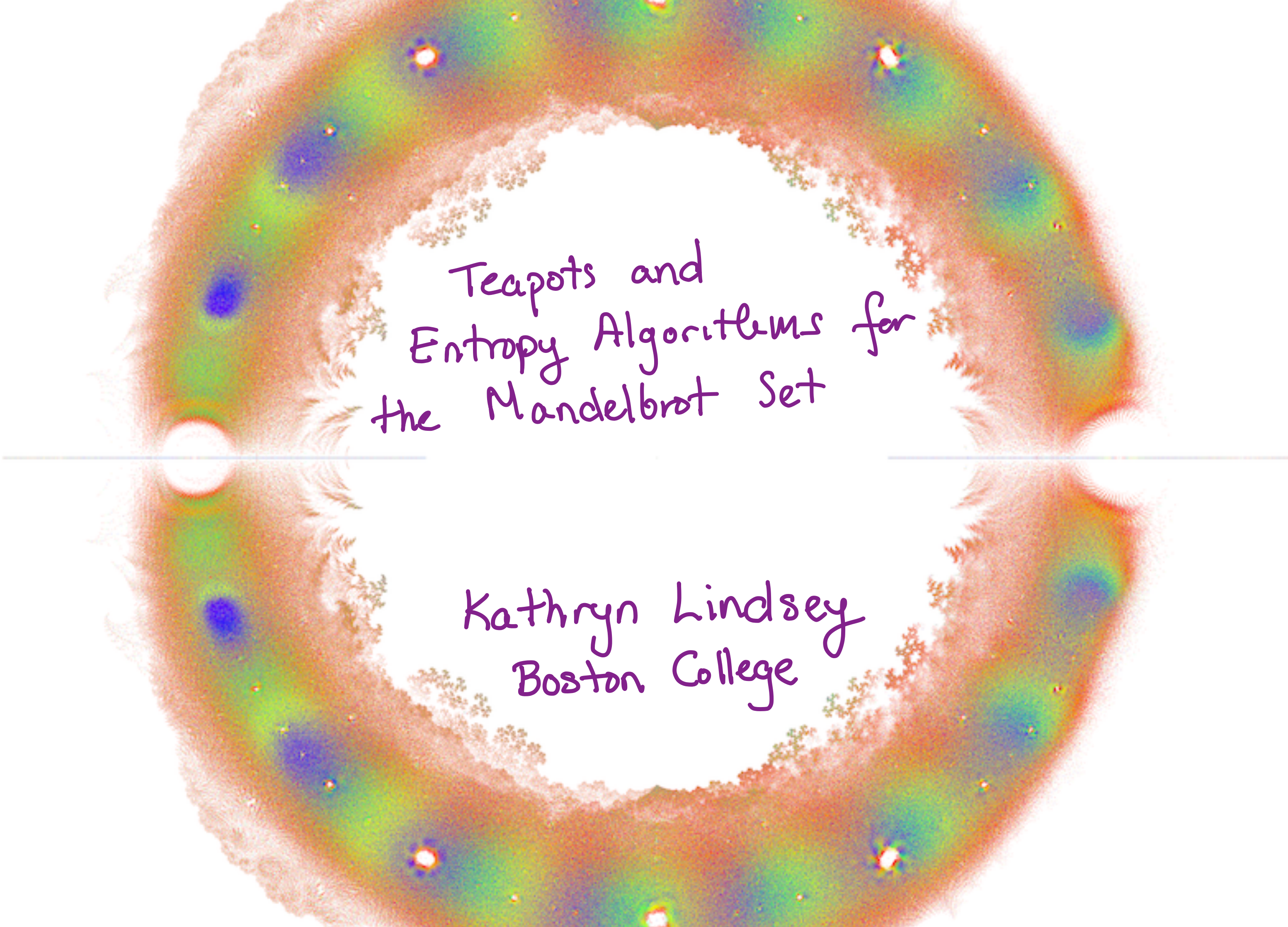

Teapots and Entropy Algorithms for the Mandelbrot Set

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Boston College

This talk is based on joint work with

Chenxi Wu

U. Wisconsin

Giulio Tiozzo

U. Toronto

All results are joint with one or both of them.
Builds on prior joint work with Harry Bray & Diana Davis.

Outline

1) Set up the question, define core entropy and roots of the Markov polynomial

2) Context: A: prior work core entropy $\longleftrightarrow$ geom. of $\mathcal{M}$.
B: uniformly expanding models

3) Statement of main results

4) A proof sketch

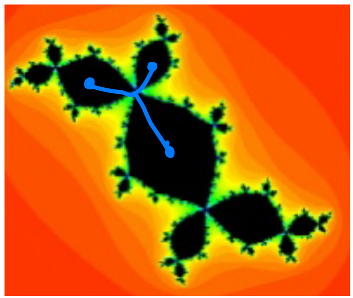
I. Introduce the question – define core entropy and roots of the Markov polynomial.

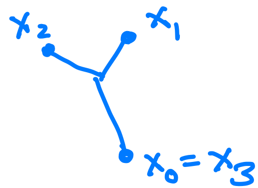
<u>Key object: Hubbard trees</u>

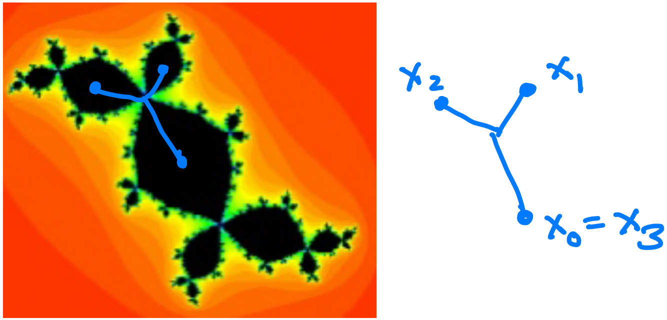
Let f be a $\boxed{\text{postcritically finite (PCF)}}$ quadratic polynomial.

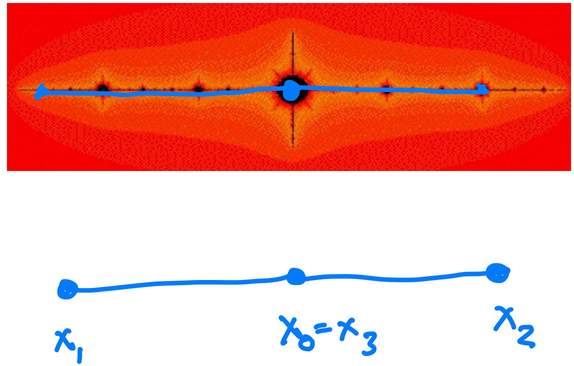
The $\boxed{\text{Hubbard tree } T_f}$ is

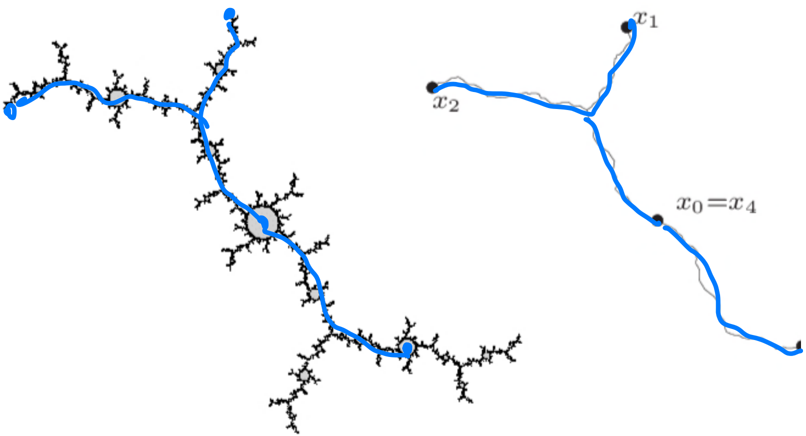
- a finite, connected tree embedded in $K(f)$,
- it contains the postcritical set,
- it is forward-invariant
- it is the minimal such set.






x_2
x_1
$x_0 = x_3$


x_1
$x_0 = x_3$
x_2


x_1
x_2
$x_0 = x_4$

Restrict dynamics to the Hubbard tree

- T_f is forward-invariant under f $\Longrightarrow$ consider the restriction $f|_{T_f}$.
- The restricted dynamical system $f|_{T_f} : T_f \to T_f$ determines f (up to conformal equivalence

$\Longrightarrow$ f acting on T_f is the "core" part of $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

Definition (Thurston):

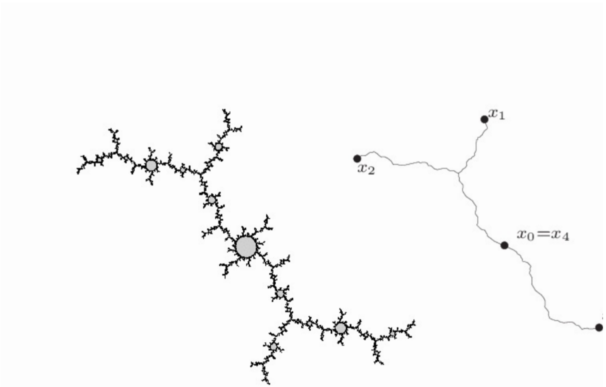
The $\boxed{\text{core entropy}}$ of f_c := topological entropy of $f_c|_{T_{f_c}}$

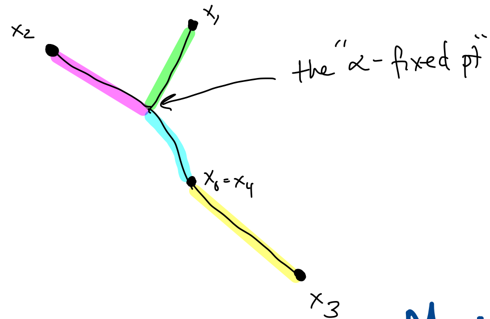
$\boxed{\text{growth rate}}$:= $e^{\text{core entropy}}$

Markov partitions & subshifts of finite type

Cut Hubbard tree at postcritical set $\Rightarrow$ Markov partition

$\Rightarrow$ topological conjugacy $f_c|_{T_{f_c}} \cong$ subshift of finite type.





Incidence matrix M_c:

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

leading eigenvalue ≈ 1.395.

Fact: Growth rate = the leading eigenvalue of the incidence matrix

Call the characteristic polynomial the **Markov polynomial**

$$\boxed{Mar(c) := \det(M_c - \lambda I)}$$

Review of concepts:

PCF parameter c

$\longrightarrow$ Hubbard tree T_{f_c}

$\longrightarrow$ Markov partition (by cutting at postcritical set)

$\longrightarrow$ incidence matrix M_{f_c}

$\longrightarrow$ set of roots of Markov poly (eigenvalues)

$$\boxed{Z(c)} := \{ \lambda \in \mathbb{C} : \det(M_{f_c} - \lambda I) = 0 \}.$$

(growth rate = largest root)

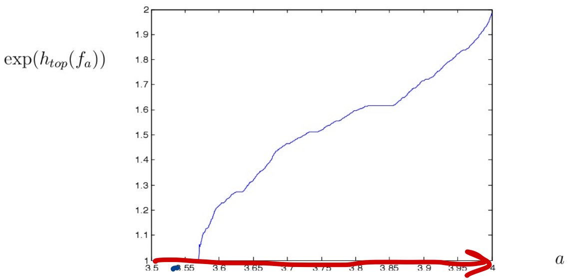
Now we're ready for

The Question:

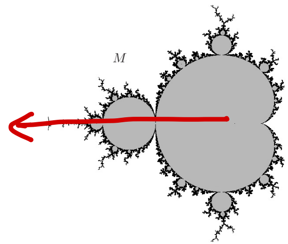
How does $c \mapsto Z(c)$ reflect the geometry of Mandelbrot set?

II. A. Putting the question in context: prior work relating (just) core entropy $\Leftrightarrow$ geometry of $\mathcal{M}$

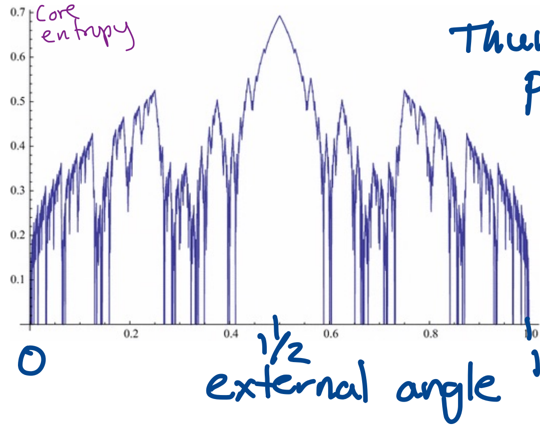
1) Milnor & Thurston proved core entropy is nondecreasing as you move out along the real vein of $\mathcal{M}$



Milnor-Thurston's plot



2) Each rational external angle θ determines a PCF parameter f_{c_θ}.
Thurston suggests
$\theta \mapsto \text{core entropy}(f_{c_\theta})$
is continuous



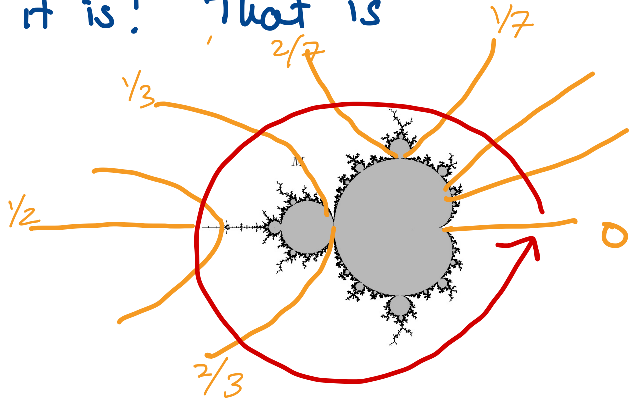
Thurston's plot.

3) Tiozzo and Schleicher prove it is! That is

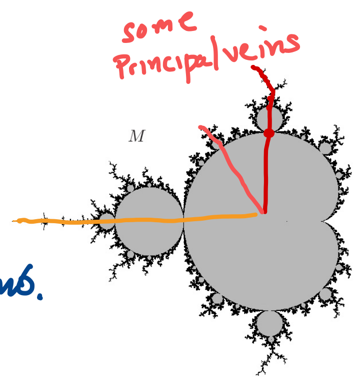
$$\mathbb{Q}/\mathbb{Z} \to \mathbb{R}$$

$$\theta \mapsto \text{core entropy}(f_{c_\theta})$$

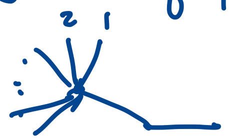
extends to a <u>continuous</u> function $\mathbb{R}/\mathbb{Z} \to \mathbb{R}$



4) Tiozzo shows core entropy is nondecreasing along <u>principal veins</u>



(The p/q-limb consists of params c where f_c has rotation # p/q about α-fixed pt.
The p/q principal vein connects to tip of p/q limb.
Hubbard tree is a q-pronged star



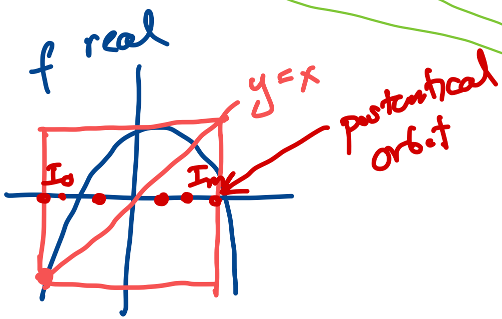
)

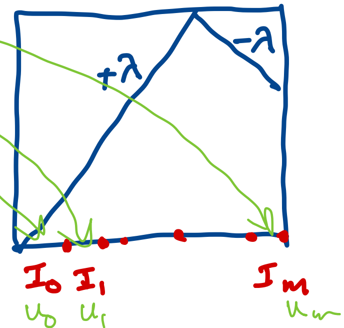
II. B. Putting the question in context: uniformly expanding models

Def: f uniformly expanding means f is piecewise linear with slope $f'(x) = \pm\lambda$.

Fact: Roots of Markov poly (eigenvalues of M_f) are stretch factors of uniformly expanding models of f. Eigenvectors give lengths of intervals.

$$[u_0, \ldots, u_m]\begin{bmatrix} M_f \end{bmatrix} = \lambda \begin{bmatrix} M_f \end{bmatrix}$$



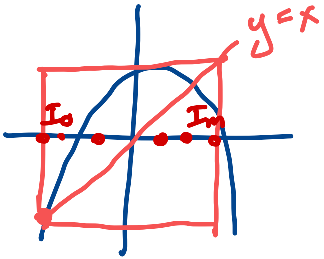


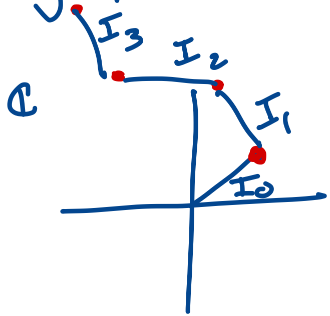
uniform expander model.

Length of $I_k = u_k$

lengths

If $\lambda \in \mathbb{C}$, $\Rightarrow$ complex embedding of interval





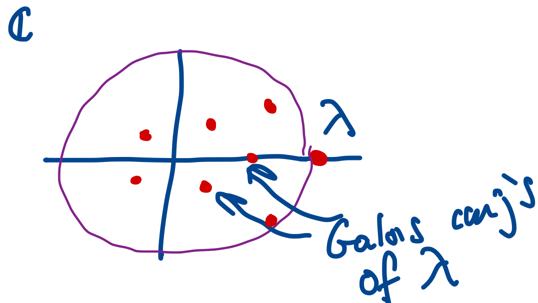
a slope $\pm\lambda \in \mathbb{C}$ map sends

$$I_k \longmapsto I_{k+1}$$

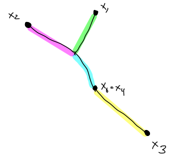
Open question: complete characterization of set of growth rates of PCF quadratics.

Easy necessary condition: If λ is growth rate of PCF real interval map, then λ is a $\boxed{\text{weak Perron number.}}$

λ Weak Perron: λ is a real, positive algebraic integer whose norm is $\geq$ norm of its Galois conjugates



Why necessary? Perron-Frobenius theorem says these are the types of numbers that arise as leading eigenvalues of matrices of 0's and 1's.



$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Theorem (Thurston, Entropy in Dim 1):
$\{$weak Perron numbers$\}$ = $\{$growth rates of real PCF polys$\}$

no restriction on degree.

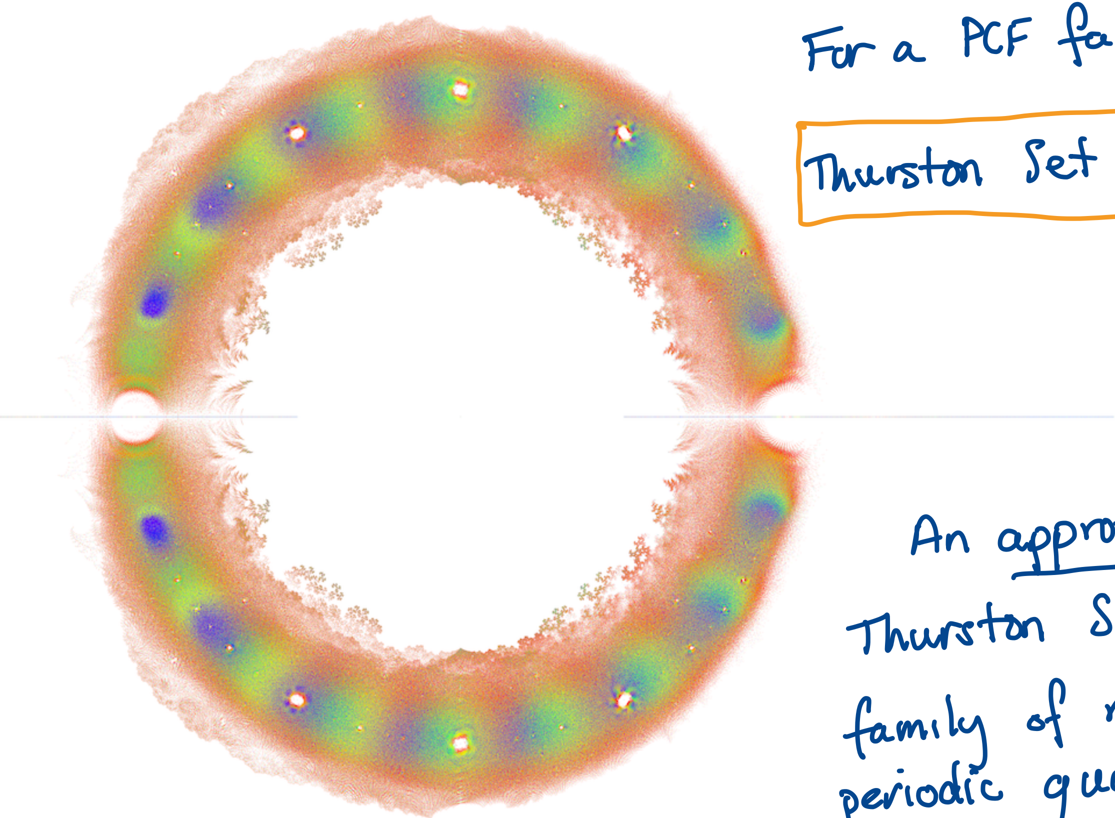
Still open: Which weak Perron #s arise as growth rates of polys of degree 2? (or d?) (or some dynamical behavior?)

Def:

For a PCF family $\mathcal{Y} = \{f_c\}_c$

$$\boxed{\text{Thurston Set}(\mathcal{Y})} := \overline{\bigcup_c Z(c)}$$

roots of Markov polys

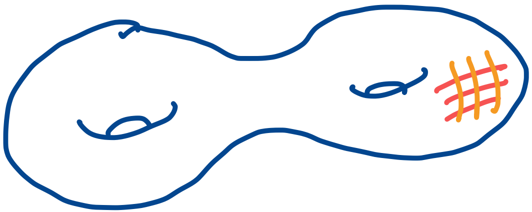


An approximation of the Thurston Set for the family of real, strictly critically periodic quadratic polys.

(color ~ critical period 2 to 23)

(An aside: this is a "1-D version" of a famous open question in 2-D dynamics.

pseudo-Anosov surface diffeomorphisms



essentially, a surface S with a map $f: S \to S$ for which you can choose coordinates so that

f expands the "unstable" direction by factor of λ

f contracts the "stable" direction by factor of $\frac{1}{\lambda}$

λ is called the <u>growth rate</u> or <u>dilatation</u>

What are the growth rates of pseudo-Anosovs ?)

... back to our 1-D case ...

III Teapots: a tool for investigating <u>all</u> roots of Markov poly.

Def: The p/q-teapot is

$$\Upsilon_{p/q} := \{ (z, \lambda) \in \mathbb{C} \times \mathbb{R} : \exists f_c \in p/q\text{-principal vein s.t. } \lambda = \text{growth rate}(f_c),\ z \in \underbrace{Z(f_c)}_{\text{roots of the Markov poly}} \}$$

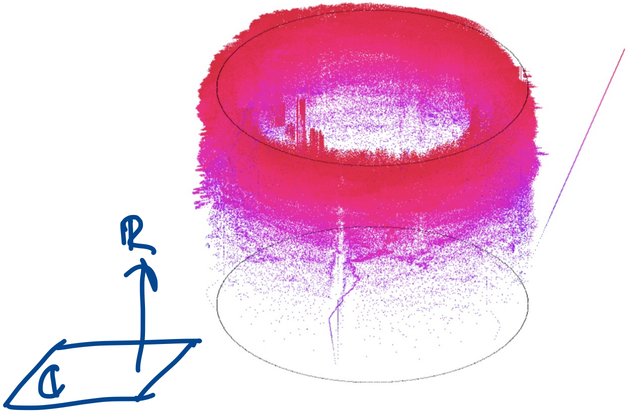


FIGURE 2. The Master Teapot $\Upsilon_{1/3}$ for the $1/3$-vein. We plotted the roots associated to all critically periodic parameters with simplified itinerary of period up to 20, obtaining $\sim 2.8 \times 10^6$ points.

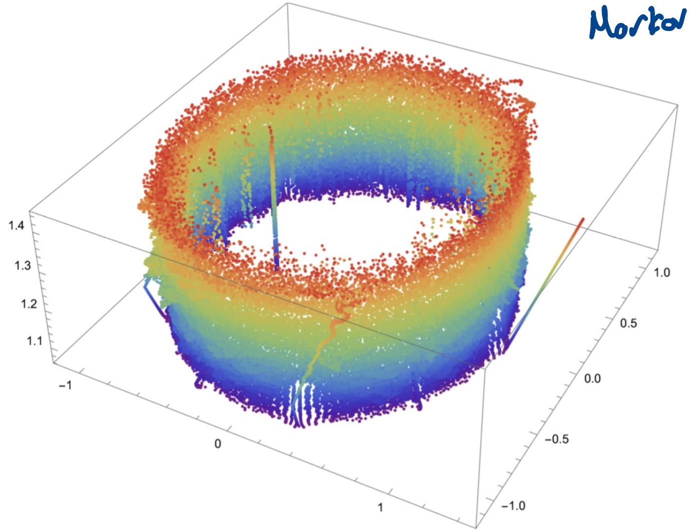


FIGURE 1. The Master Teapot $\Upsilon_{1/5}$ for the $1/5$-vein.

1

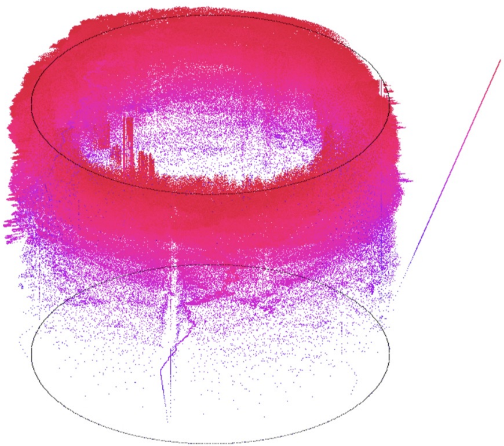
(Thurston defined the teapot for real, critically periodic quadratics; we defined it for veins)

Main Theorem 1: "Persistence Theorem" for principal veins

Horizontal slices of $\Upsilon_{p/q}$ grow monotonically inside the unit cylinder.

i.e. if $(z,\lambda) \in \Upsilon_{p/q}$ with $|z| \le 1$, then $\{z\} \times [\lambda, \lambda_q] \subset \Upsilon_{p/q}$

growth rate of tip (leading root of $x^q - x^{q-1} - 2 = 0$)



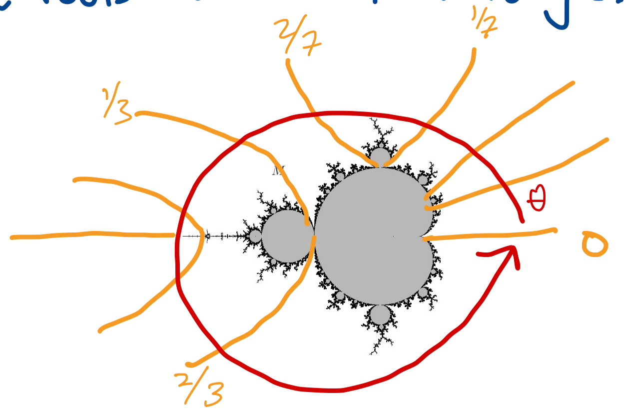
$\Upsilon_{1/3}$

FIGURE 2. The Master Teapot $\Upsilon_{1/3}$ for the 1/3-vein. We plotted the roots associated to all critically periodic parameters with simplified itinerary of period up to 20, obtaining $\sim 2.8 \times 10^6$ points.

Theorem: The unit cylinder is in $\Upsilon_{p/q}$, i.e. $S^1 \times [0, \lambda_q] \subset \Upsilon_{p/q}$.

Theorem: For real, critically periodic family, an algorithm for certifying that a point is not in teapot.

Main Theorem 2: Continuity of large roots with external angle.



Theorem setup:

For rational θ, define

$$Z^+(\theta) := S^1 \cup \{z \in Z(f_{c_\theta}) : |z| \geq 1\}.$$

Z^+ is a map $\mathbb{Q}/\mathbb{Z} \to \mathrm{Com}^+(\mathbb{C})$

compact subsets of $\mathbb{C}$, equipped with Hausdorff metric

$$d_{Haus}(A,B) = \inf\{\varepsilon > 0 : A \subset N_\varepsilon(B),\ B \subset N_\varepsilon(A)\}$$

Theorem statement:

The map $Z^+ : \mathbb{Q}/\mathbb{Z} \to \mathrm{Com}^+(\mathbb{C})$ admits a continuous extension from $\mathbb{R}/\mathbb{Z} \to \mathrm{Com}^+(\mathbb{C})$.

Main theorem 3 :

For any PCF parameter, the following 2 polynomials have the same roots off the unit circle:

- the polynomial from Thurston's entropy algorithm
- the Markov polynomial

If, furthermore, the parameter is critically strictly periodic and belongs to the principal vein, a 3rd polynomial with the same roots of the unit circle is

- the principal vein kneading polynomial (we defined).

(Why need this? Markov poly is simplest, but not stable under perturbation of external angle. Thurston invented his entropy algorithm to deal with this. Our principal vein kneading poly is "stable" along principal veins.)

IV Proof Sketch: $z^+ : \theta \longmapsto \left\{ \begin{array}{l} \text{eigenvalues } |\cdot|>1 \\ \text{of } M_\theta \end{array} \right\} \cup S^1$ is continuous.

$\|$

$\left\{ \begin{array}{l} \text{roots of Thurston poly} \\ \text{of norm} \geq 1 \end{array} \right\}$

Form a directed graph

(n,n)

$\vdots$

$(2,2)$ $(2,3)$ $\dots$

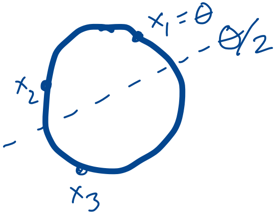
$(1,1)$ $(1,2)$ $(1,3)$ $\dots$ $(1,n)$

Graph whose vertices are (x_i, x_j) pairs.

If (x_i,x_j) is nonseparated add edge $(x_i,x_j) \to (x_{i+1}, x_{j+1})$

If (x_i,x_j) is seprated, add $\to (1, x_i+1)$ and $\to (1, x_{j+})$

Thurston entropy algorithm:



$x_i = 2x_{i-1} \mod 1$

Say (x_i, x_j) is separated if in interiors of different parts of partition.

" equal if $x_i = x_j$

" nonsep. otherwise

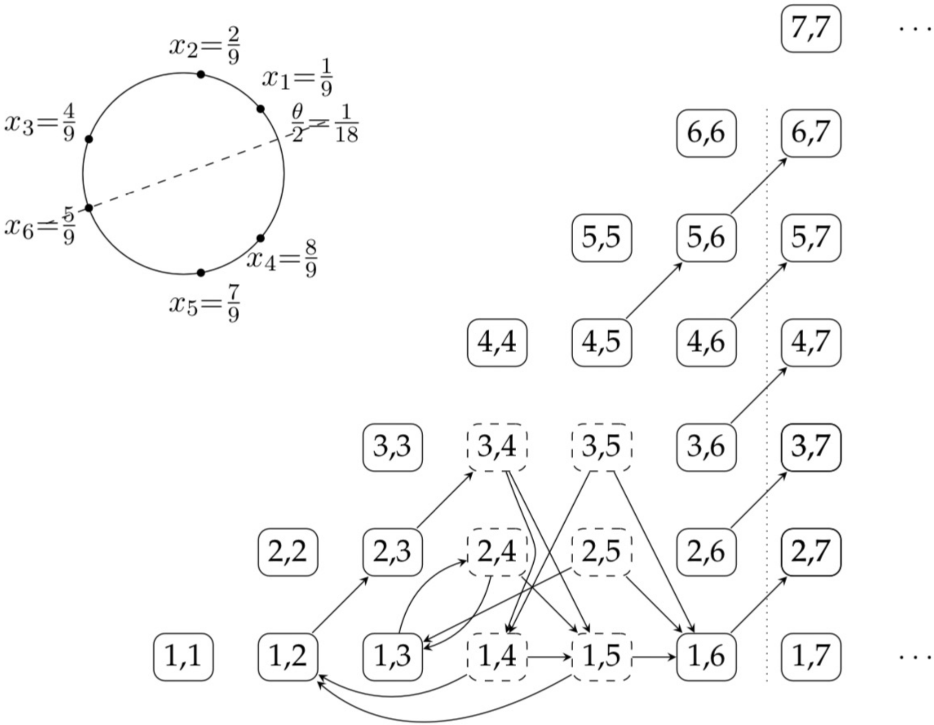


FIGURE 5. The infinite graph $\Gamma_{1/9}$. Angle $1/9$ is strictly periodic with period 6; the vertical dotted line indicates the edge of a "fundamental domain" for $\equiv_{6,0}$. Vertices that are separated are indicated with a dashed boundary; non-separated and equivalent vertices have a solid boundary. The angle diagram in the upper left is helpful for determining which vertices of Γ are separated.

Represent this directed graph as an incidence matrix.

Fact: growth rate of f_{c_0} = inverse of leading eigenvalue of this matrix.

(P_{Th} is the char. poly.)

Tiozzo introduced "infinite wedges" T' (or finite) and used spectral determinant of infinite graphs - a power series (whose coeffs depend on the # of cycles of various lengths in the graphs - and for the graphs we are interested in, a finite portion of the graph determines each coeff.)

Generalizes the characteristic polynomial.

In periodic (or preperiodic case), look at larger and larger covers of the finite model graph.

Th: (Roughly stated) The characteristic polys of a k-fold cover of the finite graph model = (a cyclotomic) $\times$ (char. poly of finite model).

These charc. polys of large covers $\rightarrow$ spectral det.

Upshot: For PCF maps, inside the open unit disk spectral det and Thurston poly have same roots.

Then argue that the coeffs of spectral det. vary continuously with θ. ($\rightarrow$ argue roots move continuously).

$\boxtimes$

Thank you all!

Thank you, Prof. Hubbard, for your support and mentorship over the years.

Extra

Plot/Approach 1: Galois conjugates of growth rates

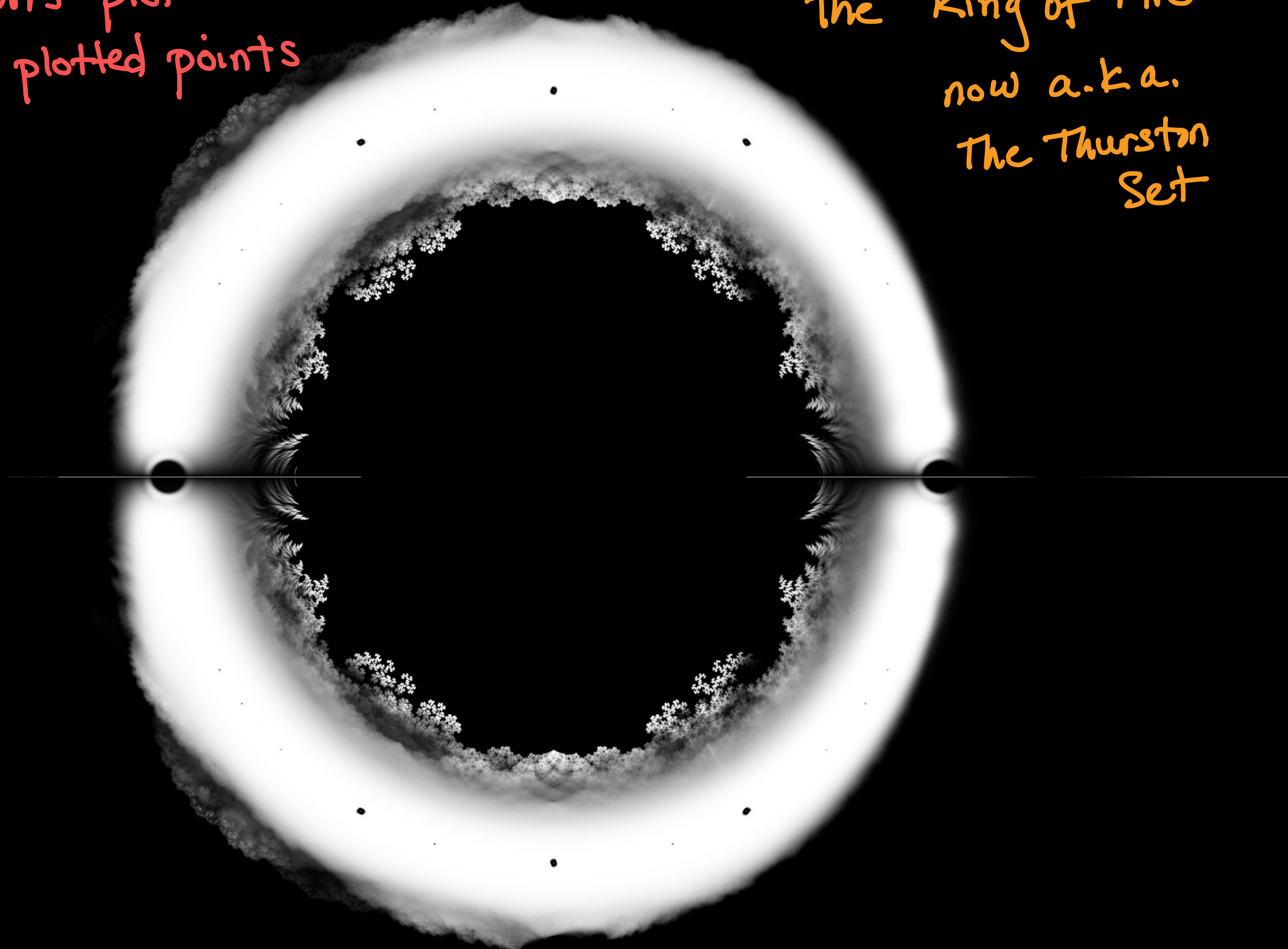
In the year before is death in 2012, W. Thurston did an experiment:

1. Get a large list of real PCF quadratic polynomials
2. For each, calculate its growth rate λ and also all the Galois conjugates of λ.
3. Plot all these numbers in $\mathbb{C}$.

The result of his experment was...

Thurston's plot
white = plotted points

The "Ring of Fire"
now a.k.a.
The Thurston
Set



Plot/approach 2: connectedness locus for an IFS family.

For each $\lambda \in \mathbb{D}$, define $L_\lambda :=$ limit set of IFS generated by $x \mapsto \lambda x + 1$ and $x \mapsto \lambda x - 1$.

Define $\mathcal{Z} = \{ \lambda \in \mathbb{D} : 0 \in L_\lambda \}$.

(Show Mathematica)

Plot/approach 3: Roots of power series with ± 1 coefficients

Consider the set of all power series of the form

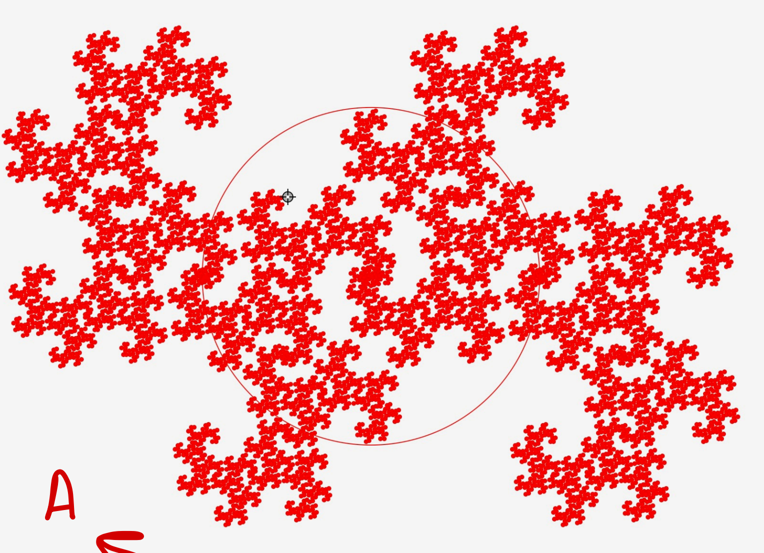
$$\sum_{i=0}^{\infty} a_i x^i, \qquad a_i \in \{+1, -1\} \quad \forall i.$$

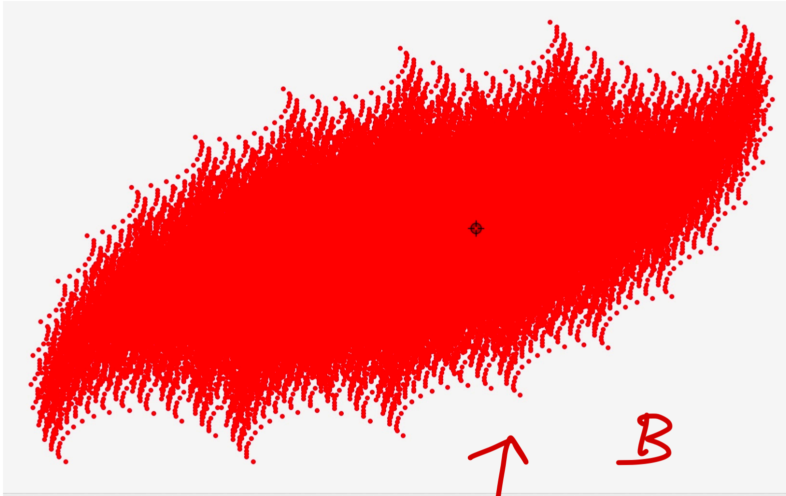
A useful connection: For $\lambda \in \mathbb{C}$ with $|\lambda| < 1$,

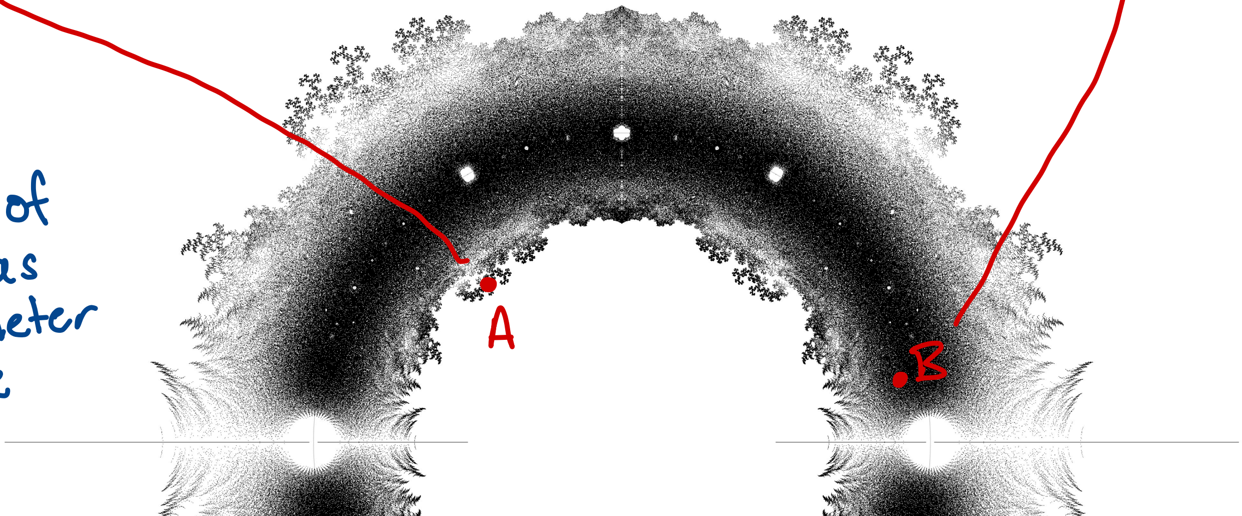
$\lambda \in$ Thurston set $\iff$ $0 \in$ limit set of the IFS generated by

$$f_1(x) = \lambda x + 1$$

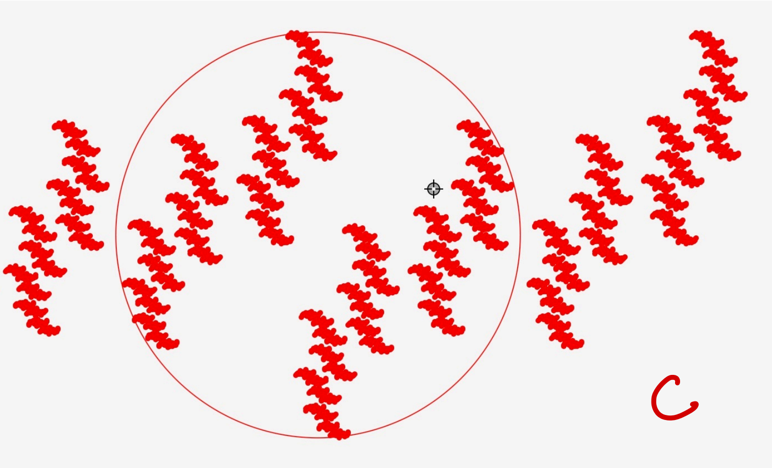
$$f_2(x) = \lambda x - 1.$$







Think of this as parameter space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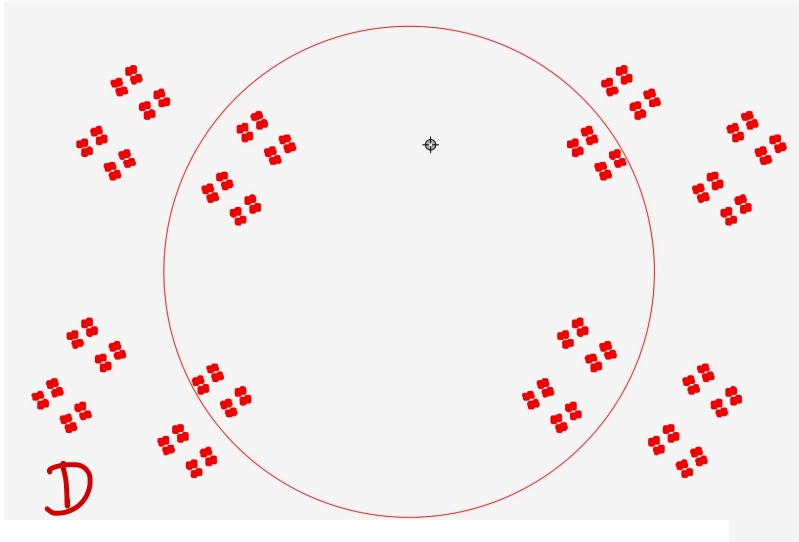
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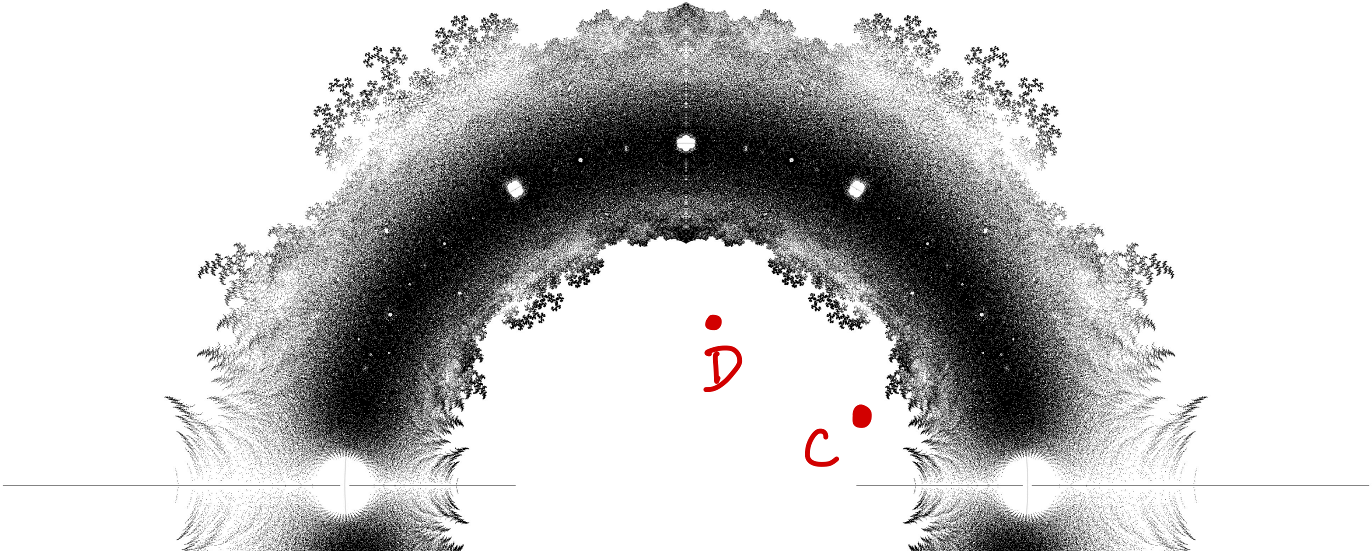
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Some recent results about Thurston set:

- It is contained in an annulus (Solomyak)
- It contains (an open neighborhood of) the unit circle. (Tiozzo)
- The part in $\mathbb{D}$ coincides with the set of all roots of all power series with ± 1 coefficients. (Calegari-Koch-Walker)
- The "holes" you see are "fake", artifacts of finite approximation. (Bray, Davis, L., Wu)
- It does have holes in the fractaly bits (Calegari-Koch-Walker)
- It is locally connected (Tiozzo)