asymptotic M-set?? Suggestions welcome!)





SHELL COMPONENTS

Hyperbolic components of $M \longrightarrow$ Shell components



[F-Keen'21] If S is a **shell component** of period p, the multiplier map

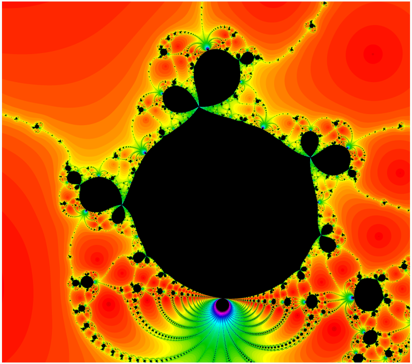
$$\rho : S \to \mathbb{D}; \quad \rho(\alpha) = (T_\alpha^p)'(\text{att. } p\text{−cycle})$$

is a universal covering of infinite degree. It extends continuously to $\overline{S} \setminus \{\alpha^*\}$, where α^* is the (unique) **virtual center** in ∂S.

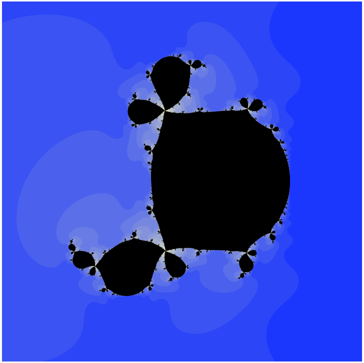
Virtual centers satisfy $T_{\alpha^*}^p(1/\alpha^*) = \infty.$

THE STRAIGHTENING THEOREM

Theorem [ABF'25]
Let (f, U, V, u, v) be a TL-map.
Then f is **hybrid equivalent** to T_α for some $\alpha \in \mathbb{D} \setminus \{1\}$.
If $K(f)$ is connected, α is unique, and $\alpha = 0$ iff $u = v$.



Newton's method of $z + ae^z$.



T_α period 3.

THE PARAMETER VERSION

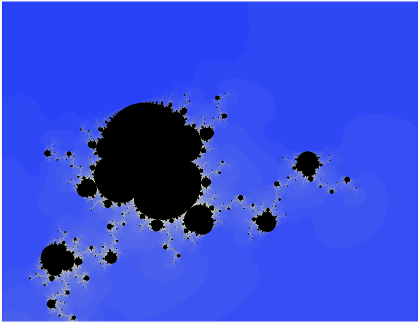
Theorem [ABF'25]

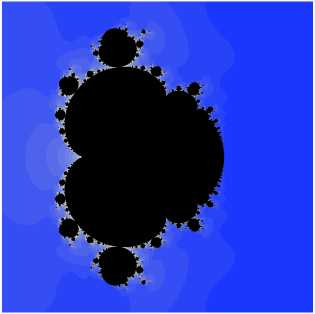
Let $(f_\lambda, U_\lambda, V_\lambda, u_\lambda, v_\lambda)$ be a "proper" holomorphic family of TL-maps, over $\lambda \in \Lambda$ a topological disk.

Let

$$M_f = \{\lambda \in \Lambda \mid K(f_\lambda) \text{ is connected}\}$$

Then, M_f is homeomorphic to the Tandelbrot set M.





THE PARAMETER VERSION

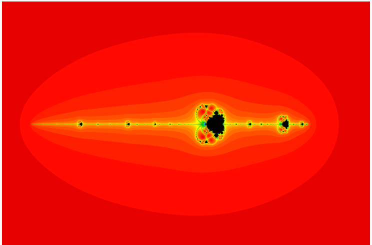
Theorem [ABF'25] (Existence of TL-restrictions)

Let $(f_\lambda)_{\lambda\in\Lambda}$ natural family of finite type meromorphic maps, $\lambda_0 \in \Lambda$.

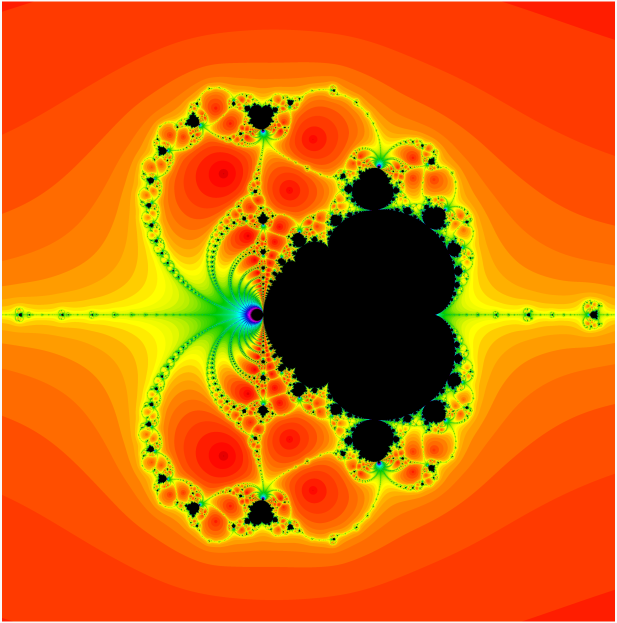
v_{λ_0} active asymptotic value (whose forward orbit does not persistently contain a critical point).

f_{λ_0} has at least one simple pole p which is not a singular value, and that its tracts (above quasidisks) are quasidisks.

Then arbitrarily close to λ_0, there exists λ_k such that $f_{\lambda_k}^{n_k}$ has a tangent-like restriction for some $n_k \in \mathbb{N}$.



THE PARAMETER VERSION



Thank you for your attention and HAPPY BIRTHDAY!