Rescaling Limits of Rational Maps.

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Rational maps of degree $d \geq 1$

$$f : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$$

$$z \longmapsto \frac{a_d z^d + \cdots + a_0}{b_d z^d + \quad + b_0} = \frac{P(z)}{Q(z)} \; ; \quad P \neq 0 \wedge Q \neq 0$$

is a degree d rational map if

$$\max\{\deg P, \deg Q\} = d$$

P, Q are relatively prime.

otherwise, f is a degenerate rational map. of degree d

Space of Rational Maps

$\overline{\mathrm{Rat}_d}$:= rational maps, maybe degenerate

$$\overline{\mathrm{Rat}_d} \xrightarrow{\sim} \mathbb{P}_{\mathbb{C}}^{2d+1}$$

$$\frac{a_d z^d + \cdots + a_0}{b_d z^d + \quad + b_0} \longmapsto [a_0 : \text{—} : a_d : b_0 : \text{—} : b_d]$$

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Ψ

$f_0(z) = c\,\dfrac{(z-z_0)\cdots(z-z_k)}{(z-z_0)\cdots(z-z_k)} \cdot \tilde{f}_0(z)$ $\quad 0 \leq \deg \tilde{f} < d$, $\tilde{f}_0$ is the reduction of f_0.

Degenerating sequence/family

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$\{f_t\}_{t\in\mathbb{D}_r}$ is a degenerate family if

$$\mathbb{D}_r \ni t \longrightarrow f_t \in \overline{\mathrm{Rat}_d} \cong \mathbb{P}^{2d+1}_{\mathbb{C}} \text{ is holomorphic.}$$

$$f_t \in \partial \mathrm{Rat}_d \quad \text{iff} \quad t=0$$

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$f_n \to \tilde{f}_0$ (resp. $f_t \to \tilde{f}_0$) uniformly in compact subsets of $\hat{\mathbb{C}} \setminus \{z_0 - z_k\}$

Moduli Space / Degenerations

$\mathrm{rat}_d := \mathrm{Rat}_d / \text{conjugacy by } PSL_2(\mathbb{C})$. complex orbifold
(Silverman)

We are interested in $\{f_n\} \subseteq \mathrm{Rat}_d$ s.t $[f_n] \to \infty$

ie. there is no scaling $\{M_n\}$ s.t

$$M_n^{-1} \circ f_n \circ M_n \to f \in \mathrm{Rat}_d.$$

Rescaling Limits.

Let $\{f_n\}$ (resp. $\{f_t\}$) be degenerating:

$\tilde{g} \in \mathrm{Rat}_{\tilde{d}}$, $\tilde{d} \geq 2$ is called a rescaling limit of period $q \geq 1$ if there exists $\{M_n\} \subseteq \mathrm{Rat}_1$ (resp $\{M_t\}$) such that

$$M_n^{-1} \circ f_n^{q} \circ M_n \longrightarrow \tilde{g}$$

$$(\text{resp. } M_t^{-1} \circ f_t^{q} \circ M_t \longrightarrow \tilde{g})$$

uniformly off a finite set.

Examples

$$f_t(z) = z^2(z + t^{-1}).$$

$$M_t(z) = tz \quad \Rightarrow \quad M_t^{-1} \circ f_t \circ M_t(z) \to z^2.$$

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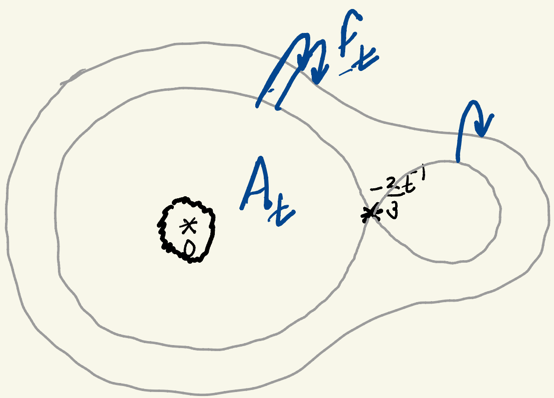


$\omega = 0$, $\omega' = -\frac{2}{3}t^{-1}$ escapes.

$\text{Modulus}(A_t) \to \infty$ as $t \to 0$

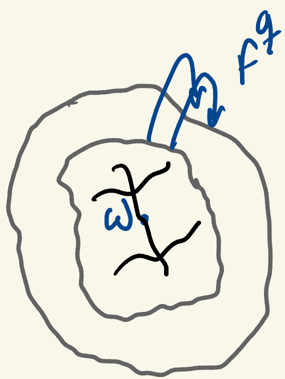
Examples (Branner-Hubbard).

f cubic

Critical Points ω, ω'

$f^n(\omega') \to \infty$

$\omega \in$ Periodic component of $K(f)$ of period q.



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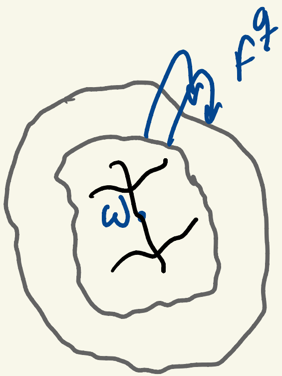


BH-wringing produces $\{g_t\}_{t\in\mathbb{D}^*}$, $\{M_t\}_{t\in\mathbb{D}^*}$:

$f = g_{t_0}$ for some t_0

$M_t^{-1} \circ g_t^q \circ M_t \to Q_{c_0}(z) = z^2 + c_0$

Quadratic Rational Maps

Let $\{f_n\} \subseteq \mathrm{Rat}_2$.

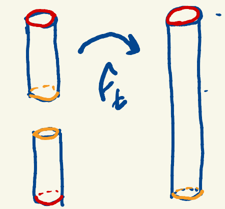
(Epstein 2000) if, for some $p \geq 2$, $\{f_n\}$ has a period p cycle with bndd multiplier, then (modulo passing to a subsequence) $\{f_n\}$ has a period $2 \leq q < p$ Rescaling Limit $\tilde{g}$: with a parabolic fixed point.

(cf Rees-Stimson 1993)

Mc Mullen Maps

$$f_t(z) := z^3 + \frac{t}{z^3}$$

$|t| \ll 1$



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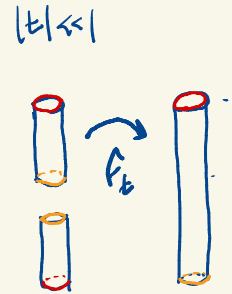
Period 1 rescaling: $M_t = id$, $M_t^{-1} \circ f_t \circ M_t(z) \to z^3$.



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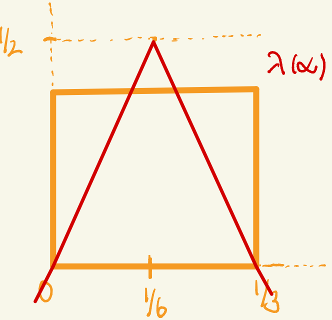

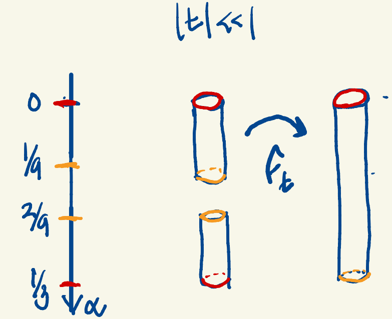

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$$f_t(t^\alpha z) \sim \begin{cases} t^{\lambda(\alpha)} z^3, & \text{if } \alpha < 1/6 \\ t^{\lambda(\alpha)} z^{-3}, & \text{if } \alpha > 1/6 \end{cases}$$

Period 1 : $f_t(t^{1/4} z) \sim t^{1/4} z^{-3} \Rightarrow M_t(z) = t^{1/4} z$, $M_t^{-1} \circ f_t \circ M_t \to z^{-3}$

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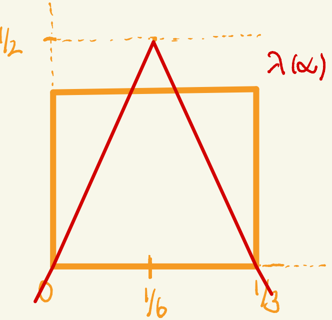

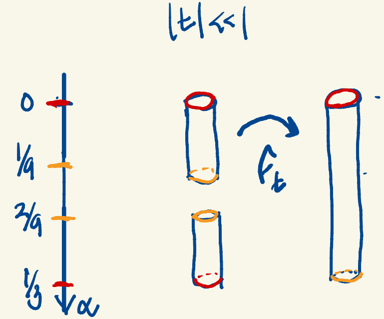

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Period q: if $\lambda^q(\alpha_0) = \alpha_0$, then $f_t^q(t^{\alpha_0} z) = t^{\alpha_0} z^{\pm 3^q}$

Scalings

$\{M_n\}, \{L_n\}$ scalings ⚠ Passing to subsequences.

$$M_n^{-1} \circ L_n \to M \in PSL_2(\mathbb{C}) \quad \text{(equivalent / dependent)}$$

or $$M_n^{-1} \circ L_n \to \tilde{a} \in \overline{\mathbb{C}} \quad \text{(independent)}$$

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Fact. $PSL_2(\mathbb{C}) \backslash \{f \in \overline{\mathrm{Rat}_d} : \deg \tilde{f} \geq 1\} = \{PSL_2(\mathbb{C}) \cdot f : \deg \tilde{f} \geq 1\}$

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is compact.

$\Rightarrow$ Given $\{f_n\}$ $\exists$ $\{M_n\}$ s.t. $M_n \circ f_n \to \tilde{f}$ with $\deg \tilde{f} \geq 1$

Scaling Dynamics.

Take $\{M_n\}$ scaling, $\exists \{H_n\}$ s.t

$$H_n^{-1} \circ f_n \circ M_n \to \tilde{g} \text{ with } \deg \tilde{g} \geq 1.$$

$$\{M_n\} \xrightarrow[\tilde{g}]{} \{N_n\}$$

(From Rivera-Letelier 2000)

Scaling Dynamics.

Take $\{M_n\}$ scaling, $\exists \{N_n\}$ s.t

$$N_n^{-1} \circ f_n \circ M_n \to \tilde{g} \quad \text{with} \quad \deg \tilde{g} \geq 1.$$

$\exists \{L_n^{(1)}\}, \dots, \{L_n^{(k)}\}$ s.t

$$M_n^{-1} \circ f_n \circ L_n^{(i)} \to \tilde{g}_i \quad \text{with} \quad \deg \tilde{g}_i \geq 1$$

$$\sum_{i=1}^{k} \deg \tilde{g}_i = d$$

$$\begin{tikzcd}
\{L_n^{(1)}\} \arrow[dr, "\tilde{g}_1"] & & \\
\vdots & \{M_n\} \arrow[r, "\tilde{g}"'] & \{N_n\} \\
\{L_n^{(k)}\} \arrow[ur, "\tilde{g}_k"'] & &
\end{tikzcd}$$

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Limiting Dynamics

Trees of spheres

Projective line (Berkovich)
over Non-Archimedean Fields:

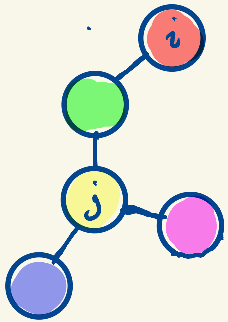
Puiseux Series

Complex Robinson Field * (Luo 2021
Favre-Gong 2025)

Trees of Spheres.

$$\mathcal{C} := \{\{M_n^{(1)}\}, \ldots, \{M_n^{(k)}\}\}$$

$$\mathcal{T}_{\mathcal{C}} := \bigsqcup_{i=1}^{k} \mathbb{C}_i /\sim$$



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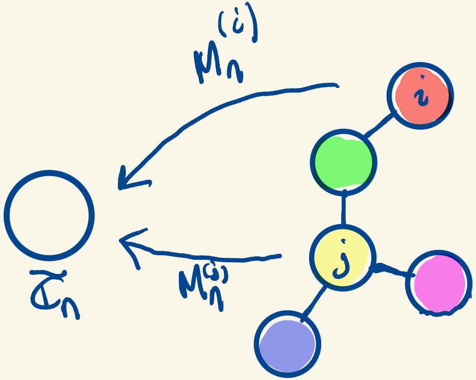


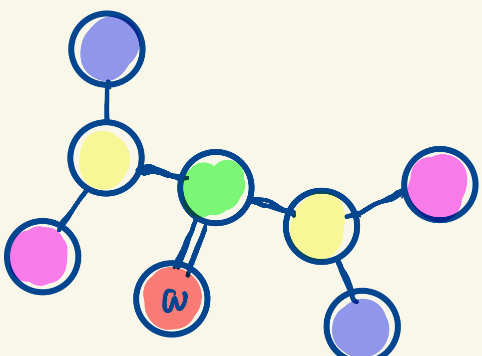
$z_n \in \overline{\mathbb{C}_n}$ converges $w \in \overline{\mathbb{C}_j}$.

$$M_n^{(j)^{-1}}(z_n) \xrightarrow{\text{iff}} w$$

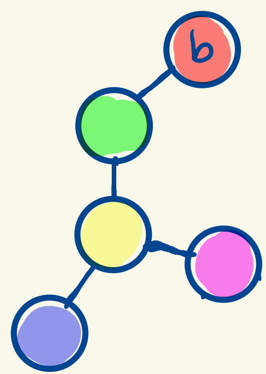
Maps between trees of spheres.

$\ell' = \{\{L_n^{(1)}\}, \ldots, \{L_n^{(\ell)}\}\}$ the "preimage" of ℓ.

$\Upsilon_{\ell'}$ = associated tree of spheres



$\Upsilon_{\ell'}$



Υ_{ℓ}

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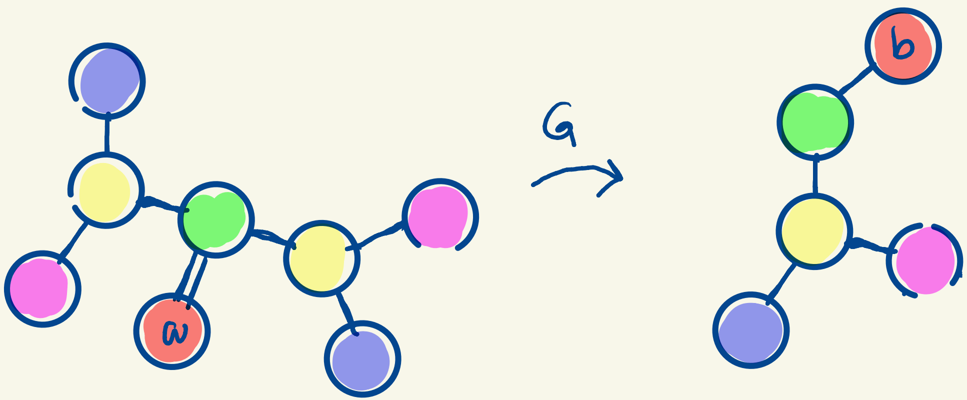
$$\tilde{g}_{a \to b} = \lim M_n^{(b)^{-1}} \circ f_n \circ L_n^{(a)}$$

$$f_n : \overline{\mathbb{C}}_n \to \overline{\mathbb{C}}_n$$

$$\big\downarrow n \to \infty$$

$$G : \Upsilon_{\mathcal{E}'} \to \Upsilon_{\mathcal{E}}.$$



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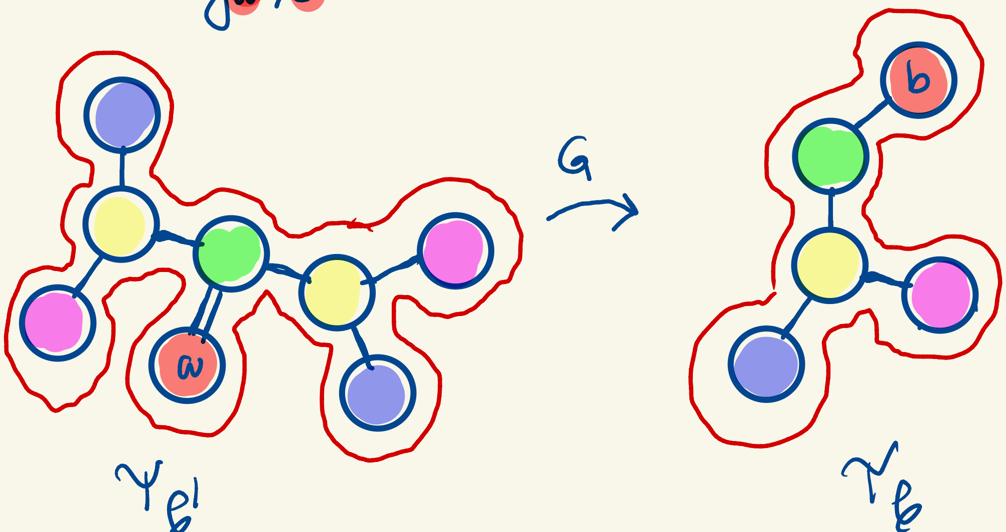

G is a cover of trees of spheres (Arfeux 2017)

Counting Rescaling Limits.

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Proof. (à la Arfeux 2017) Topological properties of branched coverings of trees of spheres

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(Shishikura Tree) ④ Construct f_0 with N non-Jordan periodic components.

Puiseux series dynamics.

$$f_t(z) = \frac{a_d(t) z^d + \cdots + a_0(t)}{b_d(t) z^d + \cdots + b_0(t)} \in \mathbb{C}((t)).(z).$$

$$\xi = \sum_{\lambda \in \Lambda} \alpha_\lambda t^\lambda, \quad \Lambda \text{ bndd below} \quad , \quad |\xi|_0 = e^{-\mathrm{ord}_0 \xi}.$$

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$\mathbb{C}((t))$ $\leq$ $\mathbb{C}\langle\langle t \rangle\rangle$

Formal Laurent Series — Formal Puiseux Series

$\Lambda \subseteq \mathbb{Z}$ — $\Lambda \subseteq \{ j/m : j \in \mathbb{Z} \}$ some m

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Formal Laurent Series — Formal Puiseux Series — algebraically closed, complete

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$f = \{f_t\} : \mathbb{P}^1_{\mathbb{L}} \to \mathbb{P}^1_{\mathbb{L}}$ is a rational map of degree d.

Scaling $M = \{M_t\} \in PSL_2(\mathbb{L})$

Berkovich Line

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with Gel'fand topology: $|\cdot|_x \to |p|_x$ is continuous.

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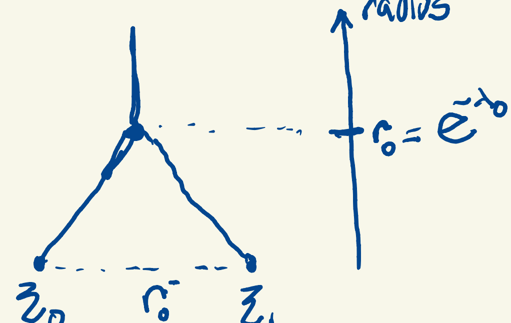
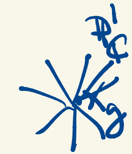
Balls are nested or disjoint:

$z_0 - z_1 = O(t^{\lambda_0})$.



Berkovich Space and Rescalings.

x_g Gauss point $\leftrightarrow$ Unit ball: it has $\mathbb{P}^1_{\mathbb{C}}$-branches.



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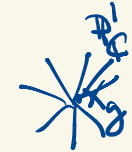

Branch points of $\mathbb{P}^{1,an}_{\mathbb{L}}$ (type II pts) $\leftrightarrow$ $PSL_2(\mathbb{L})$ orbit of x_g

$\leftrightarrow x = [M_t]$ where $L_t \sim M_t$ if $L_t^{-1} \circ M_t \big|_{t=0} \in PSL_2(\mathbb{C})$

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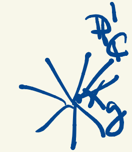

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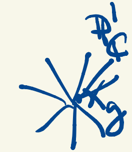

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$\tilde{g}$ is the local action on the branches at x: $T_x f$.

Periodic Points.

Rk. $\{f_t\}$, $z(t) \circlearrowleft f_t$ fixed-point multiplier $\mu(t)$:
family.

$$|\mu(t)|_0 > 1 \iff \mu(t) \to \infty$$

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~ 2000

(Rivera-Letelier) **Def.** x type II fixed point is repelling if $\deg T_x f \geq 2$
2003

i.e. $x = [M_t] \circlearrowleft \{f_t\}$ is repelling iff $M_t^{-1} \circ f_t \circ M_t \big|_{t=0} = \tilde{g}$

s.t. $\deg \tilde{g} \geq 2$

Fatou-Julia.

[Hsia] 2000

Given $f \in \mathbb{k}(z)$, $x \in \mathcal{F}(f) \subseteq \mathbb{P}^{1,an}_{\mathbb{k}}$ if $\exists\, U \ni x$ s.t. $|\mathbb{P}^{1,an}_{\mathbb{k}} \setminus \bigcup f^n(U)| \geq 3$

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(K.) 2015 x repelling periodic type II (rescaling)

$T_x f^q : \mathbb{P}^1_{\mathbb{C}} \circlearrowleft$ not PCF (rescaling limit not PCF)

$\Rightarrow$ $\exists$ critical pt $\omega \in \mathbb{P}^1_{\mathbb{L}}$ s.t $f^n(\omega) \xrightarrow{n \to \infty}$ orbit of x

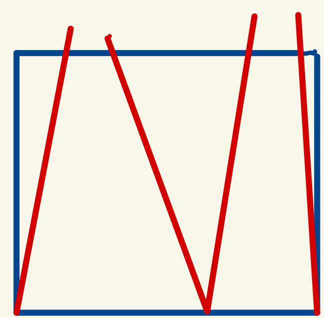
Repelling Periodic Points

Th (Luo 2021). Suppose multipliers of period p pts of $f_t : \mathbb{P}^1_{\mathbb{C}} \circlearrowleft$ are bounded as $t \to 0$, for all $p \geq 1$.

$\Rightarrow$ f_t has at most 2 non-monomial Resc. Lim. (polynomial).

Modulo change of coordinates, $f = \{f_t\} : \mathbb{P}^{1,an}_{\mathbb{L}} \circlearrowleft$ is "Bernoulli":
(c.f Favre-Riv.Let)

$J(f) \subseteq [x_0, x_1]$

$f \circlearrowleft$ 

slopes in $\mathbb{Z}$

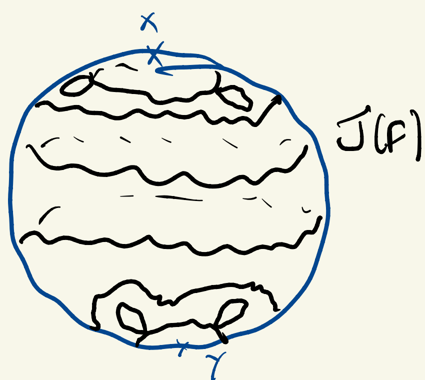
Th (Luo). $\{[f_{t_n}]\}$ contained in a hyperbolic component $\mathcal{H} \subseteq \mathrm{rat}_d$.

$[f_{t_n}] \to +\infty$ s.t period p multipliers are bounded.

$\Rightarrow$ $J(f_{t_n})$ is nested.

Def $f : \mathbb{P}^1_{\mathbb{C}} \circlearrowleft$ has nested Julia set if:

$\exists\, x, y \in F(f)$ s.t all components J_0 of $J(f)$ separate x, y.



Quadratic | Bicritical and Newton.

(K. 2014) At most two rescaling limits if $\{f_t\} \subseteq \overline{\mathrm{Rat}_2}$

(Nie-Pilgrim 2022) At most two rescaling limits if $\{f_t\} \subseteq \overline{\mathrm{Bicrit}_d}$

(Nie 2023) At most $d-1$ rescaling limits if $\{H_t\} \subseteq \overline{\mathrm{Newton}_d}$

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(Nie-Pilgrim 2022) At most two rescaling limits if $\{f_t\} \subseteq \overline{\mathrm{Bicrit}_\delta}$

(Nie 2023) At most $\delta-1$ rescaling limits if $\{N_t\} \subseteq \overline{\mathrm{Newton}_\delta}$

(Epstein 2000, Nie-Pilgrim 2022) $\mathcal{H}$ bicritical hyperbolic component of type D and periods ≥ 2.
$\Rightarrow$ $\mathcal{H}$ is bounded.

(Nie-Pilgrim 2020) for Newton maps of degree 4.


Facultad de Matemáticas
Edificio Rolando Chuaqui

Ultrafilters:

No more subsequences: non principal ultrafilter ω: in $\mathbb{N}$

$\omega \subseteq \mathcal{P}(\mathbb{N})$ with the property. Given $(x_n) \subseteq X$ compact.

$\exists! \, x \in X$ s.t. $\forall V$ neighborhood of x, $\{n \in \mathbb{N} : x_n \in V\} \in \omega$:

$$\lim_{\omega} x_n := x.$$

x

Robinson Field.

Consider $\varepsilon_n \downarrow 0$, let

$$\mathcal{A}^{\varepsilon} := \{ (z_n) \in \mathbb{C}^{\mathbb{N}} : |z_n|^{\varepsilon_n} \text{ is bounded} \}$$

$$|(z_n)|_{\omega} := \lim |z_n|^{\varepsilon_n}$$

$$\mathfrak{m} := \{ (z_n) : |(z_n)|_{\omega} = 0 \}$$

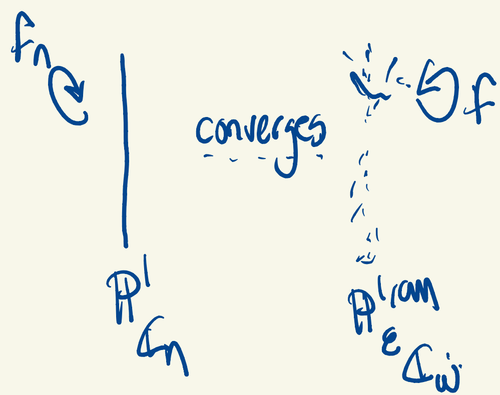
$${}^{\varepsilon}\mathbb{C}_{\omega} := \mathcal{A}^{\varepsilon} / \mathfrak{m} \text{ is an alg. closed, complete field}$$

Rational Map $\circlearrowright \mathbb{P}^1_{{}^{\varepsilon}\mathbb{C}_\omega}$

(Luo, Favre-Gong) Let $f = \{f_n\} \subseteq \mathrm{Rat}_d$ degenerating.

Then $\exists\, \varepsilon_n \downarrow 0$ s.t. $f \in {}^{\varepsilon}\mathbb{C}_\omega(z)$ and $f : \mathbb{P}^{1,an}_{{}^{\varepsilon}\mathbb{C}_\omega} \circlearrowleft$ is non-trivial.

Furthermore



Measures (De Marco-Faber, Favre, Favre-Gong)
∧ (Baker-Rumely, Favre-Rivera-Letelier).
∧ (Freire-Lopes-Mañé ∧ Lyubich).

$$\mu_{f_n} = \lim_{k\to\infty} \sum_{x \in f_n^{-k}(x_0)} \delta_x \longrightarrow \mu_f = \lim_{k\to\infty} \sum_{x\in f^{-k}(x_0)} \delta_x$$

$$\cap \qquad\qquad \cap$$

$$\mathcal{M}^1(\mathbb{P}^1_{\mathbb{C}_n}) \qquad\qquad \mathcal{M}^1(\mathbb{P}^{1,an}_{\mathbb{C}_\omega}).$$

Measures and Iterations

(K.-Nie 2024) $\{f_n\} \subseteq \overline{\mathrm{Rat}_d}$ s.t. $f_n^k \to g_k \in \overline{\mathrm{Rat}_{d^k}}$ $\forall k$.

$$\Rightarrow \quad \underset{\substack{\cap \\ M^1(\hat{\mathbb{C}})}}{\mu_{f_n}} \to \underset{\substack{\cap \\ M^1(\hat{\mathbb{C}})}}{\mu} \quad \text{where } \mu \text{ is determined by } (g_k)$$

Appendix: Def Ultrafilter

$\omega \subseteq \mathcal{P}(\mathbb{N})$: $\emptyset \notin \omega$, $E \in \omega$ or $\mathbb{N} \setminus E \in \omega$

$E \in \omega$, $E \subset F \Rightarrow F \in \omega$

$E \in \omega$, $F \in \omega \Rightarrow E \cap F \in \omega$

elements in ω are ω-big sets.

ω is principal if $\omega = \{E : n \in E\}$ for some n

Quadratic Rational explicit example

$$f_t(z) = t - \frac{1+z^2}{z} + \frac{t}{z^2} - at^5 \qquad \omega = 0 \to \infty \to t \to 0$$

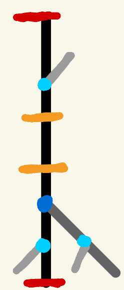
$$M_t(z) = tz \Rightarrow M_t^{-1} \circ f_t^2 \circ M_t \to 1 + \frac{z^2}{z-1} \qquad \infty^5$$

$$L_t(z) = t^3 z \Rightarrow L_t^{-1} \circ f_t^3 \circ L_t \to z^2 + a$$

Counting Rescaling Limits

(A) A Shishikura tree encodes the degeneration and Y. Luo characterizes the trees that arise.

(B) Construct one with N non-monomial PCF resc.



- $1 - \frac{1}{z^{3^2}}$ $\qquad \overset{*}{0} \to \overset{*}{\infty} \to 1 \to 0$
- $1 - \frac{1}{z^{3^4}}$

Question $\exists \{f_t\} \subseteq \overline{\mathrm{Rat}_d}$ with as many independent RL?