Rescaling limit of quadratic rational maps and a search for Berkovich spider

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JHH80 Dynamical Developments
Degenerations of Flat Surfaces and Rational Maps
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My encounter with John Hamal Hubbard

Mathematical:

1984- Douady's Bourbaki Seminar, Orsay Note, Polynomial-like mapping, …

Personal:

1986 ICM Berkeley (invited speaker: Douady, Sullivan)

1987 Conference in Cornell

1988 Max Planck Institute in Bonn, West Germany

Postdocs: Yuval Fisher and Ben Bielfeld, Visitor Tan Lei

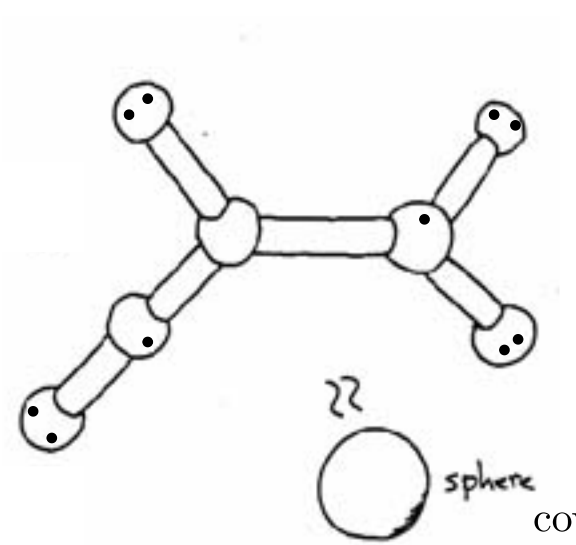
Conference and Méchoui in IHES

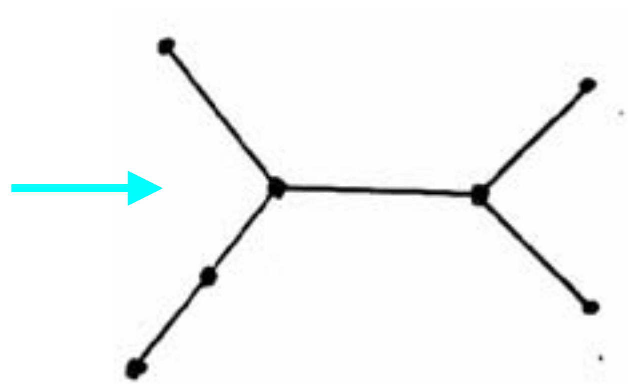
“Tropical complex dynamics”

Tree and piecewise linear dynamics appear from 1-dim complex dynamics, in several settings, all related to the “**degeneration**” of rational maps or related Riemann surfaces.

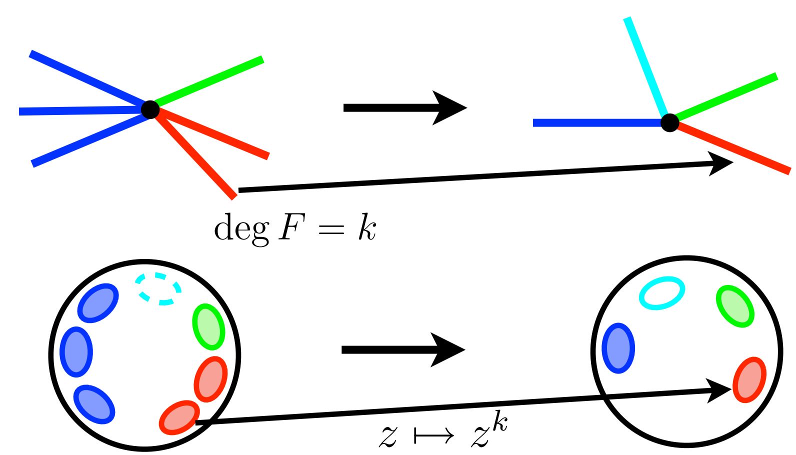
Objects:

· T metric tree ($\mathbb{R}$-tree)
· $F : T \to T$ piecewise linear map whose “derivative” $DF \in \mathbb{N}$ (or $\lambda\mathbb{N}$)
· “small sphere” S^2_q associated to a vertex q of T with marked points $x(\beta)$ corresponding to (infinitesimal) branch β at q
· mapping $g_q : S^2_q \to S^2_{F(q)}$ (branched cover) between the small spheres such that if $g_q(x(\beta)) = x(F(\beta))$ and $\deg_{x(\beta)} g_q = DF|_\beta$.





covering degree on annulus $= DF$



3 settings for "tropical complex dynamics" from degeneration

1. From configuration of multiply connected Fatou components, related to the limit of (stretching) qc deformation

   presented in 1987 Cornell conference and 1988 IHES conference

   cf. DeMarco-McMullen tree 2008, Luo's work Kiwi's talk

2. Thurston's theory of characterization of rational maps among self-branched coverings of 2-sphere with PCF condition.

   Thurston's pull-back in Teichmüller space. Key analysis is when its iteration diverges to the bdry, and show the stabilized degeneration

   For a Thurston obstruction, can associate a PL map on a tree.
   In 1988 MPI Bonn with Tan Lei, this lead to a construction of obstructed mating without Levy cycle.

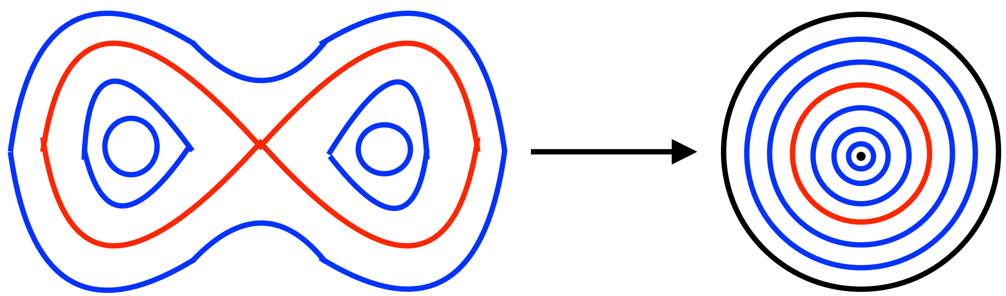
3. From degenerating families of rational maps, rescaling limits (Kiwi, Arfeux), interpreted using Berkovich space.

   cf. Kiwi's talk
   Realization problem: from tree to family? (in progress? w. Arfeux, Kiwi, Ramadas)

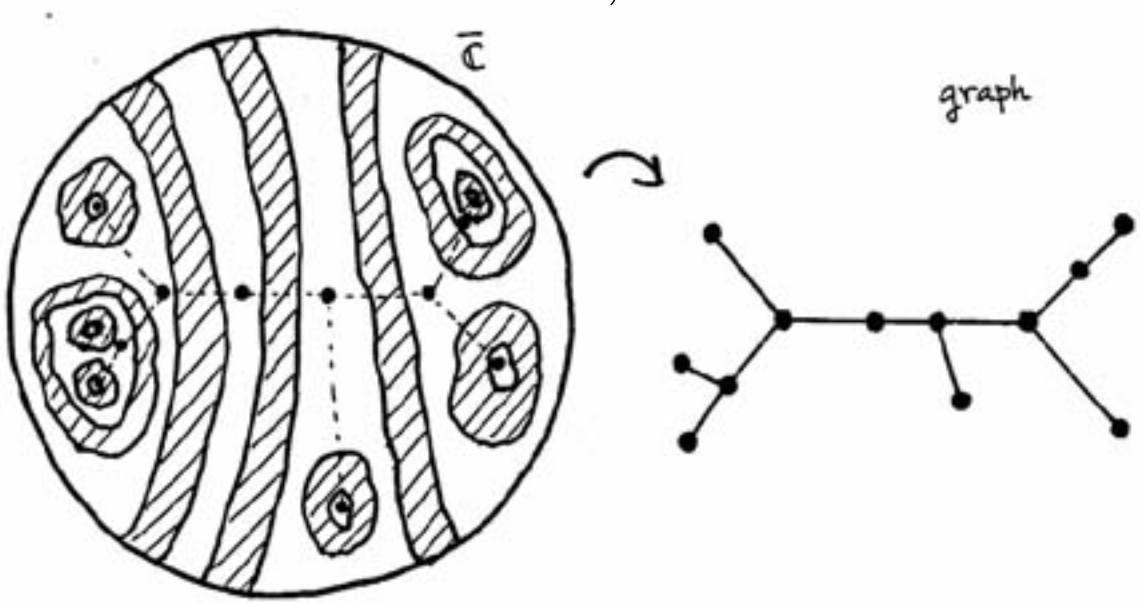
1. Tree for multiply connected Fatou components

Poor choice of the title of the paper: "configuration of Herman rings"

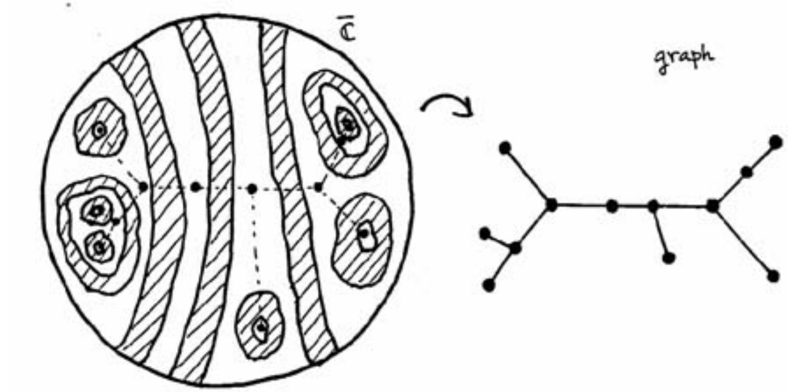
Collection of annuli $\mathcal{A} = \mathcal{A}_f$ for a rational map f: Fatou components (other than parabolic basins) can be decomposed into a collection of disjoint annuli as follows. Siegel disks and Herman rings and their inverse images are foliated by invariant curves and delete the closure of grand orbits of critical orbits and the center of Siegel disks. Super attracting basins and attracting basins are also foliated by the level curves of Böttcher coordinates and linearizing coordinates.



From the collection of annuli $\mathcal{A}$, we construct a tree as follows.



Construction of the Tree and PL map



Definition. Let $\mathcal{A}$ be a collection on *disjoint* annuli of $\widehat{\mathbb{C}}$. Each annulus A is canonically foliated by topological circles. For $x, y \in \widehat{\mathbb{C}}$, let $A[x, y]$ be the union of leaves which separate x and y. So $A[x, y]$ is either a subannulus of A or an emptyset. Define $d : \widehat{\mathbb{C}} \times \widehat{\mathbb{C}} \to [0, +\infty]$ by

$$d(x, y) = \sum_{A \in \mathcal{A}} mod\ A[x, y].$$

Then $d(\cdot, \cdot)$ (which may take value ∞) satisfies the triangle inequality. Let

$$T = T_{\mathcal{A}} = \widehat{\mathbb{C}}/_{\sim_{\mathcal{A}}},$$

where $x \sim_{\mathcal{A}} y$ if and only if $d(x, y) = 0$. Then d satisfies the triangle inequality. T is a tree and d induces a (generalized) metric.

If $\mathcal{A} = \mathcal{A}_f$ comes from a rational map f, then it induces a piecewise linear map $F : T \to T$, whose "derivative" DF is the covering degree of annuli.

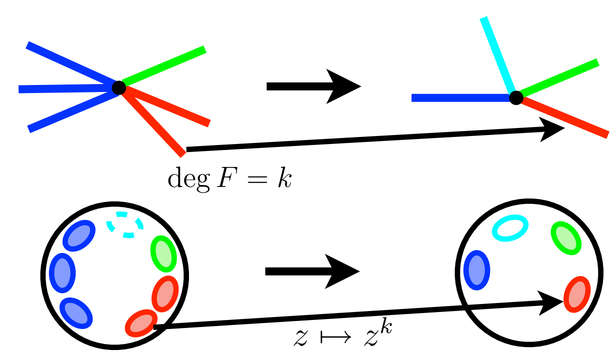
We can extract a finite skeleton, by choosing a finite forward invariant collection X of connected sets in the complement of $\mathcal{A}$ and considering only annuli separating X.

Properties of $F : T \to T$:
T is a (finite tree) with geodesic metric and F is piecewise linear with $DF \in \mathbb{N}$.
There exists periodic segments corresponding to periodic Fatou components, on such segments the return map is like id (SD and HR), $x \mapsto x + c$ (HR), and $x \to kx$ (SAB).
Dense set of points are eventually mapped on to the periodic segments.

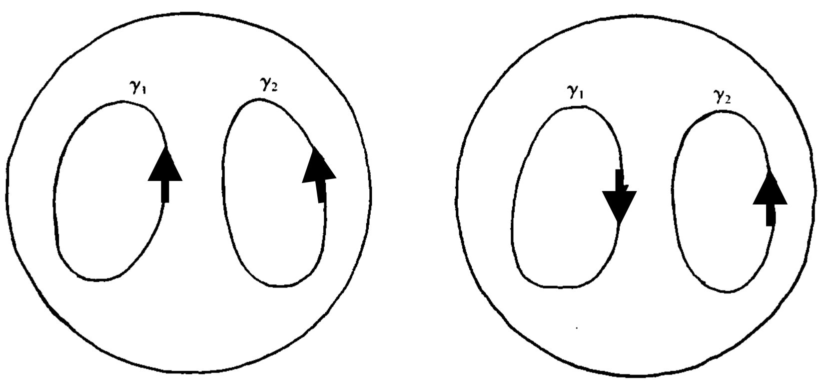
T can be considered as the "limit" of the stretching deformation of annuli.

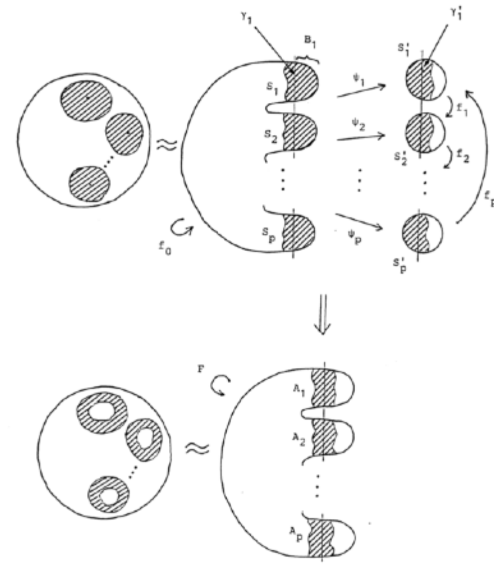
Local models on small spheres corresponding to vertices

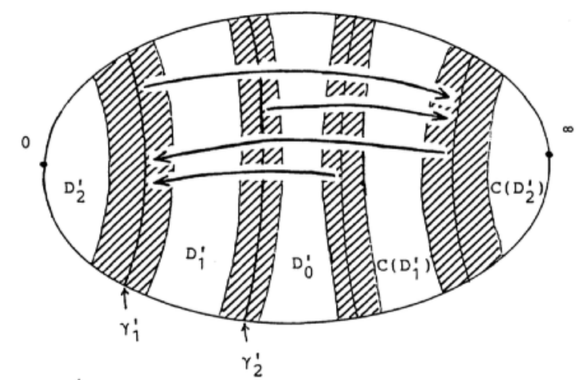
· "small sphere" S^2_q associated to a vertex q of T with marked points $x(\beta)$ corresponding to (infinitesimal) branch β at q.
· rational map $g_q : S^2_q \to S^2_{F(q)}$ between the small spheres such that if $g_q(x(\beta)) = x(F(\beta))$ and $\deg_{x(\beta)} g_q = DF|_\beta$.



Motivation for the tree: Herman rings of period 2 can be constructed by qc-surgery. There are two configurations. For which degree?

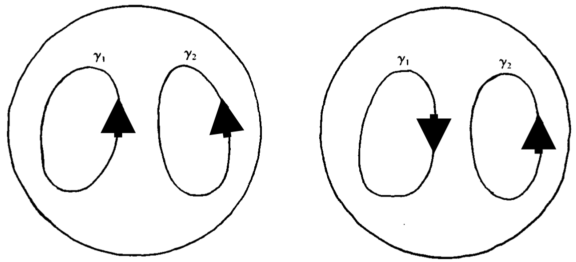
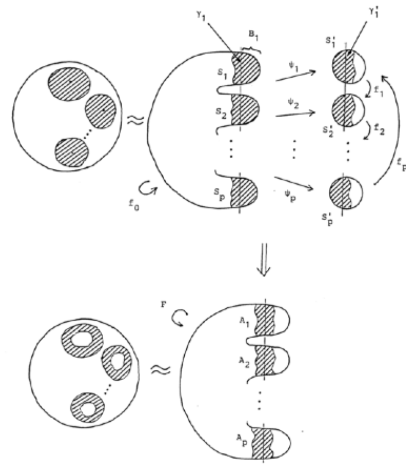
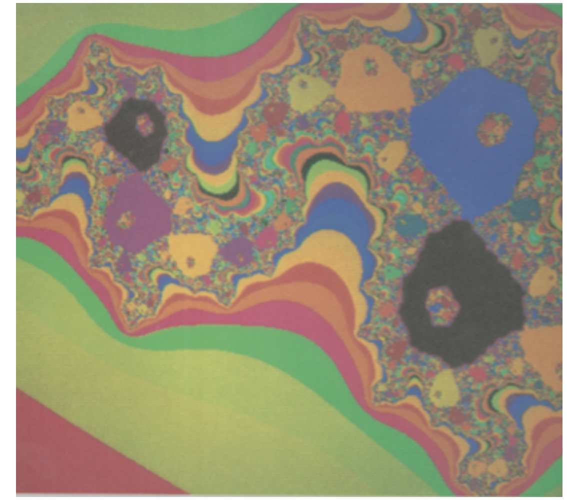






Q. Where can one find the result of surgery in the space of rational maps?

Q. How can one show that you need degree 4 for the second configuration?

Q. Where can one find the result of surgery in the space of rational maps?

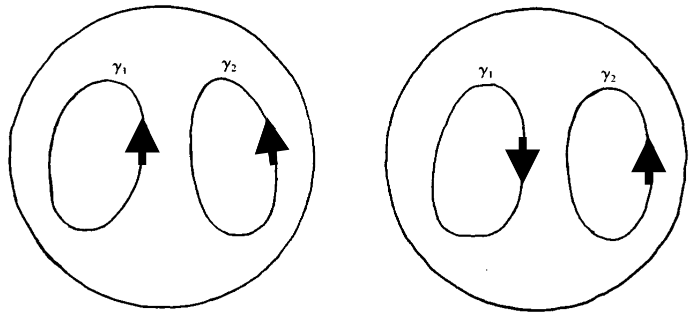
Note that the limit of stretching deformation f_t should give a period 2 cycle of Siegel disks for a rational map of lower degree (i.e. a pole and a zero collide). So assume that the limit is a quadratic polynomial with a Siegel disk of period 2.

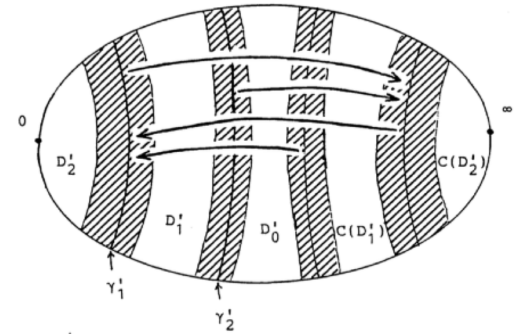
We can look for a cubic rational map of the form

$$z \mapsto z^2 \frac{z-\beta}{z-\gamma} + \alpha, \quad \text{a singular perturbation of } z^2+\alpha \text{ with SD od per 2}$$

where $f : z^2 + \alpha$ is a quadratic polynomial with a Siegel disk of period 2 with a center z_0, and β and γ should be close to z_0. Adjust β and γ so that a rescaling of f near z_0 looks like a map with a Siegel disk. (Adjusted by hand)

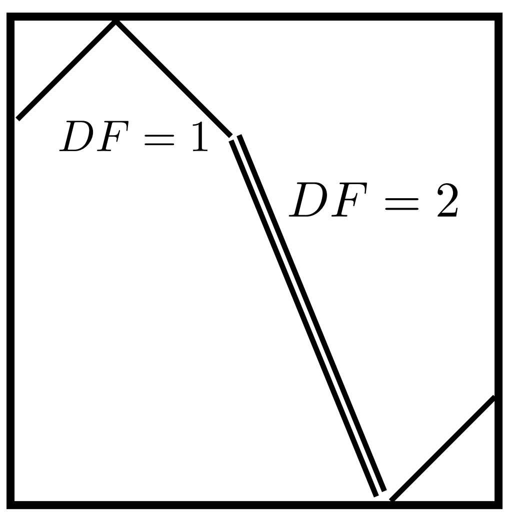
$\alpha = -0.8648749 + 0.2103377i, \beta = -0.7668679 - 0.2503722i, \gamma = -0.0809479 - 0.2492042i$

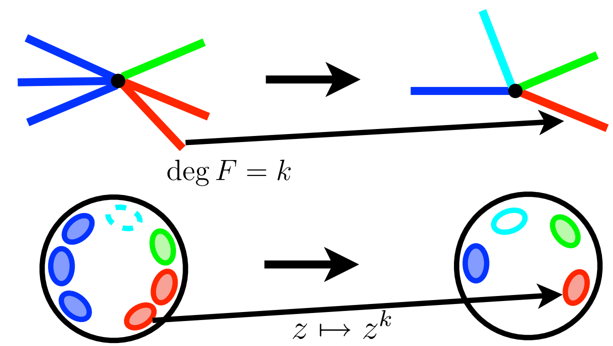




Q. How can one show that you need degree 4 for the second configuration?

graph of T





The count of critical points can be derived from Riemann-Hurwitz formula.

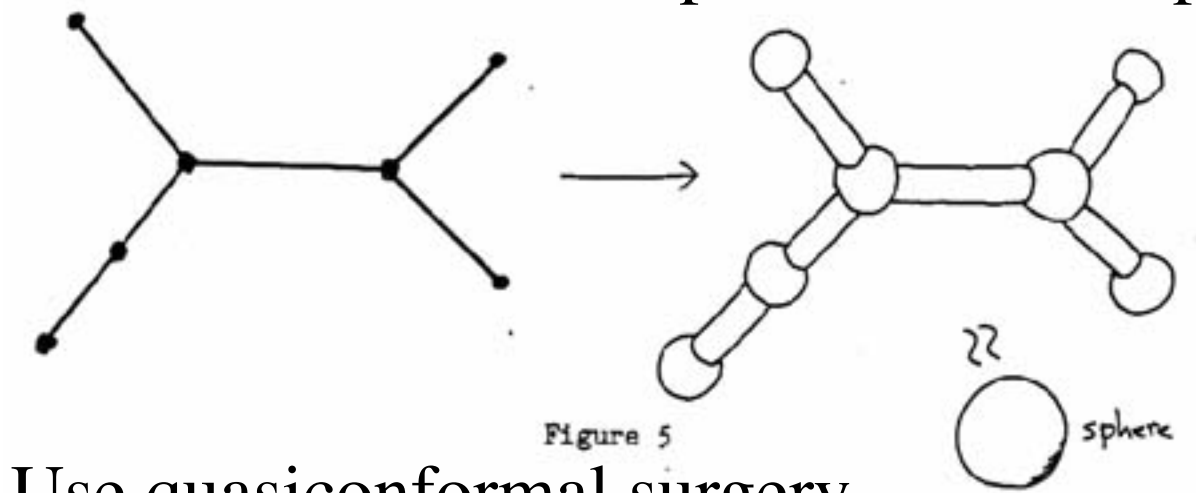
A simple folding requires 2 critical points.

A discontinuity of DF requires 1 critical point.

6 critical points needed $\implies$ $\deg \geq 4$.

Inverse problem

Construction of a rational map from a tree map and local models



If $Sing(T)$ has finite orbit, a surgery can be carried out.

Use quasiconformal surgery

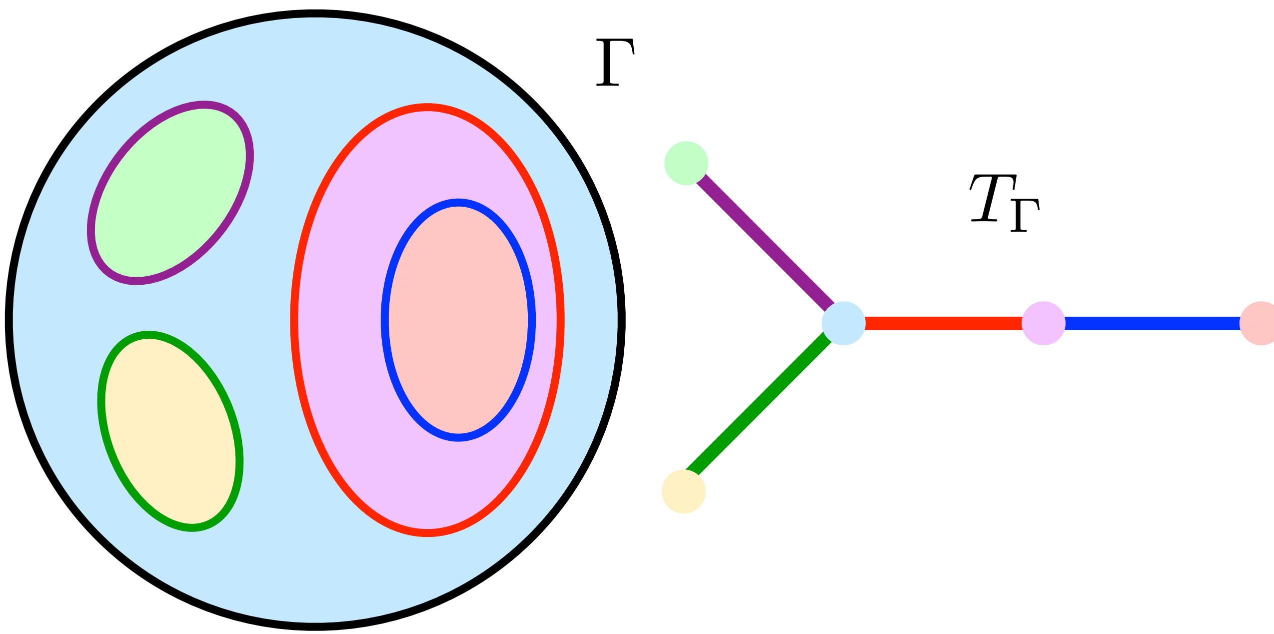
Other applications

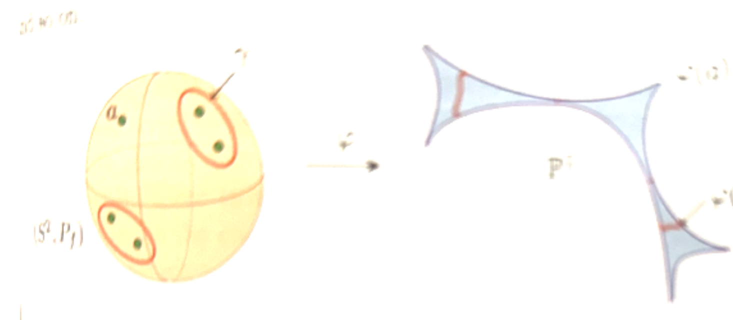
All Fatou components of the Newton's method of a polynomial is simply connected.
(Because there is only one weakly repelling fixed point.)

There exists an entire function having doubly connected wandering domain.

2. Thurston obstruction and a tree

Key ingredient in Thurston theory is the map in the Teichmüller space: $\sigma_f : Teich(S^2 - P_f) \to Teich(S^2 - P_f)$.
When f is equivalent to a rational map if and only if the orbits of σ_f stay bounded in $Teich(S^2 - P_f)$. Otherwise the corresponding sequence of Riemann surfaces go to the boundary of $Teich(S^2 - P_f)$, i.e. they degenerate and develop a system of curves which are short. So these Riemann surfaces already look like a tree.





Hubbard Teichmüller book Vol. 2

Assume that the Thurston matrix is irreducible and take the eigenvector of the matrix and assign the length to the edges by the entries of the eigenvector. Then the induced map is a piecewise linear map with derivative $DF = \lambda \cdot$ degree on the curve, where λ is the Thurston eigenvalue (≥ 1). In particular it is weakly expanding.

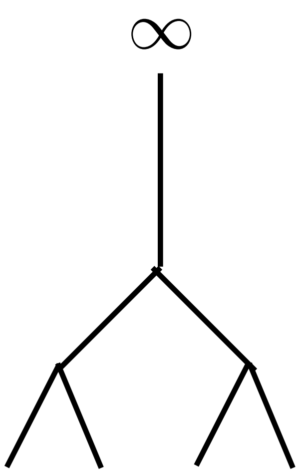
Applications

Definition for a pcf branched cover $f : S^2 \to S^2$, a multicurve $\{\gamma_0, \dots, \gamma_{n-1}\}$ ($\gamma_n = \gamma_0$) is Levy cycle if $\exists \gamma'_j \subset \gamma_{j+1}$ such that $\gamma'_j \sim \gamma_j$ and $f|_{\gamma'_j}$ is injective.

Levy cycle theorems. (Levy, Rees, Tan Lei) For a topological polynomial or quadratic branched cover, if there is a Thurston obstruction, then there exists a Levy cycle.

top. polynomial

(nesting) order is preserved
+ weak expansion
$\implies$ a cycle of degree 1

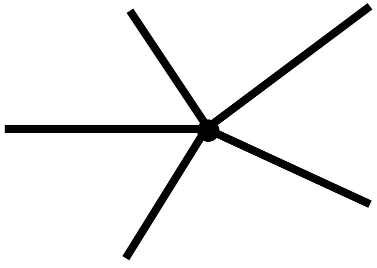


quadratic

There exist a fixed point of F,
and periodic branches around it.

If all the periodic branches are of degree 1,
they correspond to a Levy cycle.

If there is a degree 2 **periodic** branch,
it cannot be folded and the distance from
the fixed point cannot shrink, a contradiction

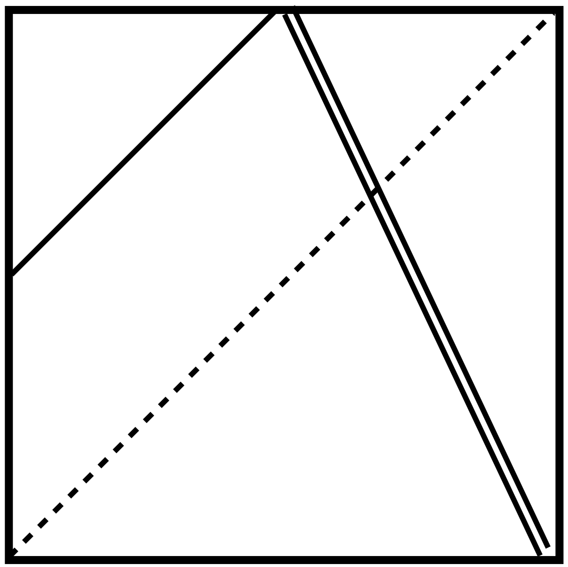


Obstructed cubic branched covering not containing Levy cycle

The map should be weakly expanding. ($\lambda \geq 1$)
The derivative DF at the fixed point should be > 1.

Let $T =$ segment.

F



Degeneration of rational maps and Trees of spheres

cf. Kiwi's talk

Rescaling limits of degenerating family of rational maps can be organized by the tree of spheres (Kiwi, Arfeux). It can be also interpreted using the Berkovich space for the field of Puiseaux series.

In quadratic case, the structure of the tree is simple and it has a centr star which is rotated, and the end vertices of the star has a parabolic fixed point for the return map.
(cf. Levy cycle Thm for Thurston tree)

Realization problem:
Given a data, can one construct a degenerating family?

Rescaling limits of degenerating family of rational maps

J. Kiwi

$Rat_d = \{\text{degree } d \text{ rational maps}\} = \mathbb{P}^{2d+1}(\mathbb{C}) \setminus H$

$d \geq 2,\ H = \{\text{resultamt}(denom, numer) = 0\}$, hypersurface

$rat_d = Rat_d / M\ddot{o}b$

Degeneration $Rat_d \ni f_t \to f \in H\ (t \to 0)$

$f_t(z) \to f(z)$ uniformly on compact sets in $\widehat{\mathbb{C}} \setminus$ (finite set)

The limit (if exists) depends sensitively on coordinate changes (or normalization). One may take a family M_t of Möbius transformation (possibly degenerating), and see if there is a reasonable limit $M_t^{-1} \circ f_t \circ M_t \to g$.

Are there any dynamical meaning?

Rescaling limit: if for some $q > 1$ and M_t the limit $M_t^{-1} \circ f_t^q \circ M_t \to g$ exists and non-trivial.

We are interested in the case where g is of degree > 1 and not postcritically finite.

Rescaling limit $M_t^{-1} \circ f_t^q \circ M_t \to g$?

Theorem (Kiwi) For $d = 2$, possible rescaling limits which are not postcritically finite are:

either $M_t^{-1} \circ f_t^q \circ M_t \to g_0$, which is of degree 2 and has a parabolic fixed point;
or $M_t^{-1} \circ f_t^q \circ M_t \to g_0$, which is of degree 2 and has a parabolic fixed point and one of critical point is preperiodic, $q' > q$ and $M_t'^{-1} \circ f_t^{q'} \circ M_t' \to g_1$ which is a polynomial.

Later we will discuss why we are interested in non PCF limits

If we mark some of the orbits (e.g. critical orbits), then as $t \to 0$, they are separated by a curve which is pinched and the cross ratio degenerates.

Key observation: If the sphere is pinched by several curves, we obtain several spheres and they are organized like a tree.

Different degenerating coordinate changes (Möbius) focus on different parts of the original sphere.

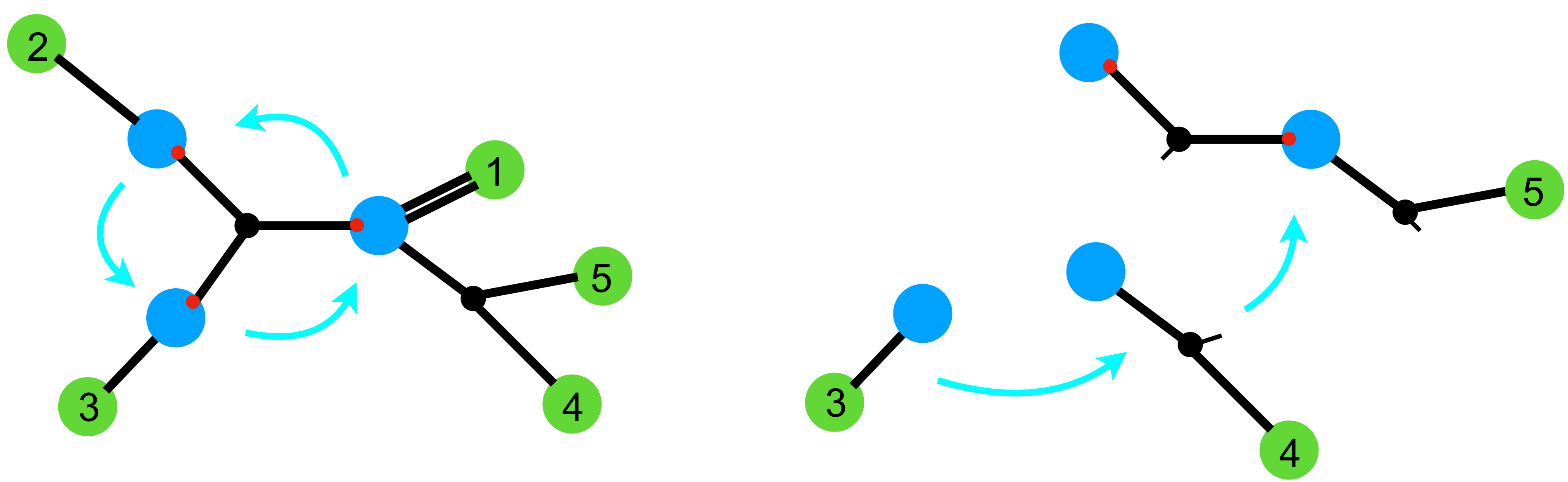
The data describing the degenerating family

Tree with piecewise linear map

A metric tree T(each edge has a positive length)
A piecewise linear map F whose "derivatives" DF are integers
(edge $\leftrightarrow$ pinching curve, DF = covering degree)
$\rightarrow$ consistency condition for the length

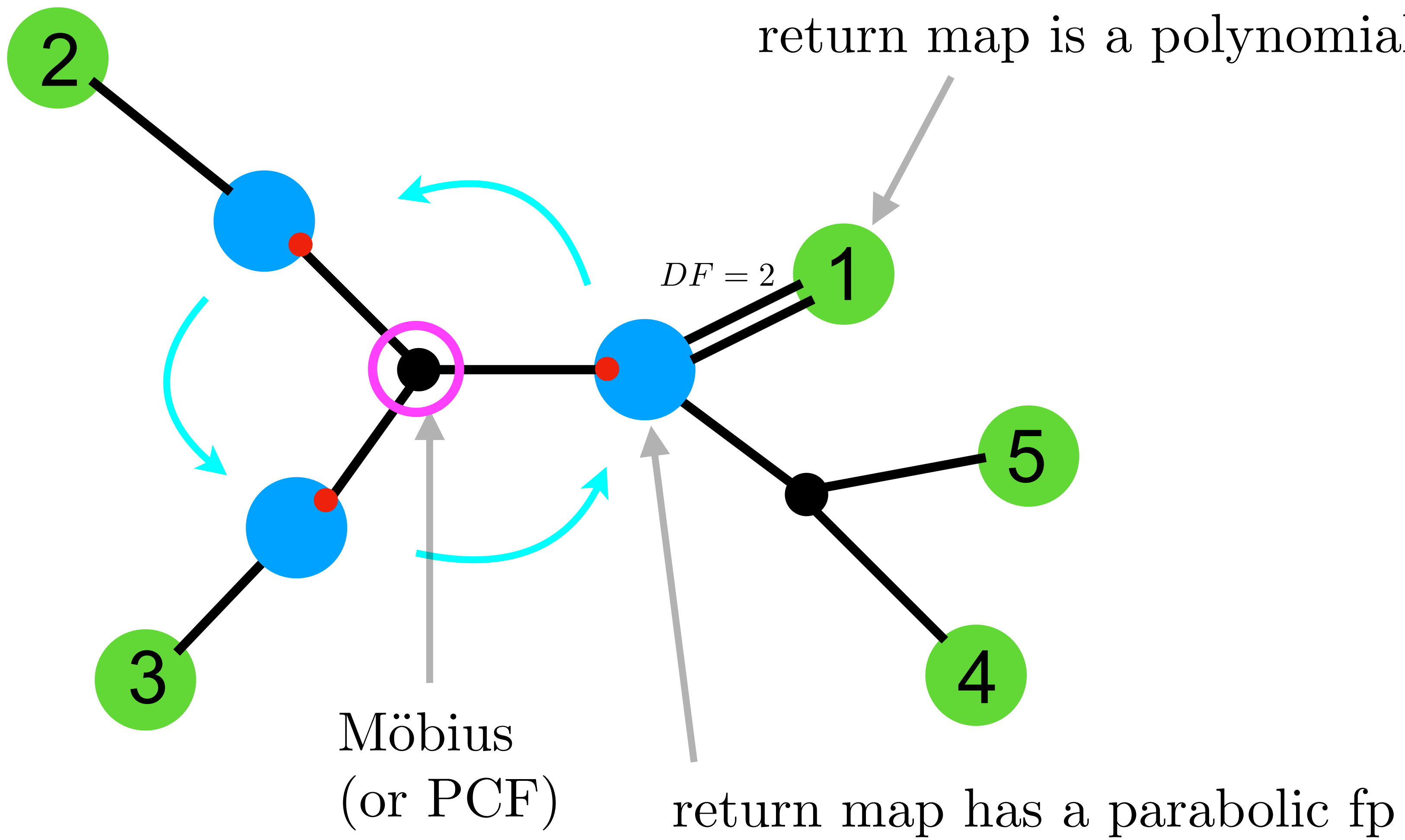
Small sphere for each vertex, rational map between them

A sphere $\widehat{\mathbb{C}}_x$ for each vertex $x \in T$
branches from $x \leftrightarrow$ marked points on $\widehat{\mathbb{C}}_x$
A rational map $g_x : \widehat{\mathbb{C}}_x \to \widehat{\mathbb{C}}_{F(x)}$
DF on a branch = local degree of g_x at the corr. marked pt



Quadratic case: Kiwi and Arfeux

Quadratic case (d=2): In the tree there should be a star on which the dynamics is a rotation. The cycle of spheres around the star has a parabolic fixed point for the return map.



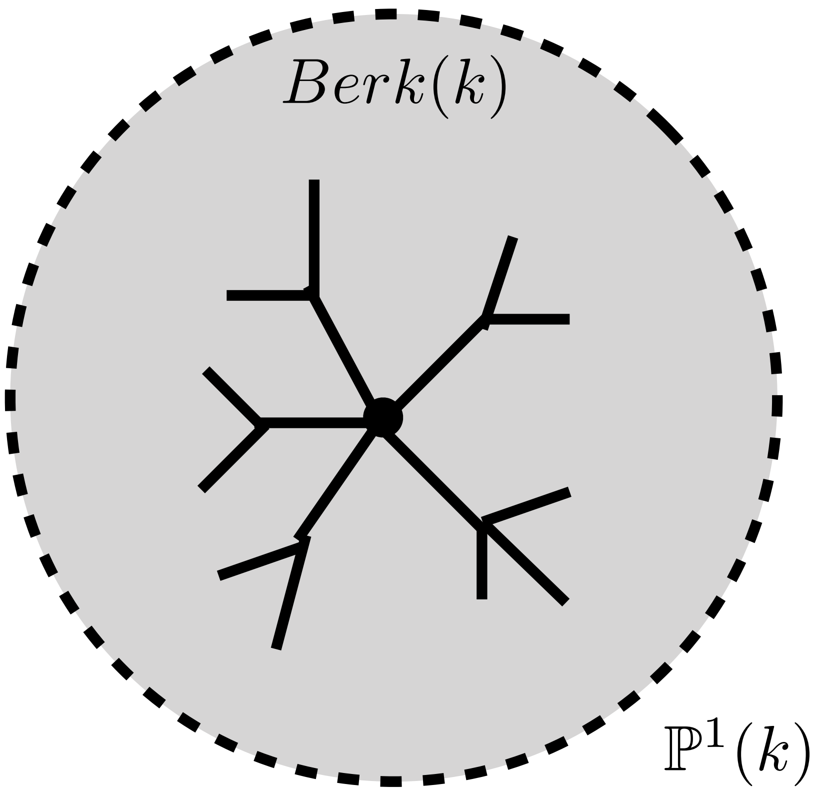
Realization problem (converse): worked out cases by Arfeux-Kiwi-S.

Berkovich space

Let k be an algebraically closed field, which is complete with respect to a non-Archimedean valuation $|\cdot|$.

The projective space $\mathbb{P}^1(k)$ is not locally compact.

Berkovich space $Berk(k) = \mathbb{P}^{1,an}(k)$ is a space which contains $\mathbb{P}^1(k)$ as the boundary. $Berk(k)$ is locally compact, Hausdorff and path-connected,



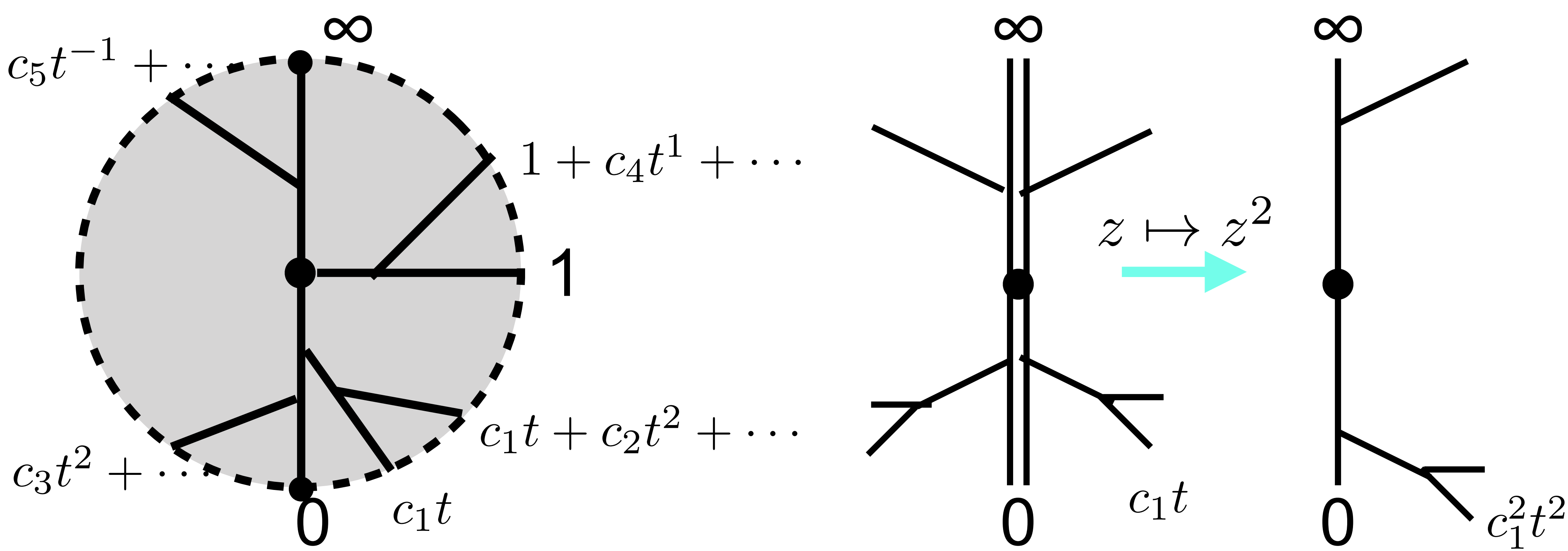
see M. Jonsson's picture

Dynamics on Berkovich space of formal Laurent series field
Kiwi & Arfeux

Let $\mathbb{C}((t))$ be the field of formal Laurent series with variable t. Let $\mathbb{L}$ be an algebraically closed field containing $\mathbb{C}((t))$ and complete with respect to a valuation $|\cdot|$.

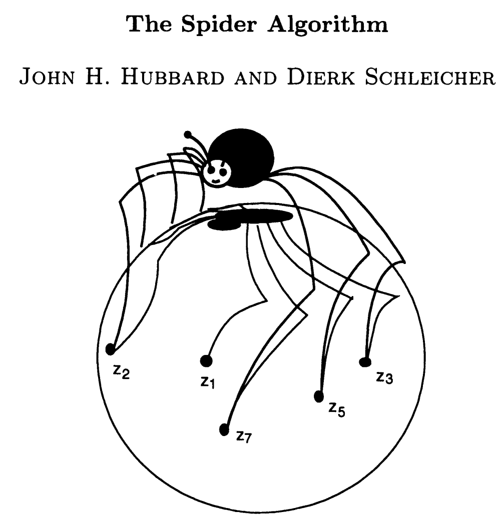
Let $f(z) = (f_t(z)) \in \mathbb{L}(z)$ be a rational map with coefficients in $\mathbb{L}$. If the series in the coefficients are convergent, then $(f_t(z))$ can be considered as a family of rational maps. (for $0 < |t| < 1$)

Then f induces a map $f : Berk(\mathbb{L}) \to Berk(\mathbb{L})$.



Berkovich spider?

The Spider Algorithm is an algorithm which implements Thurston's iteration in the Teichmüller space for pcf polynomials. It has been implemented on computer in order to find out the parameters.
A point in the Teichmüller space is represented by arcs joining marked points and infinity.



What we need for the Berkovich case is an alternative to the spider algorithm. We do have Teichmüller space for this setting. But it might be possible to set up the space of enbeddings of abstract tree (from given data) into the Berkovich space. IF a pull-back algorithm can shown to be *convergent*, then we may be able to solve the realization problem.
Arfeux, Kiwi, S. Ramadas. It might be used to study the punctures for Per_n(0). Davis, Kapiamba, Hironaka.

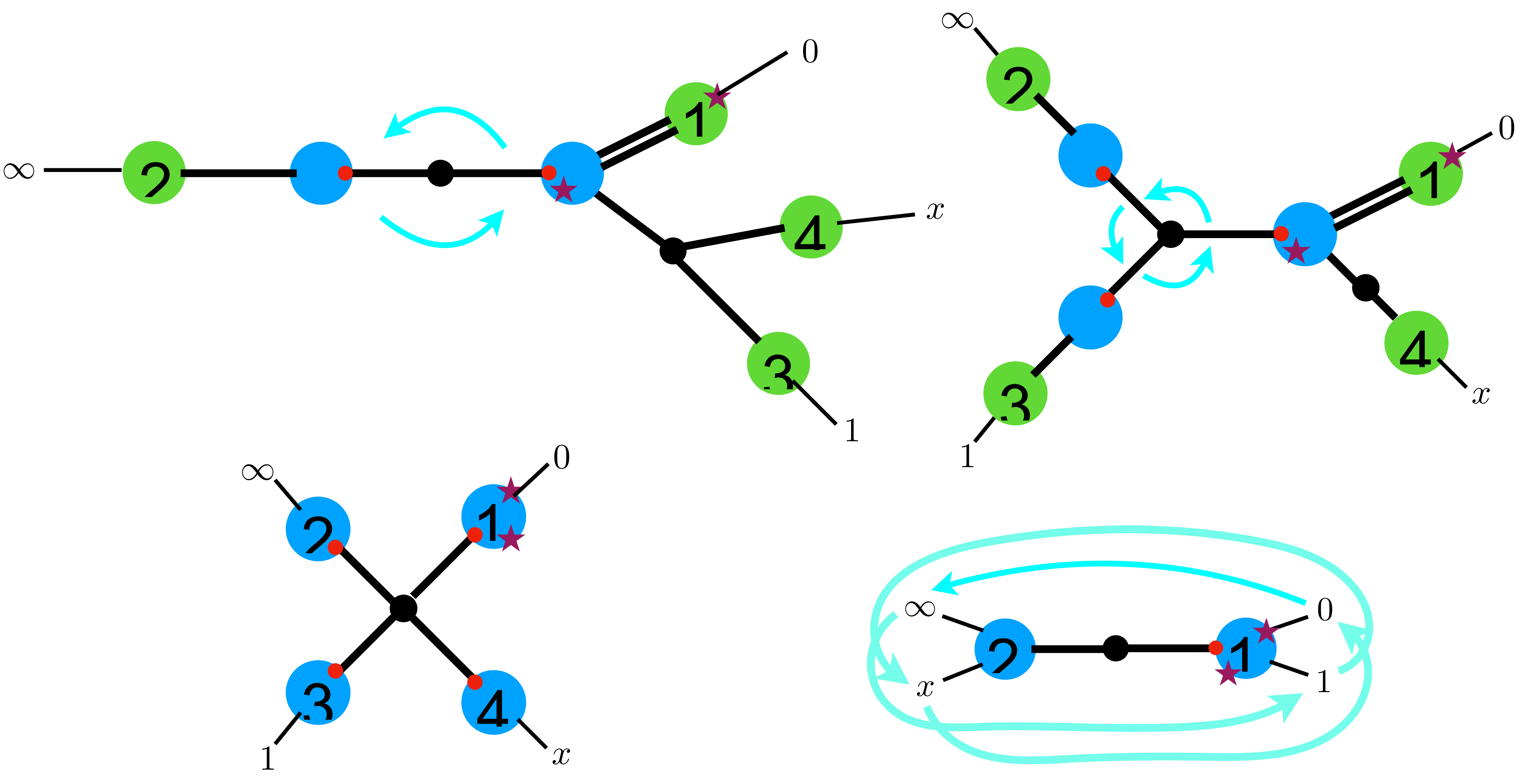
Worked out examples for the punctures of Per_4

Conditions: a star in the center with rotation

at the ends of the star, a parabolic cycle on small spheres

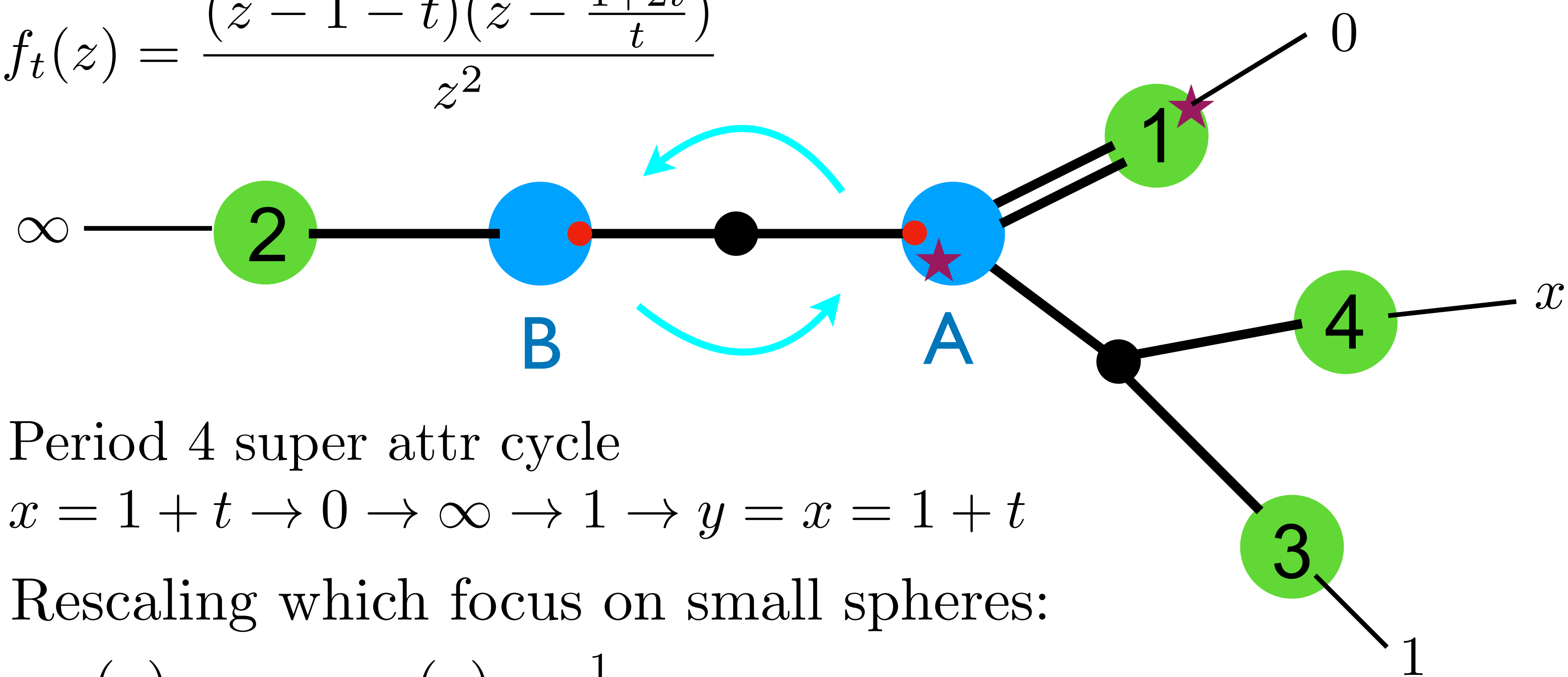
only one degree two segment

Normalization/parametrization: $x \to 0 \to \infty \to 1 \to y$ $\qquad x = y$ for Per_4



Check that the obtained family has the desired scaling limits

$$f_t(z) = \frac{(z-1-t)(z-\frac{1+2t}{t})}{z^2}$$



Period 4 super attr cycle

$x = 1+t \to 0 \to \infty \to 1 \to y = x = 1+t$

Rescaling which focus on small spheres:

$\varphi_A(z) = z, \;\; \varphi_B(z) = \frac{1}{tz},$

$\varphi_0(z) = \frac{z}{t^2}, \; \varphi_\infty(z) = \frac{1}{t^5 z}, \; \varphi_1(z) = \frac{z-1}{t^4}, \; \varphi_x(z) = \frac{z-x}{t^3}$

Maps between small spheres and the return map:

$\varphi_B \circ f_t \circ \varphi_A^{-1} \to \frac{z^2}{1-z}, \;\; \varphi_A \circ f_t \circ \varphi_B^{-1} \to 1-z, \;\; \text{composition} \to 1 - \frac{z^2}{1-z}$

$\varphi_\infty \circ f_t \circ \varphi_0^{-1} \to z^2, \;\; \varphi_1 \circ f_t \circ \varphi_\infty^{-1}, \varphi_x \circ f_t \circ \varphi_1^{-1}, \varphi_0 \circ f_t \circ \varphi_x^{-1} \to -z,$
composition $\to -z^2$

Happy birthday, Hamal!

And many thanks to the organizers:
Xavier Buff, Jasmin Raissy, Sarah Koch, Corentin Boissy