Connected components of strata of meromorphic differentials with residue conditions

based on the joint work with Yiu Man Wong

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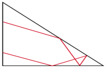
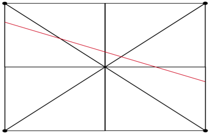
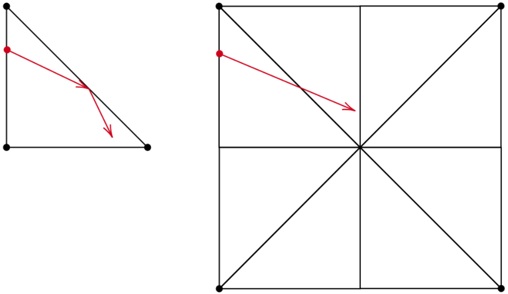




Billiards



Trajectory of billiard with angles in $\pi\mathbb{Q}$
$=$ Straight line flow in a polygon with parallel sides identified

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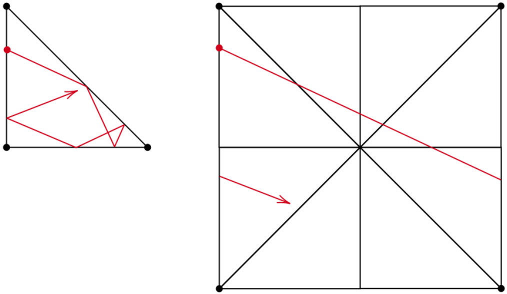

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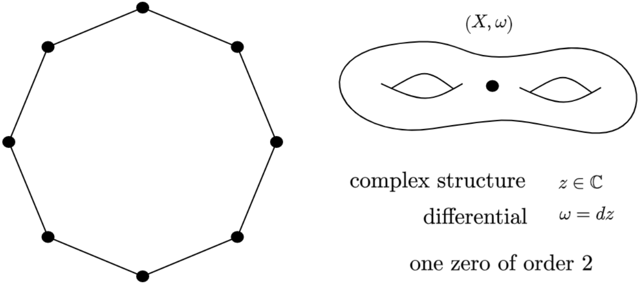
Flat surfaces

- A *flat surface* is (X, ω) : $X \in \mathcal{M}_g$ and a 1-form ω on X.



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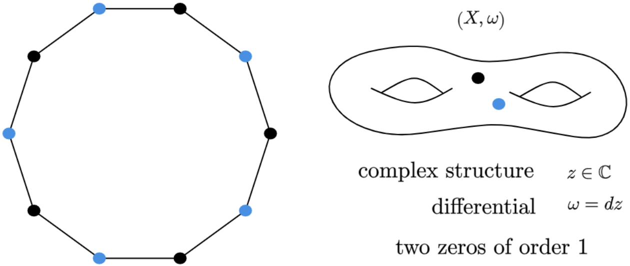


Strata of differentials

- For a partition $\mu = (a_1, \ldots, a_m)$ of $2g-2$, a **stratum of differentials** prescribed by μ is

$$\mathcal{H}_g(\mu) := \{(X,\omega) \mid \operatorname{div}\omega = \sum_i a_i z_i\}.$$

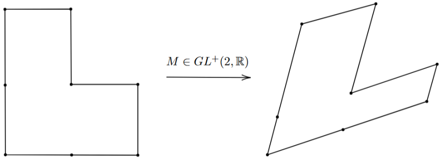
- Smooth complex orbifold (complex algebraic variety) of dimension $2g+m-1 = \dim_{\mathbb{C}} H^1(X, \boldsymbol{z}; \mathbb{C})$.
- Local coordinates are given by the period

$$\mathrm{Per}(X,\omega) : H_1(X, \boldsymbol{z}; \mathbb{Z}) \to \mathbb{C}$$

$$[\gamma] \mapsto \int_\gamma \omega$$

Strata of differentials

A stratum has a natural $\mathrm{GL}_2^+(\mathbb{R})$-action.



Theorem (Masur's criterion '92)

If $g_t \circ (X, \omega)$ *is* **recurrent**, *where* $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$, *then the vertical straight line flow is* **uniquely ergodic**.

$\mathrm{GL}_2^+(\mathbb{R})$ orbits

Slogan : Classify the $\mathrm{GL}_2^+(\mathbb{R})$-orbit closures.

Theorem (Eskin-Mirzakhani-Mohammadi '15, Filip '16)

Any $\mathrm{GL}_2^+(\mathbb{R})$*-orbit closure is an* **algebraic subvariety** *of* $\mathcal{H}_g(\mu)$*, defined by a* $\mathbb{R}$*-linear subspace in the period coordinates.*

$\mathrm{GL}_2^+(\mathbb{R})$ orbits

Setting:

- A partition $\mu = (a_1, \ldots, a_m, -b_1, \ldots, -b_n)$ of $2g-2$
- $(X, \omega) \in \mathcal{H}_g(\mu)$ has poles p_i of order b_i.
- $\boldsymbol{p} = \{p_1, \ldots, p_n\}$

Definition

For a partition $\mathfrak{R} : \boldsymbol{p} = P_1 \coprod \cdots \coprod P_k$ of the set $\boldsymbol{p}$ of the poles,

$$\mathcal{H}_g(\mu^{\mathfrak{R}}) := \{(X, \omega) \in \mathcal{H}_g(\mu) \mid \sum_{p \in P_j} \mathrm{res}_p\, \omega = 0 \text{ for each } P_j\}.$$

A *generalized stratum* $\mathcal{H}_g(\mu^{\mathfrak{R}})$ is a $\mathrm{GL}_2^+(\mathbb{R})$-invariant subvariety.

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Connected components?

multi-scale compactification

A compactification $\mathcal{H}_g(\mu) \subset \overline{\mathcal{H}}_g(\mu)$ called the moduli of **multi-scale differentials**, constructed by Bainbridge-Chen-Gendron-Grushevsky-Möller '19.

- $\mathbb{P}\overline{\mathcal{H}}_g(\mu) := \overline{\mathcal{H}}_g(\mu)/\mathbb{C}^*$ is a smooth compact orbifold.
- Normal crossings boundary.
- The boundary $\partial\overline{\mathcal{H}}_g(\mu)$ parametrizes geometric objects: multi-scale differentials.

The boundary strata is isomorphic to the product of the generalized strata, up to finite cover.

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Similar to the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$.

Classification of connected components

The first step for orbit classification is enumerating connected components.

- Konstevich-Zorich ('03) : Strata of holomorphic differentials
  - classified by **hyperellipticity** and **spin parity**.
- Boissy ('15) : Strata of meromorphic differentials
  - classified by the above two + **rotation number** ($g = 1$).

Theorem (L '24,L-Wong '25)

Assume $g > 0$ or $g = 0$ and $|\mathfrak{R}| < n$. The connected components of a generalized stratum $\mathcal{H}_g(\mu^{\mathfrak{R}})$ are classified by the above three invariants + **index** *($g = 0$), with (possibly) multiple hyperelliptic components.*

Topological invariants

Some strata contain **hyperelliptic** components consisting of hyperelliptic flat surfaces: (X, ω) with an involution $\sigma \curvearrowright X$ such that
$X/\sigma \cong \mathbb{P}^1$ and $\sigma^*\omega = -\omega$.
The involution σ permutes the poles: gives the *ramification profile* of $\Sigma \in Sym_n$ of this component.

- Σ fixes a pole of even order, or interchanges a pair of poles of the same order.
- $\mathrm{res}_q\,\omega + \mathrm{res}_{\Sigma(q)}\,\omega = 0$.

For each ramification profile Σ, there exists a unique hyperelliptic component.

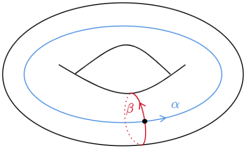
Topological invariants

For a closed curve $\gamma \subset X$, let $\operatorname{Ind}\gamma$ be the degree of a Gauss map $G_\gamma : S^1 \to S^1$ w.r.t the flat metric.

- For $g = 1$, let $\{\alpha, \beta\}$ be a symplectic basis of (X, ω) Then the **rotation number** of (X, ω) is

$$\gcd(d, \operatorname{Ind}\alpha, \operatorname{Ind}\beta)$$

$$d := \gcd(a_1, \ldots, a_m, b_1, \ldots, b_n).$$



Topological invariants

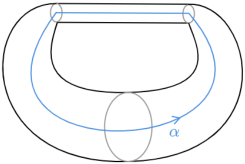
- For $g > 1$, let $\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ be a symplectic basis of (X, ω). Then the **spin parity** of (X, ω) is

$$\sum_i (\operatorname{Ind} \alpha_i + 1)(\operatorname{Ind} \beta_i + 1) \pmod 2.$$

- For $g = 0$, the **index** of (X, ω) with one pair of simple poles is

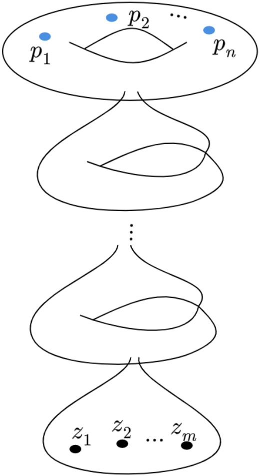
$$\gcd(d, \operatorname{Ind} \alpha)$$

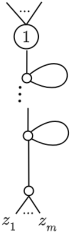
where α is a 'cross curve' of a cylinder obtained by 'plumbing' the pair of simple poles.



Method:degeneration to the boundary $\partial\overline{\mathcal{H}}_g(\mu^{\mathfrak{R}})$

For each component, we find a boundary element represented by the following diagram.



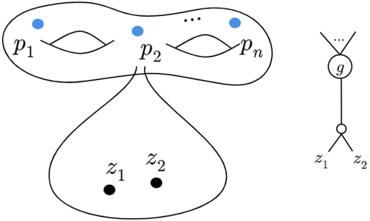


Method:degeneration to the boundary $\partial\overline{\mathcal{H}}_g(\mu^{\mathfrak{R}})$

Each connected component can be approached by
'breaking up a zero' and **'bubbling handles'**
from a connected component of a one-dimensional stratum.

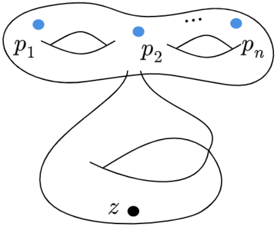
Method:degeneration to the boundary $\partial\overline{\mathcal{H}}_g(\mu^{\mathfrak{R}})$

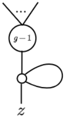
Merging zeroes : inverse of breaking up a zero.



Method:degeneration to the boundary $\partial\overline{\mathcal{H}}_g(\mu^{\mathfrak{R}})$

Unbubbling a handle : inverse of bubbling a handle.





Base cases: one-dimensional strata

Possibilities for one-dimensional $\mathbb{P}\mathcal{H}_g(\mu^{\mathfrak{R}})$. Note that $\dim \mathbb{P}\mathcal{H}_g(\mu^{\mathfrak{R}}) = 2g + m + n - |\mathfrak{R}| - 2$.

- A-signature : $g = 0$, $m = 3$ and all poles are residueless.
- B-signature : $g = 0$, $m = 2$ and all poles are residueless except a pair.
- C-signature : $g = 0$, $m = 1$ and all poles are residueless except three poles.
- D-signature : $g = 0$, $m = 1$ and all poles are residueless except two pairs.
- E-signature : $g = 1$, $m = 1$ and all poles are residueless.

We classified the connected components for all above cases except A-signature.

Equatorial net

For one-dimensional strata, the period coordinates gives local patches

$$Per_{(X,\omega)} : \mathbb{P}\mathcal{H}_g(\mu^{\mathfrak{R}}) \to U \subset \mathbb{P}^1$$

by

$$(X, \omega) \mapsto \left[\int_{\gamma_1} \omega, \int_{\gamma_2} \omega \right] \in \mathbb{P}^1$$

where $[\gamma_1], [\gamma_2]$ generate $H_1(X \setminus \boldsymbol{p}, \boldsymbol{z}, \mathbb{Z})/\mathfrak{R}$.
Choosing different generators $\gamma_1, \gamma_2 =$ post-composing an element of $PSL_2(\mathbb{Z})$.
The preimage $N := Per^{-1}(\mathbb{R})$ is well-defined real codimension one submanifold, called *equatorial net* of $\mathbb{P}\mathcal{H}_g(\mu^{\mathfrak{R}})$.

Equatorial net

The closure $\overline{N}$ in $\mathbb{P}\overline{\mathcal{H}}_g(\mu^{\mathfrak{R}})$ is a ribbon graph.

- $V(\overline{N})$ consists of elements of $\partial\mathbb{P}\overline{\mathcal{H}}_g(\mu^{\mathfrak{R}})$.
- For each vertex v, the edges adjacent to v are corresponding to prong-matchings of v.

Using this combinatorial description of $\overline{N}$, we can enumerate the connected components of $\overline{N}$.

Thank you!