Approximating Transcendental Thurston Theory

JHH80 Dynamical Developments: Degeneration of Flat Surfaces and Rational Maps

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Overview

Polynomial Approximations

"Every entire function is a locally uniform limit of a sequence of complex polynomials."

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Example property P: hyperbolicity - i.e., that every singular value is attracted to a periodic cycle

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- Kisaka 1995: If $f_n \to f$ locally uniformly in $\mathbb{C}$ and the Fatou set of f is the union of super-attracting basins, then $\mathcal{J}_{f_n} \to \mathcal{J}_f$ in the Hausdorff metric.

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- Mihaljević-Brandt 2011: Kernel convergence of non-escaping hyperbolic components in the general entire setting

Results

Recall: An entire function is post-singularly finite (PSF) if it has finitely many singular values, and each singular value has a finite orbit.

Theorem [M, 2024]

Given a PSF function $f(z) = \lambda e^z$, there exist PSF polynomials $f_n(z) = \lambda_n(1 + \frac{z}{n})^n$ such that

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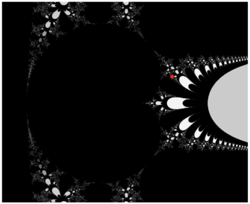
- $f_n \to f$ locally uniformly on $\mathbb{C}$, and
- $f_n|P_{f_n}$ is conjugate to $f|P_f$ for all $n \in \mathbb{N}$.

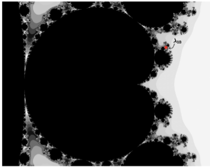
Moreover, there exists a polynomial $Q = Q(f) \in \mathbb{Z}[X]$ and integers $\ell, k \geq 1$ such that the parameter ray at angle $\frac{Q(n)}{n^\ell(n^k - 1)}$ in the space $\{z \mapsto \mu(1 + z/n)^n, \mu \in \mathbb{C}^*\}$ lands at f_n for each n.

Example

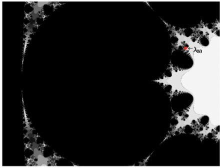


The star marks the position of $\lambda \approx 1.16302 + 0.71056i$,

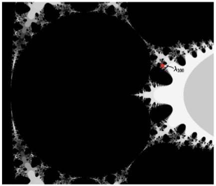
λ is in the parameter space of maps $z \mapsto c\exp(z)$



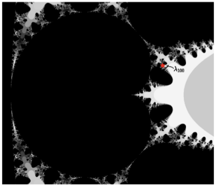
(A) $\lambda_{10} \approx 1.1176 + 0.86608i$



(B) $\lambda_{50} \approx 1.1545 + 0.74281i$



(C) $\lambda_{100} \approx 1.1575 + 0.72671i$



(D) $\lambda_{200} \approx 1.15891 + 0.71869i$

λ_n is in the parameter space of maps $z \mapsto c(1 + z/n)^n$

Results

Theorem [M-Prochorov-Reinke,2024]

Given a PSF entire map f, there exists a sequence of PSF polynomials (f_n), such that

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The proof of these results comes from Thurston theory.

Thurston Maps

Definition

A Thurston map is a branched cover $f : \mathbb{S}^2 \dashrightarrow \mathbb{S}^2$ such that

- the set of singular values S_f is finite
- the post-singular set P_f is finite ("PSF")
- f is of parabolic type

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Let $\mathrm{ess}(f)$ be the set of essential singularities of f. We say f is

- transcendental if $\mathrm{ess}(f)$ is non-empty
- planar if $\mathrm{ess}(f)$ is singleton and $f(\mathbb{S}^2 \setminus \mathrm{ess}(f)) \subseteq \mathbb{S}^2 \setminus \mathrm{ess}(f)$

Combinatorial Equivalence

Definition

Two Thurston maps f and g are combinatorially equivalent if they are topologically equivalent up to isotopy rel. P_f .

Any f that is combinatorially equivalent to a PSF entire map is said to be realized holomorphically; failing this, we say f is obstructed.

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If $|P_f| \leq 3$, then f is realized.

Realization Criterion

Let $P_f = A$. Suppose $|A| \geq 3$.

Definition

The Teichmüller space $\mathrm{Teich}(\mathbb{S}^2, A)$ is the space of hyperbolic metrics on $\mathbb{S}^2 \setminus A$.

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Formally,

$$\text{Teich}(\mathbb{S}^2, A) = \{\varphi : \mathbb{S}^2 \to \mathbb{C} \text{ homeomorphism}\}/\sim$$

Observation: A Thurston map induces a holomorphic operator $\sigma_f : \text{Teich}(\mathbb{S}^2, A) \to \text{Teich}(\mathbb{S}^2, A)$.

Theorem [Thurston]

A Thurston map with hyperbolic orbifold is realized if and only if σ_f has a fixed point in $\text{Teich}(\mathbb{S}^2, A)$.

Strategy

Theorem [M-Prochorov-Reinke, 2025]

Let f be a planar Thurston map. Suppose (f_n) is a sequence of planar Thurston maps with $P_{f_n} = P_f = A$ for all $n \in \mathbb{N}$.

Strategy

Theorem [M-Prochorov-Reinke, 2025]

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Then, if (f_n) converge combinatorially to f, then $\sigma_{f_n} \to \sigma_f$ locally uniformly on $\mathrm{Teich}(\mathbb{S}^2, A)$.

Combinatorial Convergence

Definition

Let $(f_n), f$ be planar Thurston maps satisfying $P_{f_n} = P_f = A$. Then (f_n) converges to f combinatorially if $\exists(\varphi_n) \subset \mathrm{Homeo}_0^+(\mathbb{S}^2, A)$ such that for any compact $D \subset \mathbb{R}^2$, we have

$$f_n \circ \varphi_n | D \equiv f | D$$

for all n sufficiently large.

Example

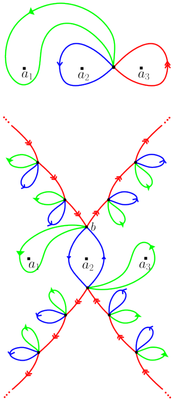

Figure: Rose graph and preimage for $f(z) = \sqrt{\ln 2}(1 - \exp(z^2))$. Here we have $P_f = \{a_1, a_2, a_3\} = \{-\sqrt{\ln 2}, 0, \sqrt{\ln 2}\}\}$

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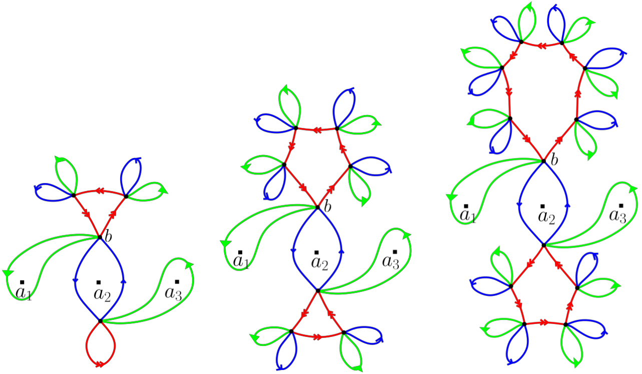

Figure: Combinatorial polynomial approximation for $f(z) = \sqrt{\ln 2}(1 - \exp(z^2))$

Idea of Proof

We start with a PSF entire map f with $P_f = A$ containing at least 3 points. Then we build a combinatorial polynomial approximation $\widetilde{f_n}$ of f

1. $\sigma_f, \sigma_{\widetilde{f_n}}$ are all locally contracting and by construction, we have
$$\sigma_{\widetilde{f_n}} \to \sigma_f$$
locally uniformly.
2. Since f is holomorphic, there exists a unique τ fixed by σ_f
3. By point 1, for each $n \in \mathbb{N}$, $\exists \tau_n$ fixed by $\sigma_{\widetilde{f_n}}$ such that
$$\tau_n \to \tau$$
4. From the τ_n, we get polynomials $f_n \to f$ locally uniformly as desired.

Problems

Characterization Problem: Given a Thurston map f, when is f combinatorially equivalent to an analytic Thurston map?

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- Bernstein-Levy: A polynomial Thurston map is realized holomorphically iff it does not have a Levy cycle
- Hubbard-Schleicher-Shishikura 2009: Levy cycle criterion holds for topological exponentials
- Prochorov 2024: Levy cycle criterion holds for a subset of all Thurston maps with $|P_f| = 4$

Conjecturally, the Levy cycle criterion holds for all planar Thurston maps.

Other results: Shemyakov 2022, D'Souza 2025.

Hubbard Trees for Polynomials

Theorem

[Douady-Hubbard 1983] Let f be a post-critically finite polynomial. Then there exists a finite tree H_f such that

- the vertex set contains $P_f \setminus \{\infty\}$
- $H_f \subset f^{-1}(H_f)$

Pullback of finite trees

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- the convex hull of A in $f^{-1}(T)$, denoted $\lambda_f(T)$, is a tree whose vertex set contains A.
- If T, T' are two trees which are isotopic rel. A, then $\lambda_f(T)$ and $\lambda_f(T')$ are isotopic rel. A.

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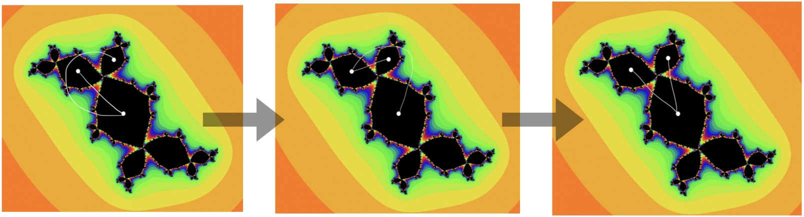

Figure: Pulling back a finite tree under the rabbit polynomial

Simplicial Complex of Finite Trees

Consider the set

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This construction gives a simplicial complex $\mathcal{T}_A$.

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Theorem [Belk-Lanier-Margalit-Winarski]

Pullback on trees induces a simplicial map $\lambda_f : \mathcal{T}_A \to \mathcal{T}_A$ for which the following holds:

- If f is realized holomorphically, then the 2-neighborhood of the Hubbard tree $[H_f]$ is a finite nucleus for λ_f.

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- If f is obstructed, then every λ_f-orbit eventually lands in $L_f \subset \mathcal{T}_A$. This collection L_f is the set of trees "dual" to the canonical obstruction of f.

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- If f has no asymptotic values, $f^{-1}(T)$ is an infinite tree.
- Otherwise, we can define an extension $\mathbb{R}^2_f \supset \mathbb{R}^2$ in which $f^{-1}(T)$ is an infinite tree.

In both cases, let $\lambda_f(T)$ be the convex hull of A in $f^{-1}(T)$.

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Let f_n be any combinatorial polynomial approximation of f.

Observation: $\lambda_{f_n}(T) \to \lambda_f(T)$ in $\mathbb{R}^2_f$.

Proposition

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If f_n is realized for n sufficiently large, then the Hubbard vertices $[H_{f_n}] \in \mathcal{T}_A$ eventually coincide.

Results

Recall that f is a transcendental planar Thurston map with $P_f = A$ and (f_n) is a combinatorial approximation of f.

Theorem [in Progress]

Tree "pullback" induces a simplicial map $\lambda_f : \mathcal{T}_A \to \mathcal{T}_{\mathcal{A}}$.

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- We know that if f is realized holomorphically, then the f_n are eventually realized holomorphically.
- We are interested in the converse statement.

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If $\langle \tau_n \rangle \to \langle \tau \rangle$ in $\mathcal{M}_{0,N}$ (where $N = |A| - 3$),
then $\tau_n \to \tau$ in $\mathrm{Teich}(\mathbb{S}^2, A)$.

Core Entropy

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Observation: Let g be an unobstructed Thurston map satisfying $g = \varphi \circ f \circ \psi$ for some $\varphi, \psi \in \mathrm{Homeo}^+(\mathbb{S}^2, A)$. Then

$$(\varphi \circ f_n \circ \psi) \text{ converges combinatorially to } g$$

Continuity of Core Entropy

Theorem [Gao-Tiozzo 2021]

Gao-Tiozzo 2021: Core entropy of degree d polynomials is a continuous function of combinatorial data.

As a consequence of our work in progress, we get

Corollary

In the deformation space of any PSF entire function f, core entropy along any combinatorially bounded slice is continuous.

Other Consequences

Combinatorial polynomial approximations allow us to study several natural objects in transcendental planar Thurston theory as limits. These include:

- Thurston's pullback operator
- Pullback on finite trees
- Pullback on closed curves and arcs
- "Hurwitz" spaces
- Correspondences descending from the pullback operator on an open subset of moduli space

Happy Birthday, Hamal!!!