

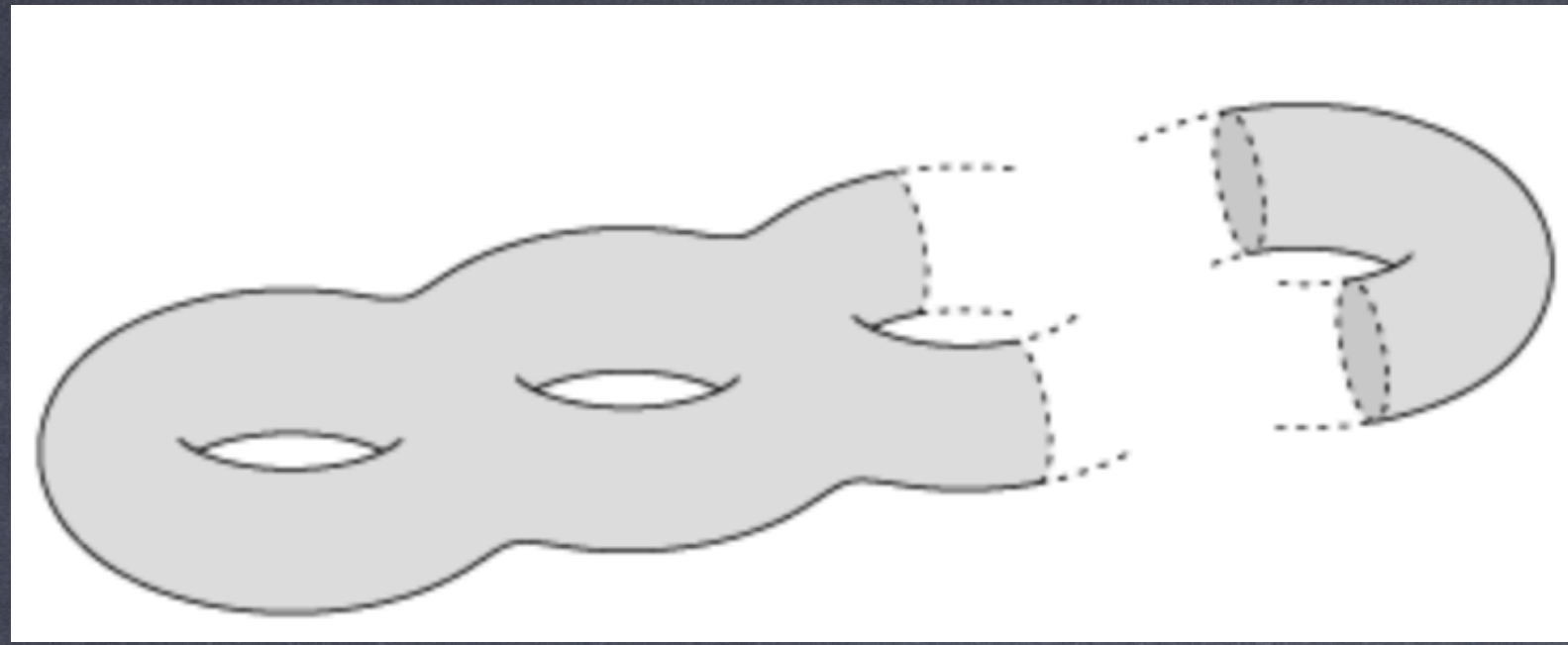
# Stretch factors and degrees

Erwan Lanneau, Institut Fourier (Grenoble UGA)  
Toulouse, June 11, 2025

Livio Liechti (Fribourg, Switzerland)

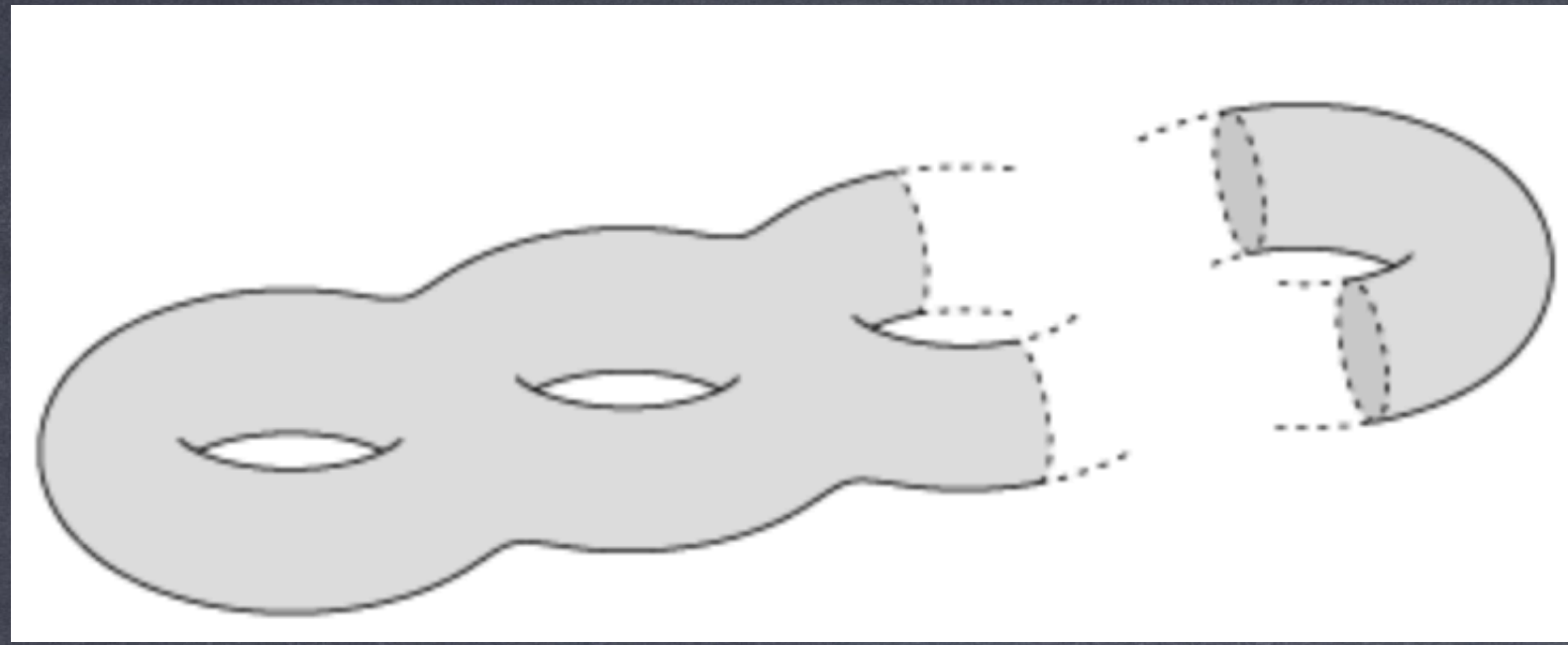
Chi Cheuk Tsang (Université du Québec à Montréal)

Pierre Dehornoy (Marseille)



$S_{g,b,n}$  compact genus  $g$  surface, with possibly  
 $b$  boundary components, and  $n$  punctures

$f: S \longrightarrow S$  pseudo-Anosov homeomorphism



$S_{g,b,n}$  compact genus  $g$  surface, with possibly  $b$  boundary components, and  $n$  punctures

$f: S \longrightarrow S$  pseudo-Anosov homeomorphism

Complexity  $\lambda(f)$  of  $f$

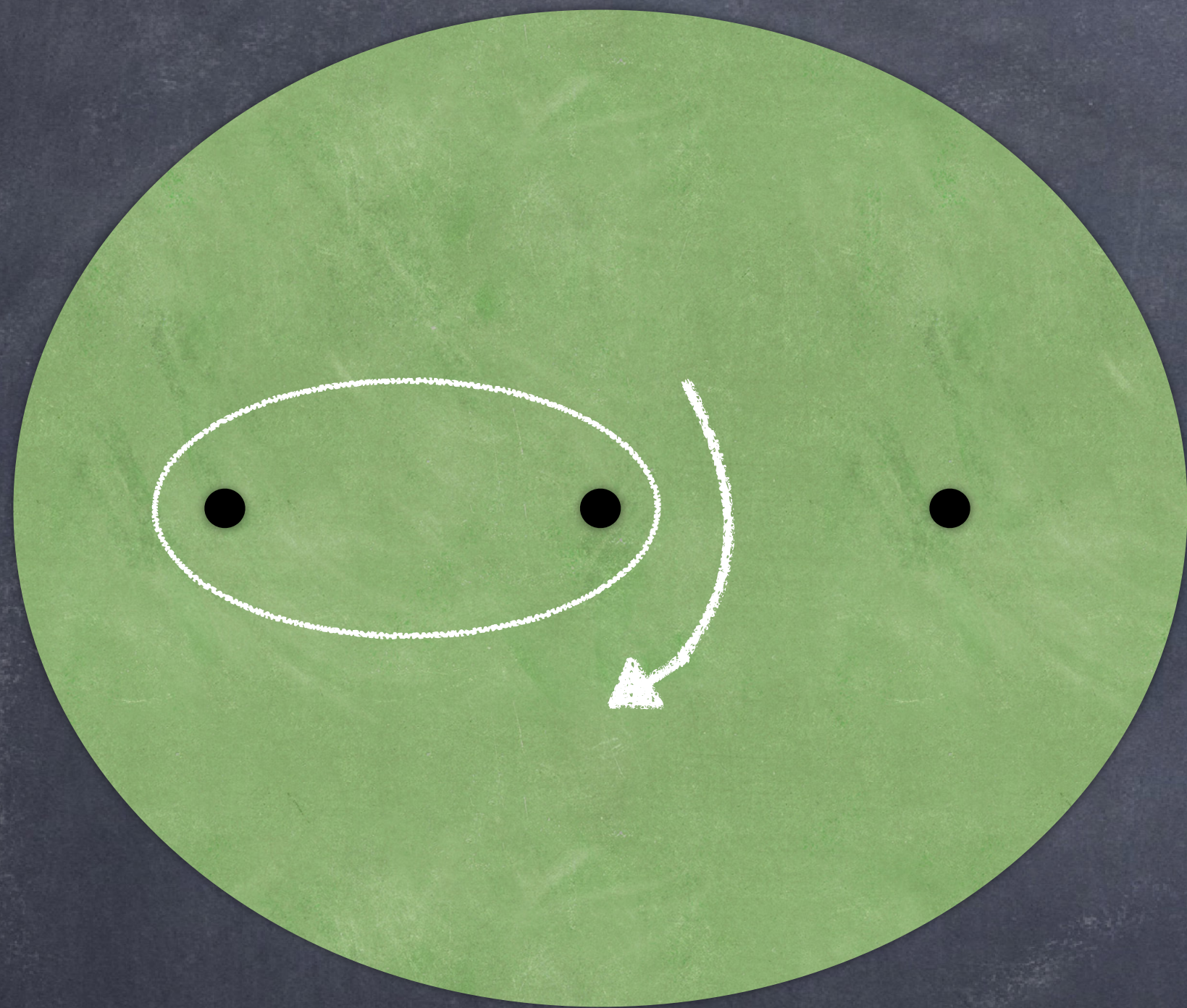
$$\lambda(f) = \lim_{k \rightarrow \infty} l_{\rho}(f^k(c))^{1/k}$$

$c$  is any simple closed curve  
 $\rho$  is any Riemannian metric

$$= \exp(h_{\text{top}}(f))$$

Realisation of algebraic integers/degrees, minimisation

# An example: braids on the disc

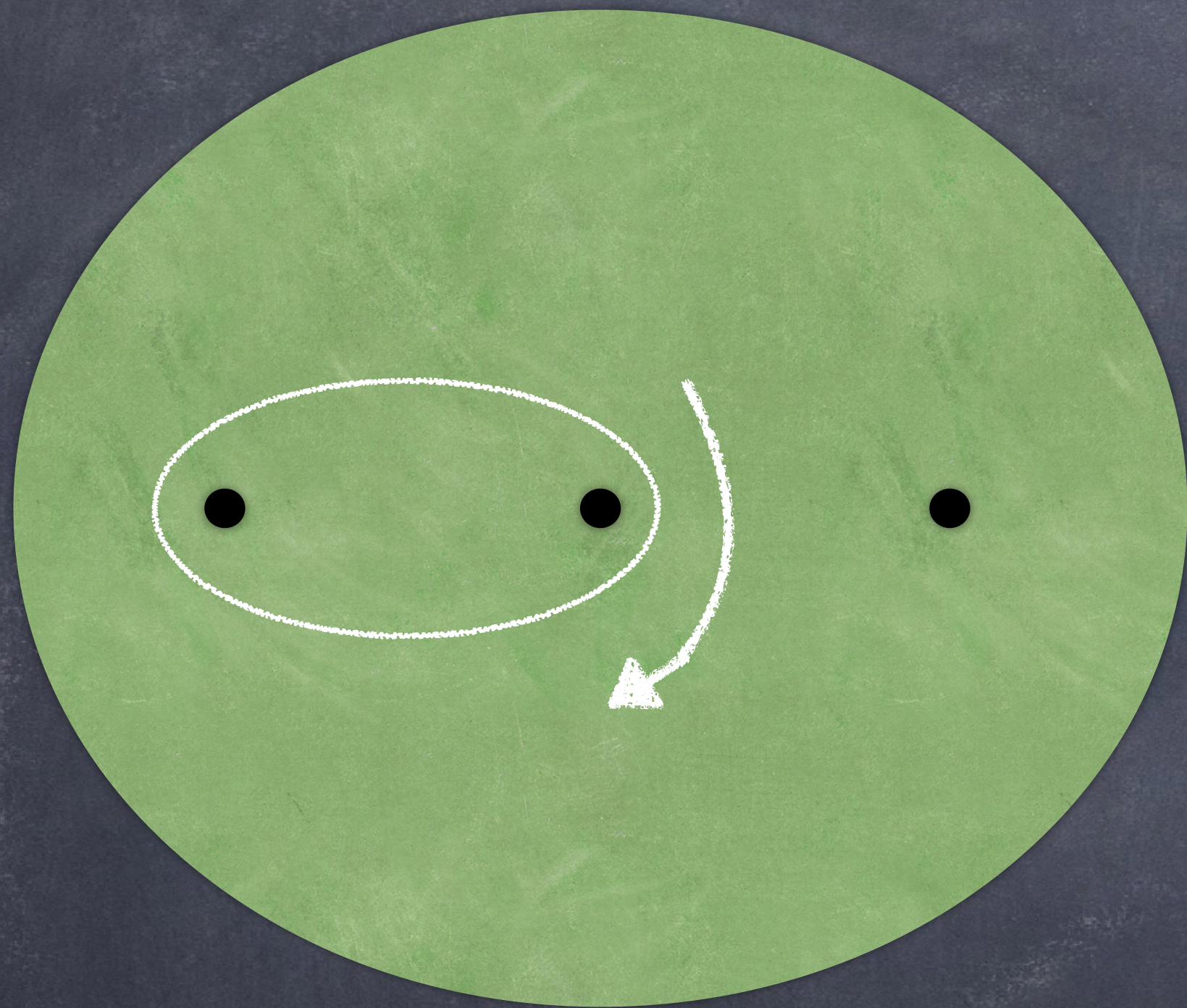


$\sigma_1$

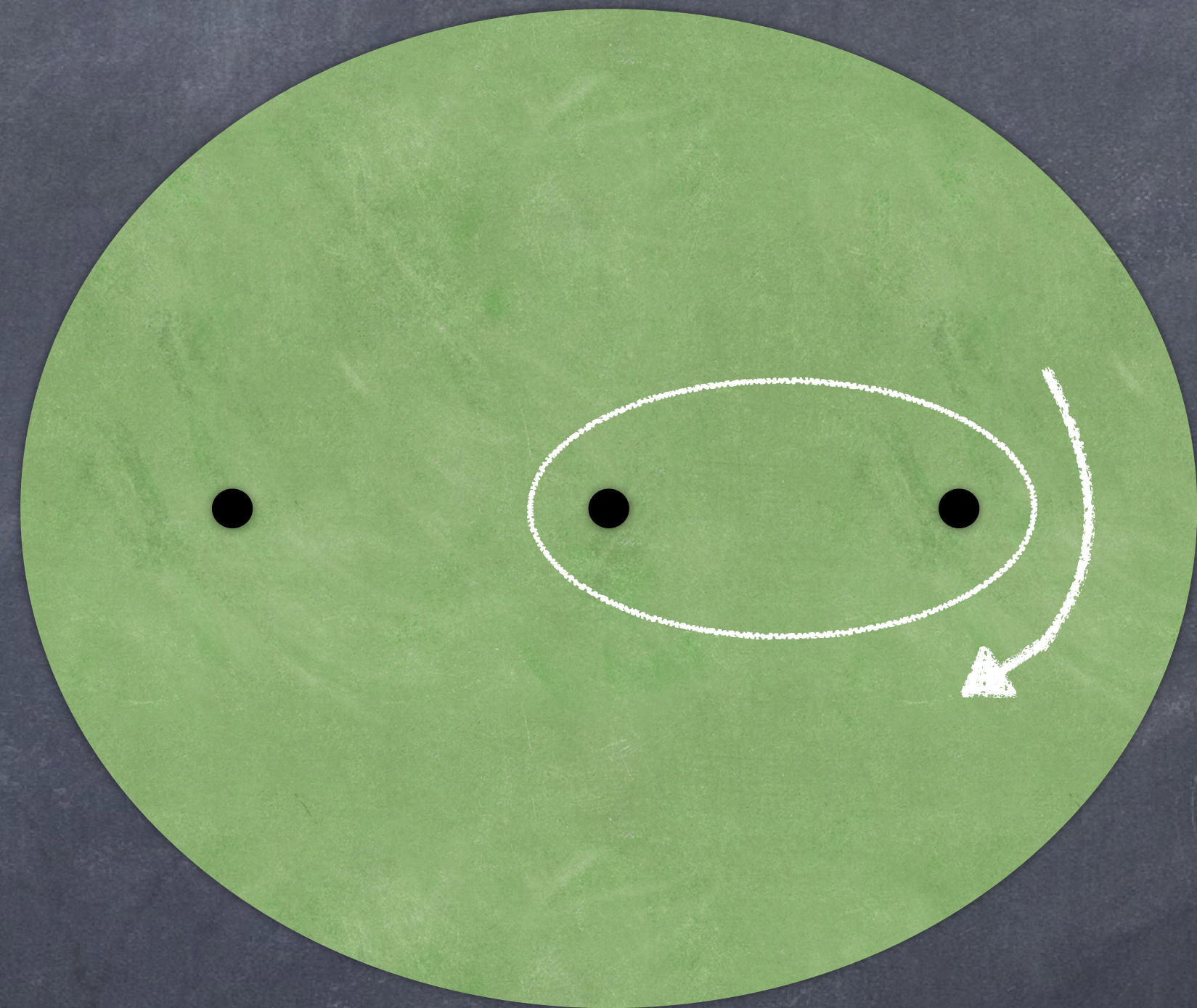
$$S_{g,b,n} = S_{0,1,3}$$

Three punctured disc

# An example: braids on the disc



$\sigma_1$



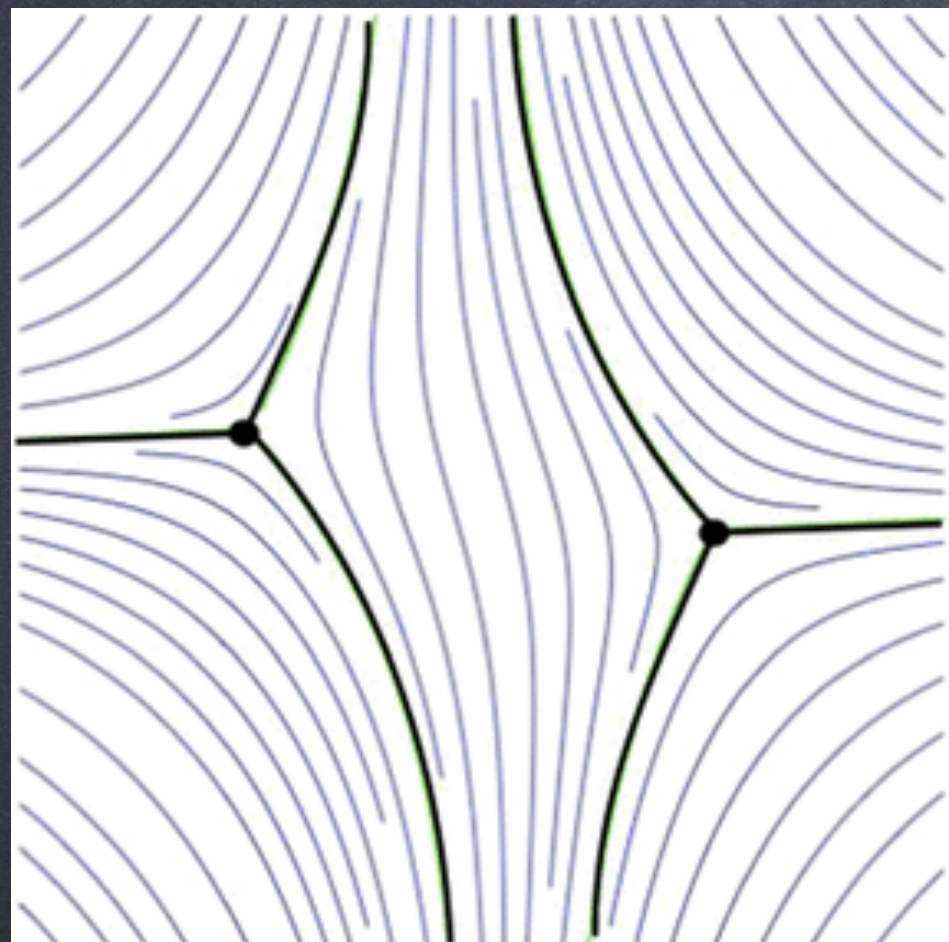
$\sigma_2$

$$S_{g,b,n} = S_{0,1,3}$$

Three punctured disc

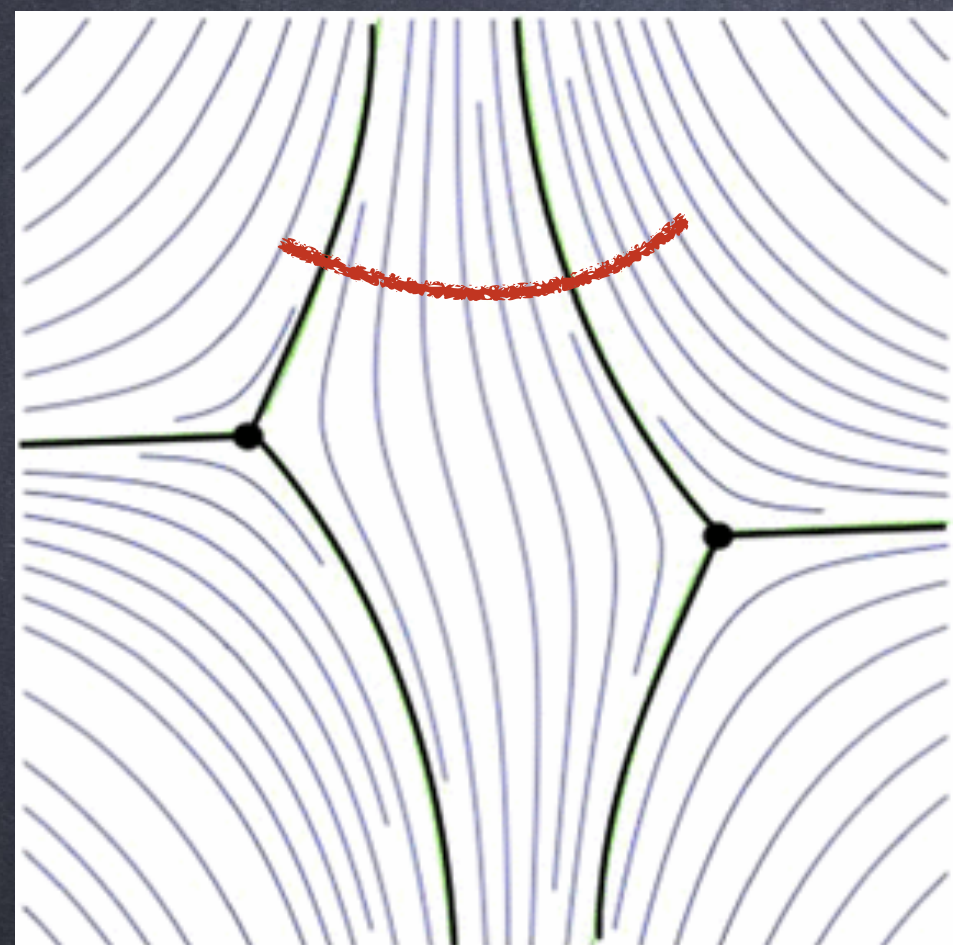
# Measured foliations

A measured foliation  $\mathcal{F}$  is a singular foliation with an invariant measure in the transverse direction. Locally it is given by  $x = \text{constant}$  with the measure  $\mu = dx$  on transverse arcs.



# Measured foliations

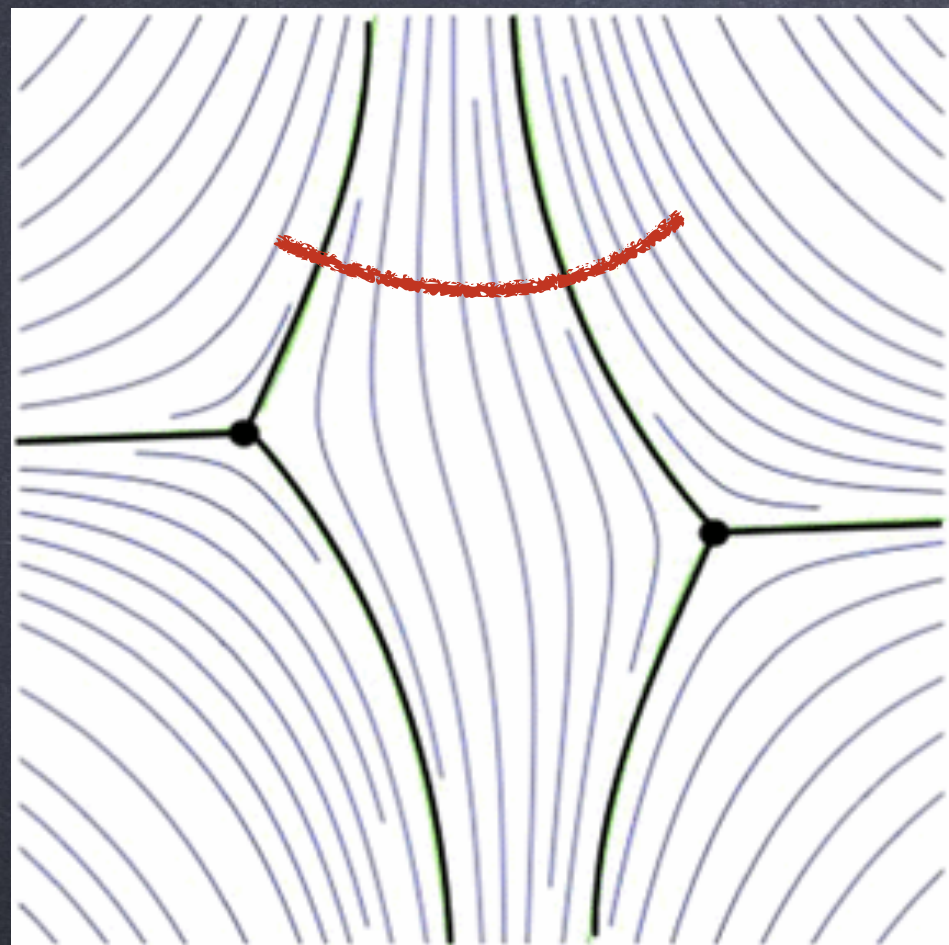
A measured foliation  $\mathcal{F}$  is a singular foliation with an invariant measure in the transverse direction. Locally it is given by  $x = \text{constant}$  with the measure  $\mu = dx$  on transverse arcs.



arc  $\alpha$

# Measured foliations

A measured foliation  $\mathcal{F}$  is a singular foliation with an invariant measure in the transverse direction. Locally it is given by  $x = \text{constant}$  with the measure  $\mu = dx$  on transverse arcs.

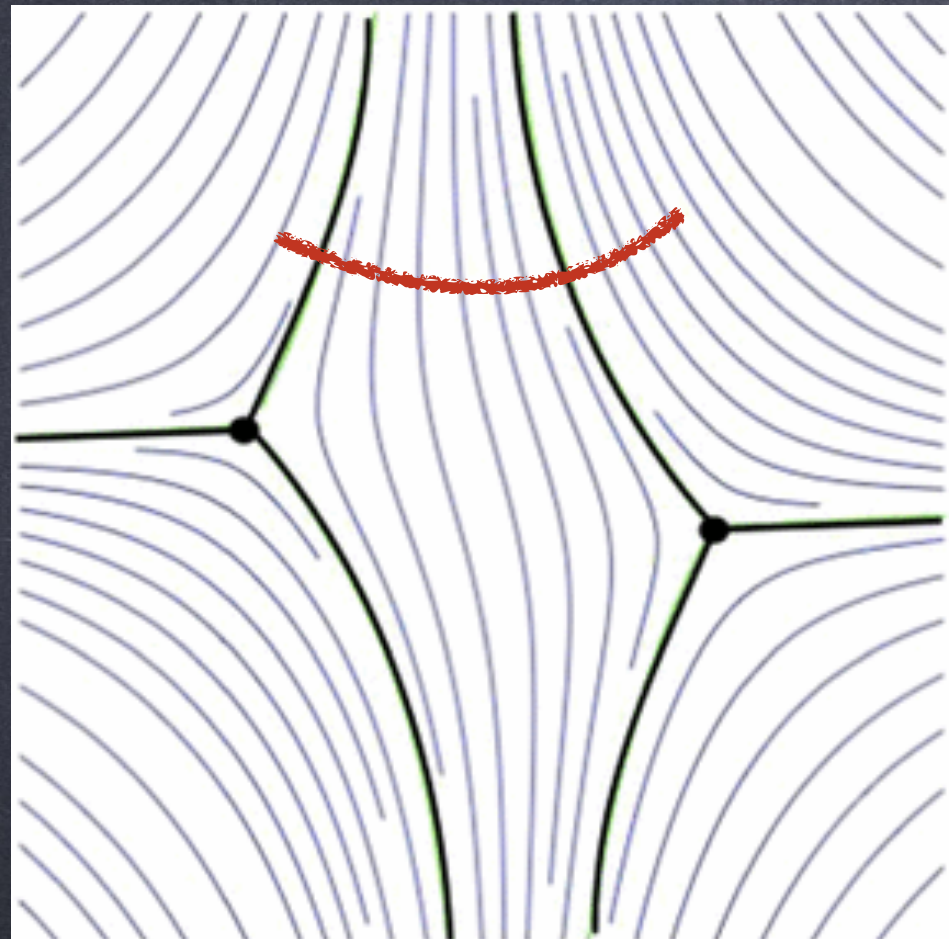


arc  $\alpha$

$$\mu(\alpha) = \int_{\alpha} dx$$

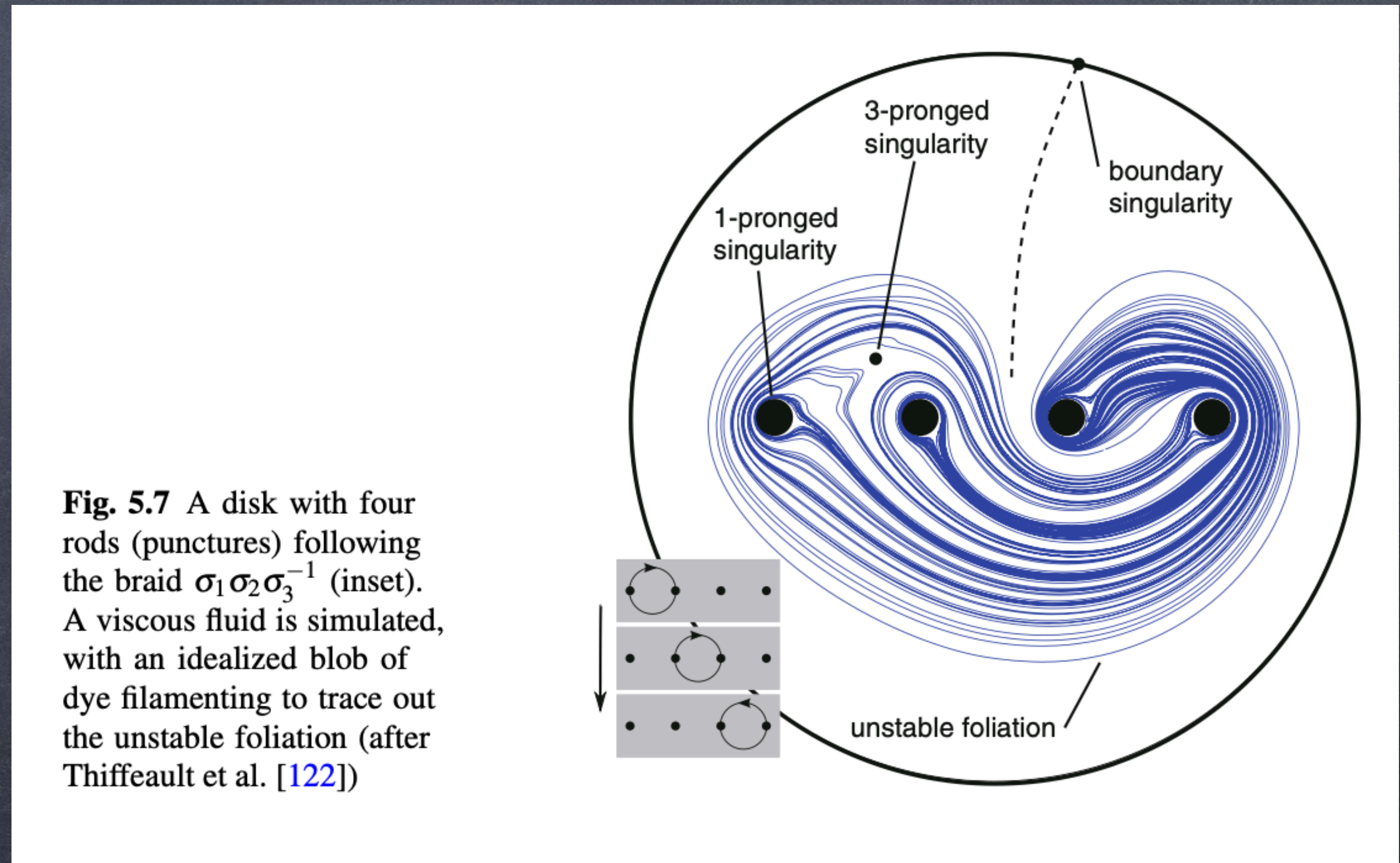
# Measured foliations

A measured foliation  $\mathcal{F}$  is a singular foliation with an invariant measure in the transverse direction. Locally it is given by  $x = \text{constant}$  with the measure  $\mu = dx$  on transverse arcs.



arc  $\alpha$

$$\mu(\alpha) = \int_{\alpha} dx$$



**Fig. 5.7** A disk with four rods (punctures) following the braid  $\sigma_1 \sigma_2 \sigma_3^{-1}$  (inset). A viscous fluid is simulated, with an idealized blob of dye filamenting to trace out the unstable foliation (after Thiffeault et al. [122])

# Pseudo-Anosov homeomorphisms

A mapping class  $\phi$  is pA if it fixes a pair of transverse measured foliations  $(\mathcal{F}^s, \mathcal{F}^u)$  such that  $\phi_*\mu_u = \lambda\mu_u$  and  $\phi_*\mu_s = \lambda^{-1}\mu_s$  for some  $\lambda > 1$ .

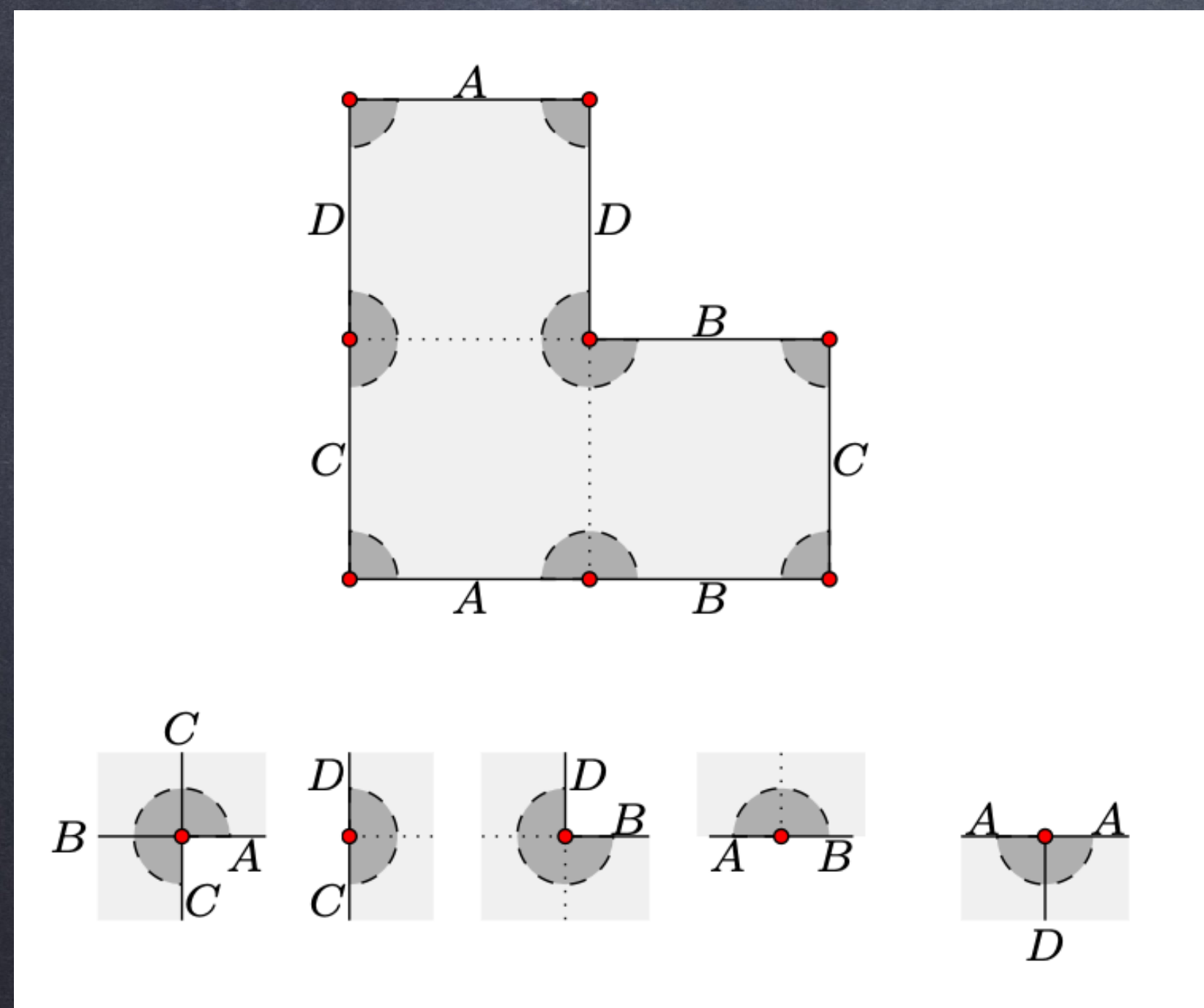
# Pseudo-Anosov homeomorphisms

A mapping class  $\phi$  is pA if it fixes a pair of transverse measured foliations  $(\mathcal{F}^s, \mathcal{F}^u)$  such that  $\phi_*\mu_u = \lambda\mu_u$  and  $\phi_*\mu_s = \lambda^{-1}\mu_s$  for some  $\lambda > 1$ .

$\lambda = \lambda(f)$  is the stretch factor of  $f$

# Affine homeomorphisms

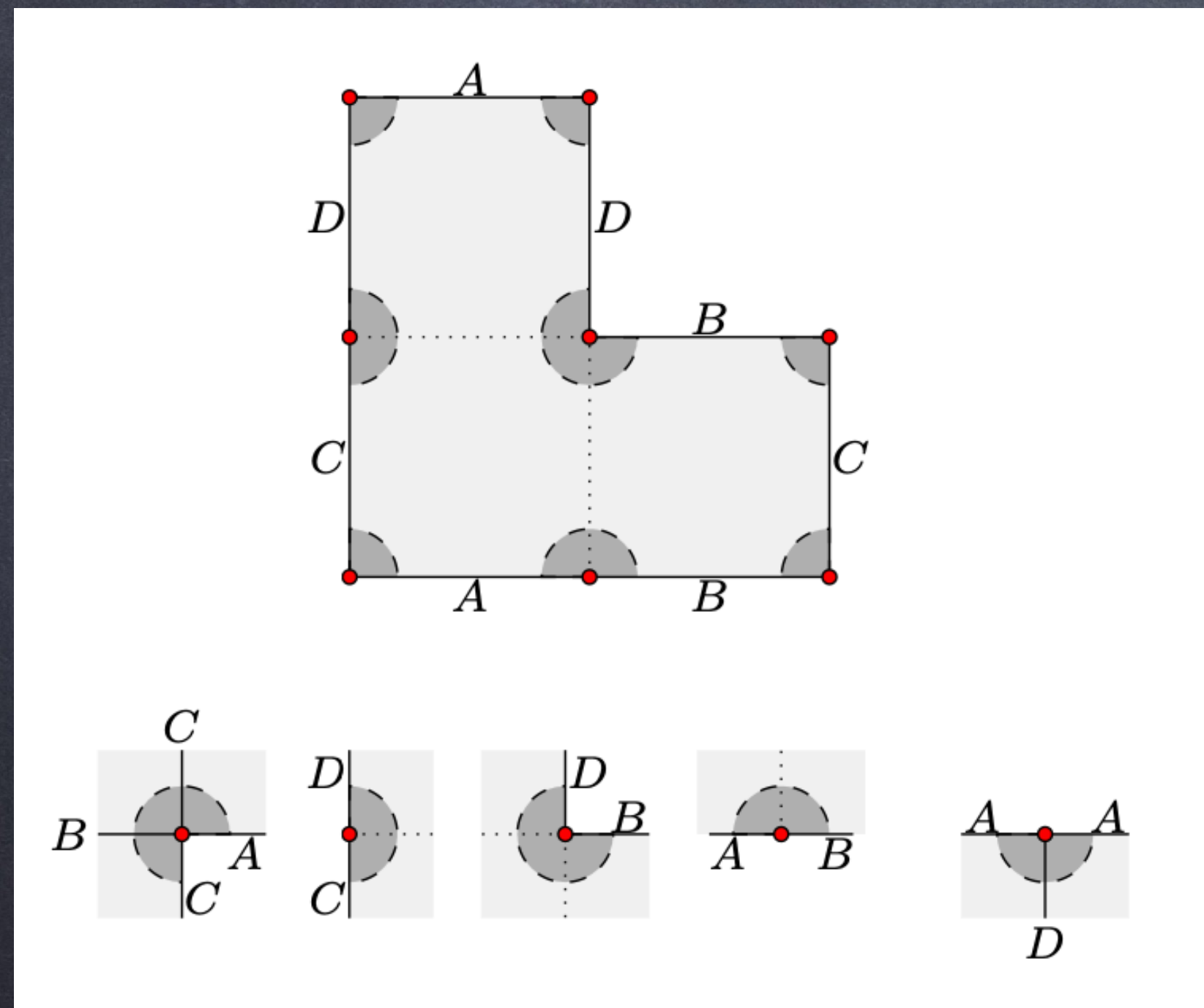
A half-translation surface  $X$  is a flat object constructed by gluing polygons along parallel sides of the same length using translations and half-translation.



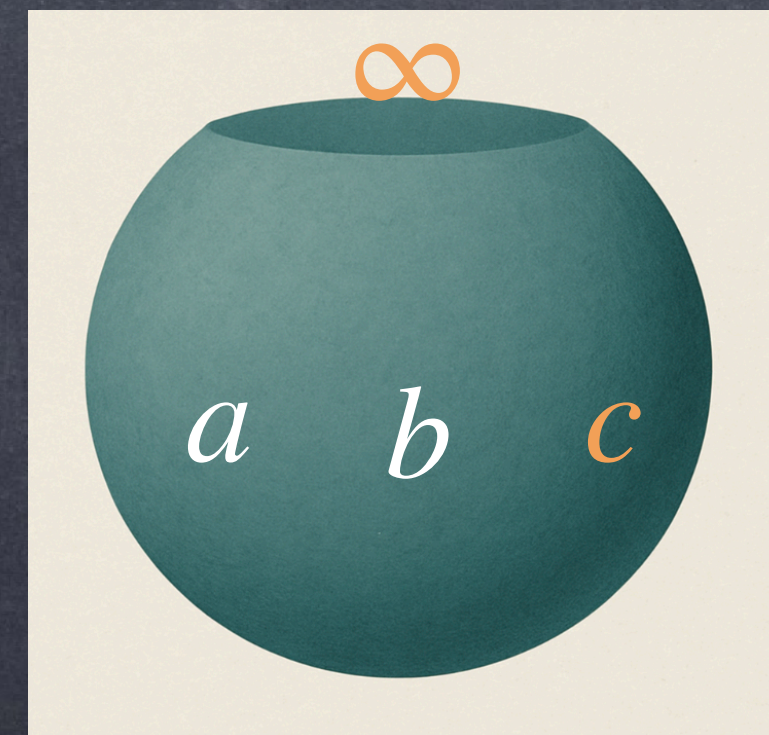
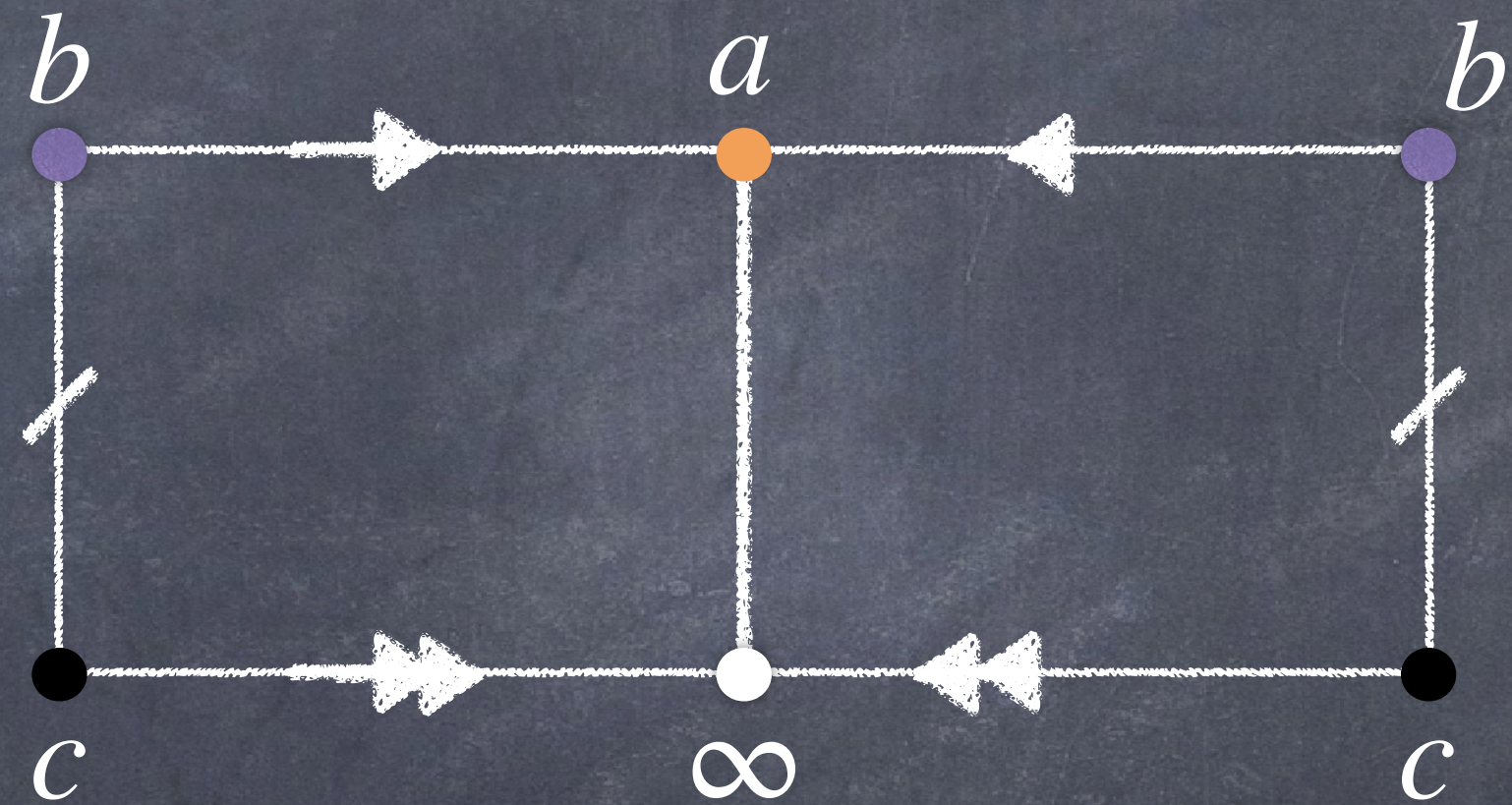
$$z \mapsto \pm z + cst$$

# Affine homeomorphisms

A half-translation surface  $X$  is a flat object constructed by gluing polygons along parallel sides of the same length using translations and half-translation.

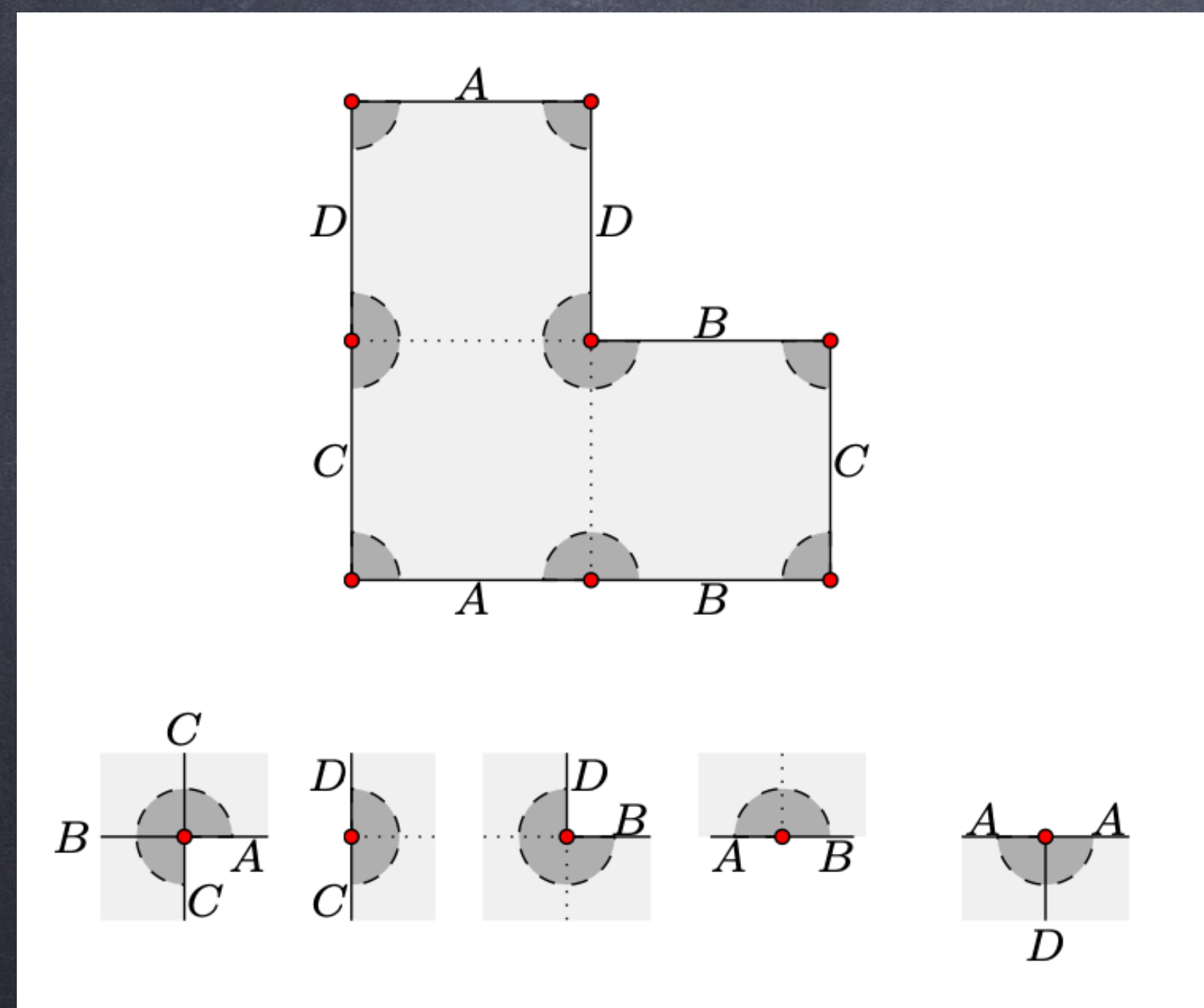


$$z \mapsto \pm z + cst$$



# Affine homeomorphisms

A half-translation surface  $X$  is a flat object constructed by gluing polygons along parallel sides of the same length using translations and half-translation.



$\phi : X \rightarrow Y$  is affine if (in the flat charts) it is  $\mathbb{R}$ -affine:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$

Since the coordinates differ only by a half-translation, the matrix is well defined in  $\text{PGL}(2, \mathbb{R})$ .

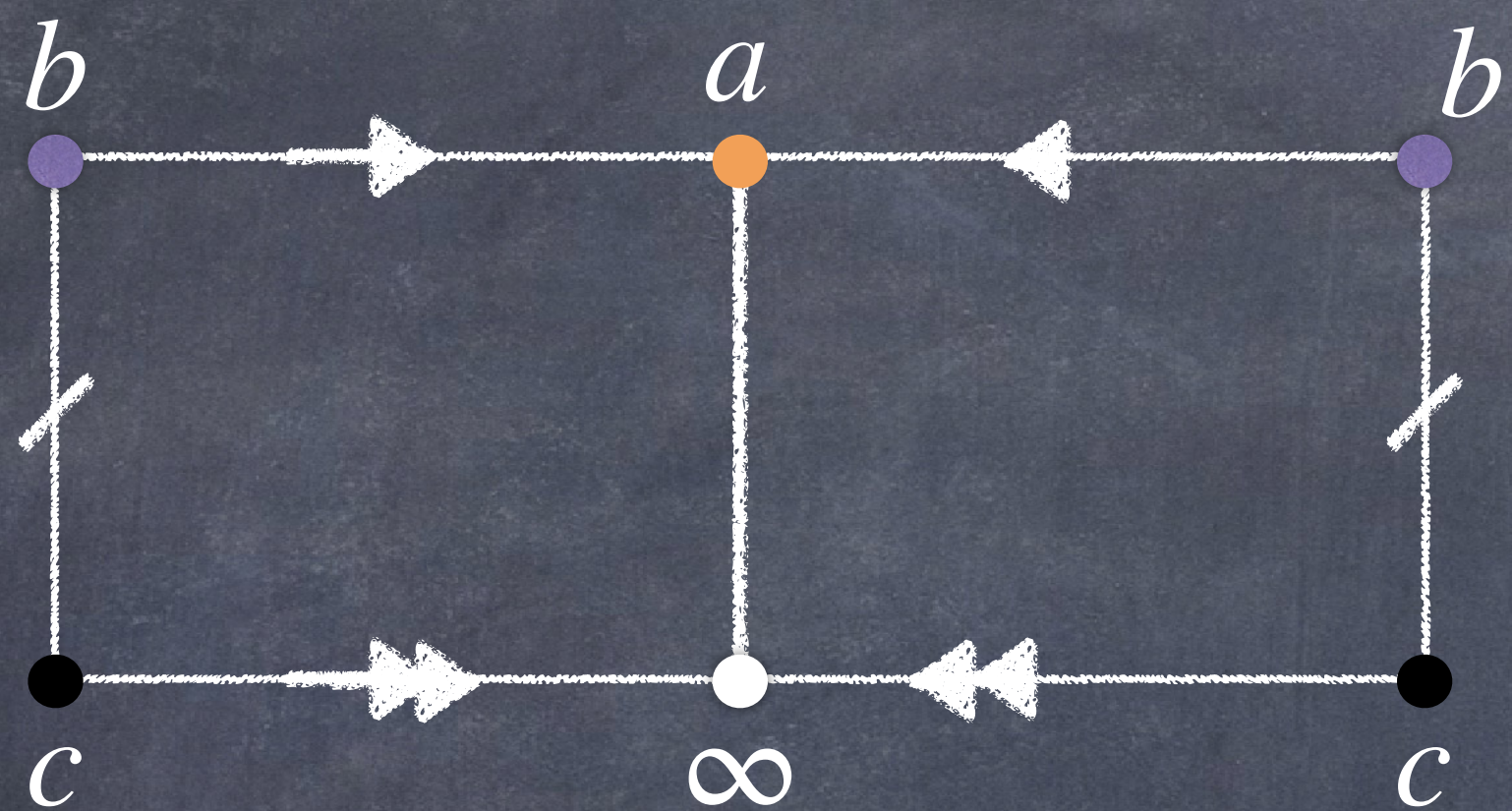
$$z \mapsto \pm z + cst$$

# Pseudo-Anosov homeomorphisms

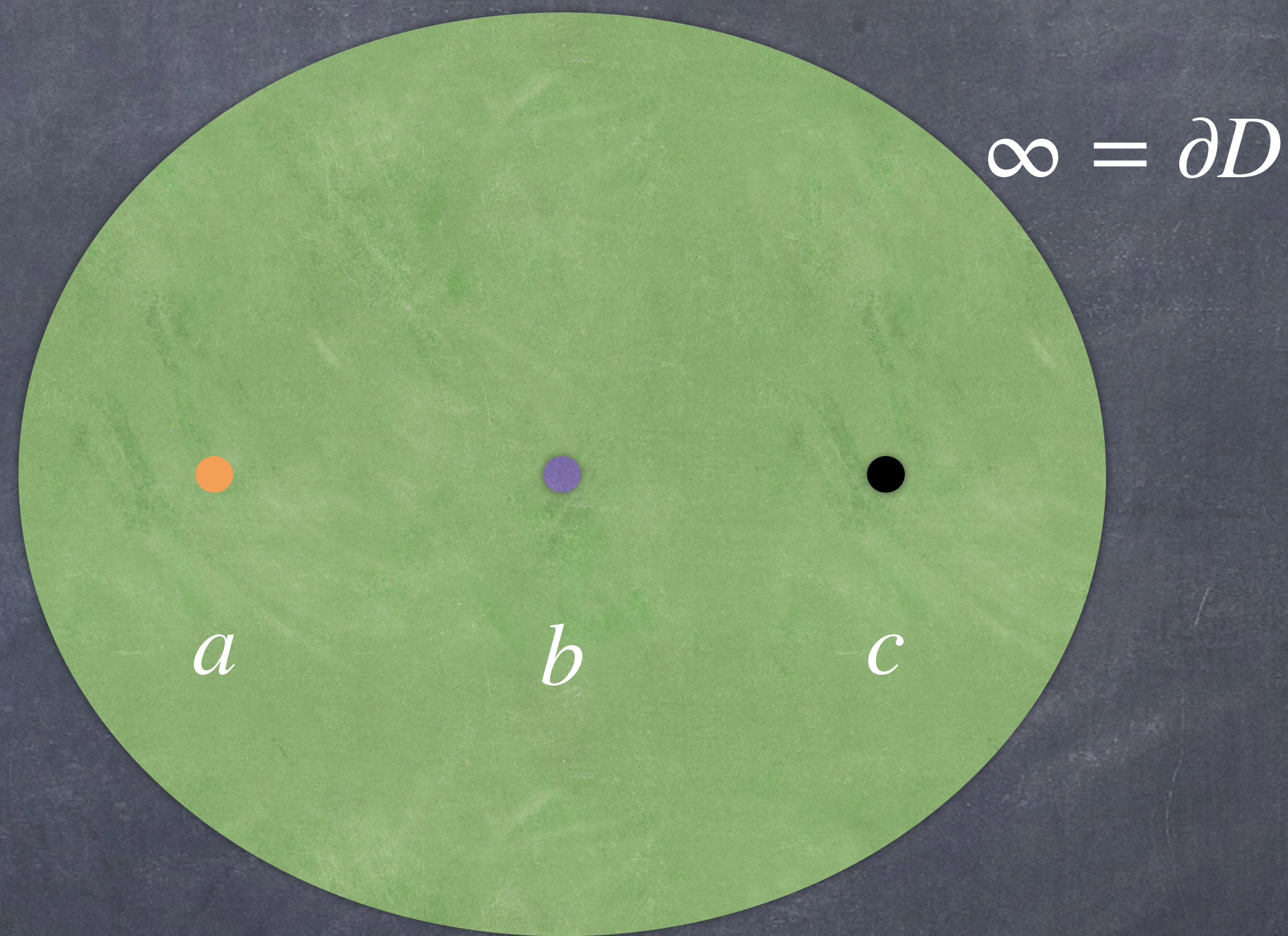
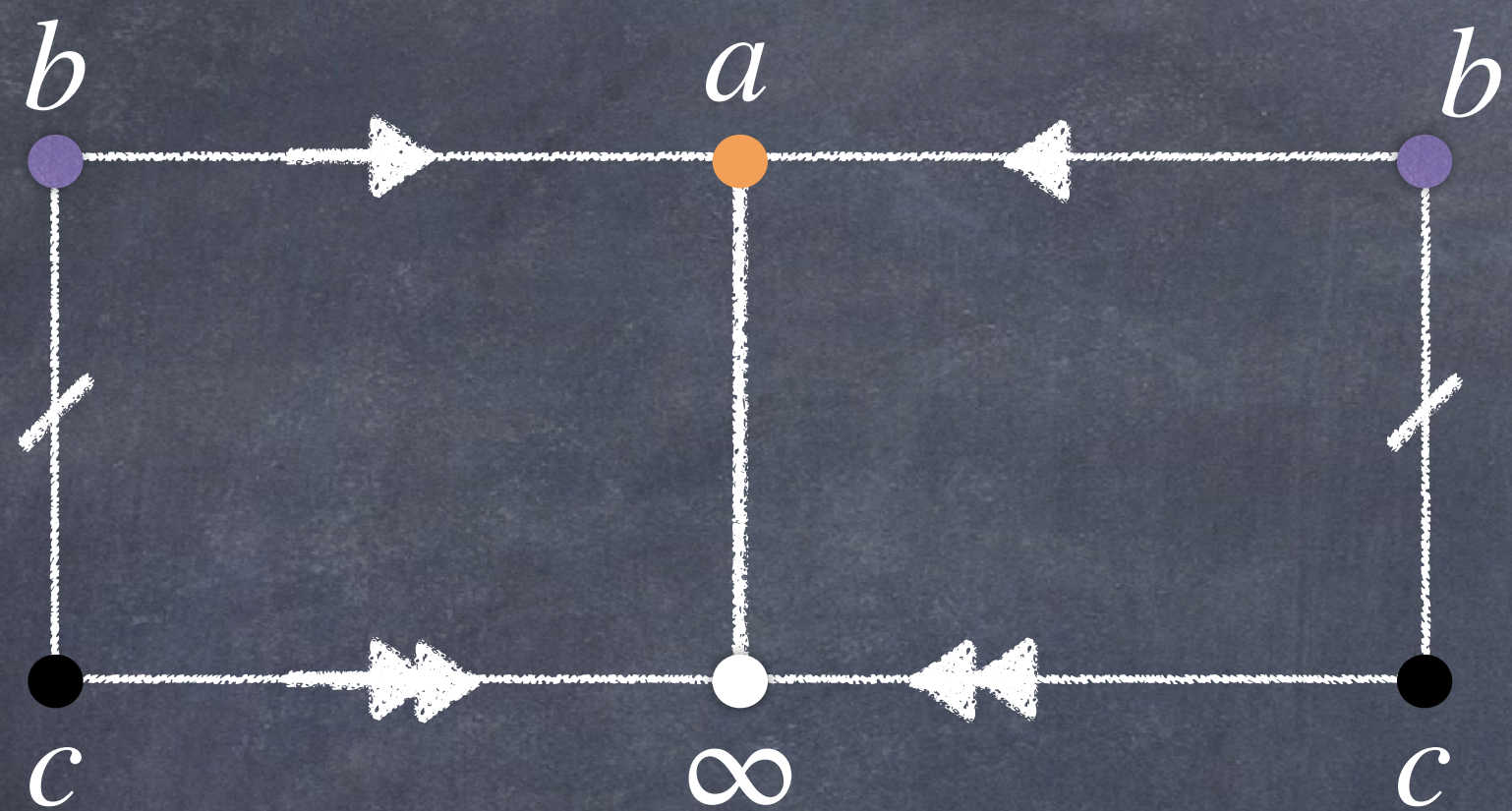
$\phi \in \text{Homeo}^+(S)$  is an affine pA if  $\phi \in \text{Aff}(X)$  for some metric  $X$  on  $S$  and  $D\phi = \pm \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$  is hyperbolic;

$\lambda(\phi) := |\lambda| > 1$  is the stretch factor of  $\phi$

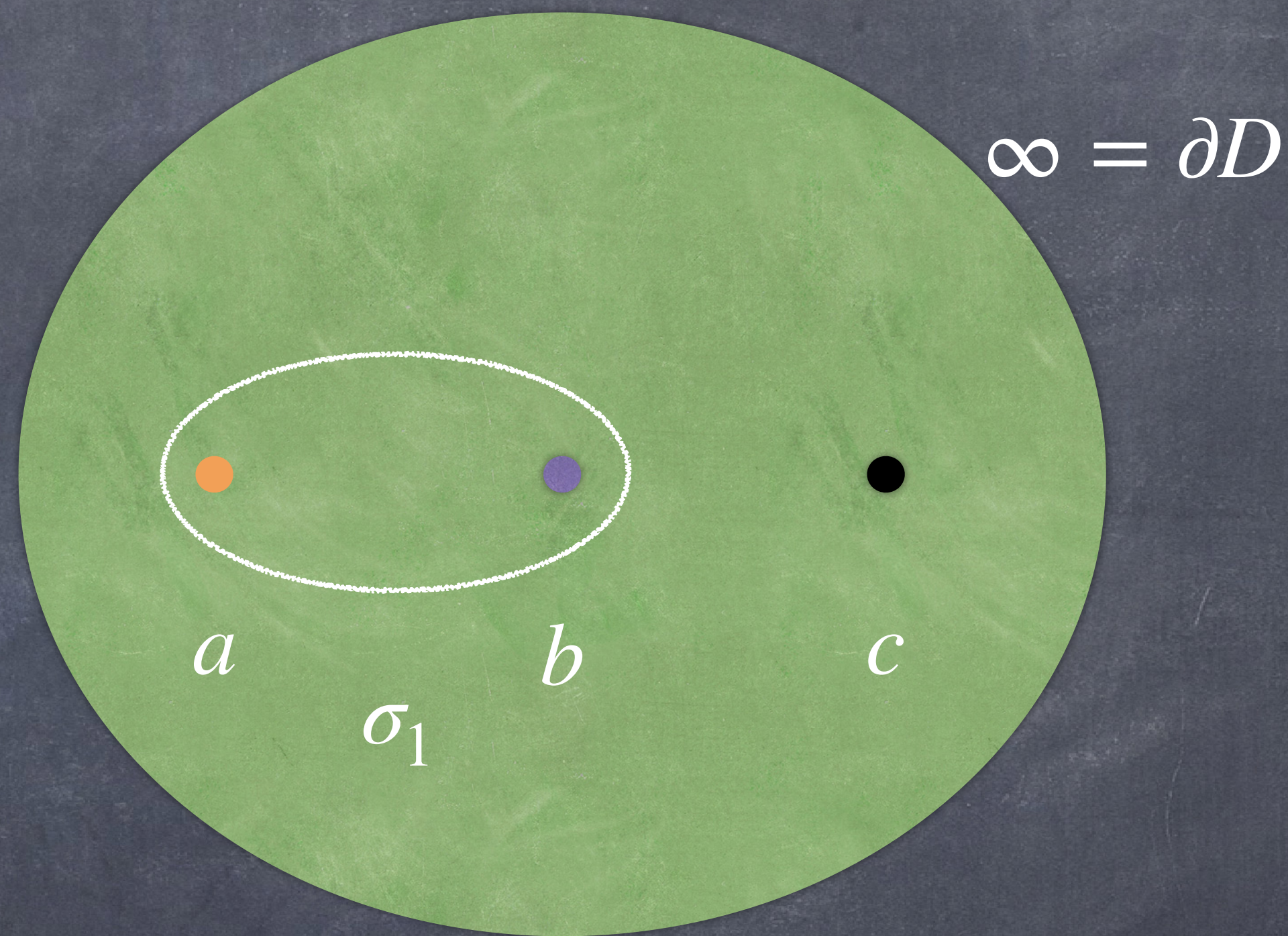
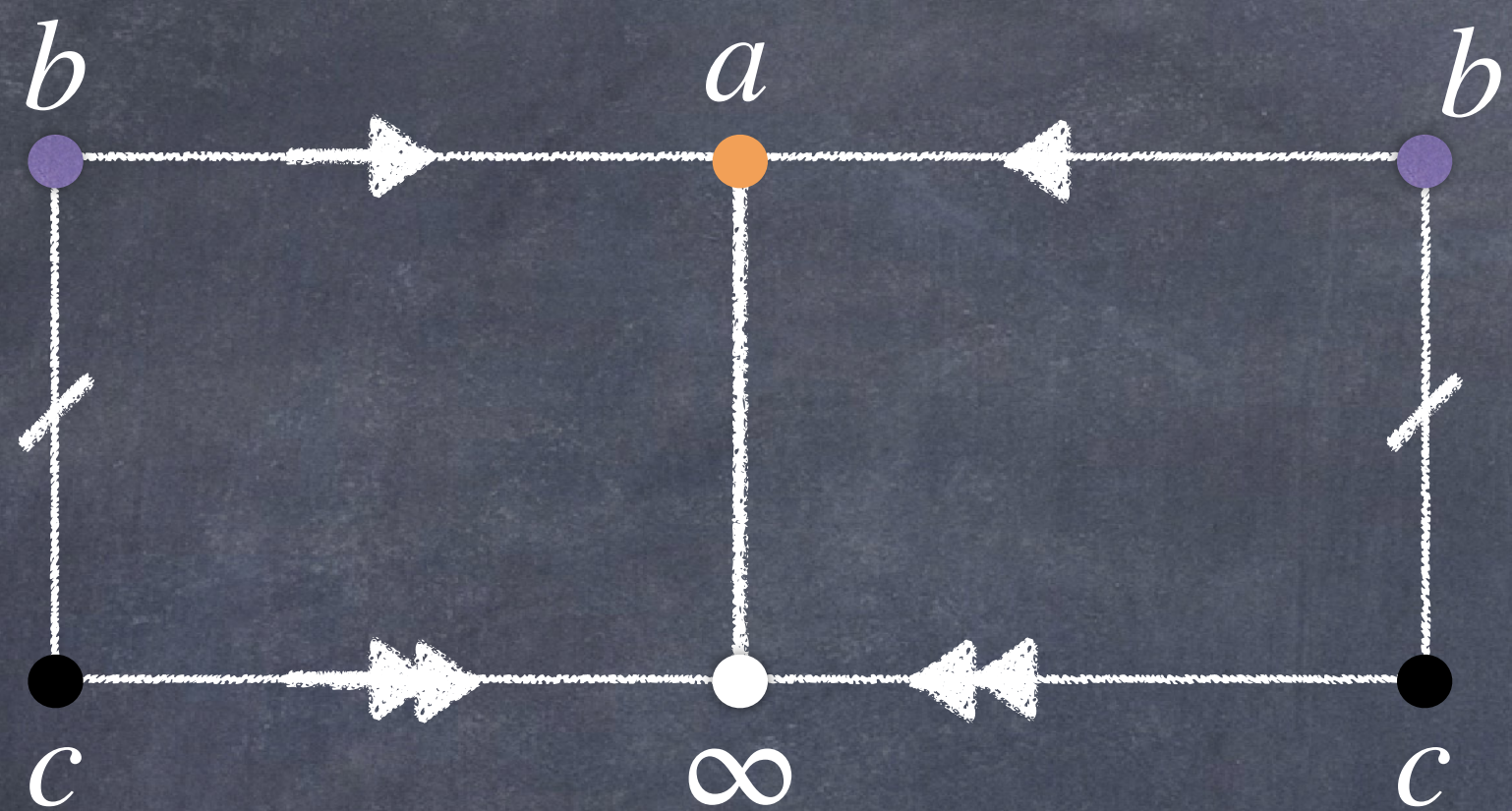
# Pseudo-Anosov homeomorphisms



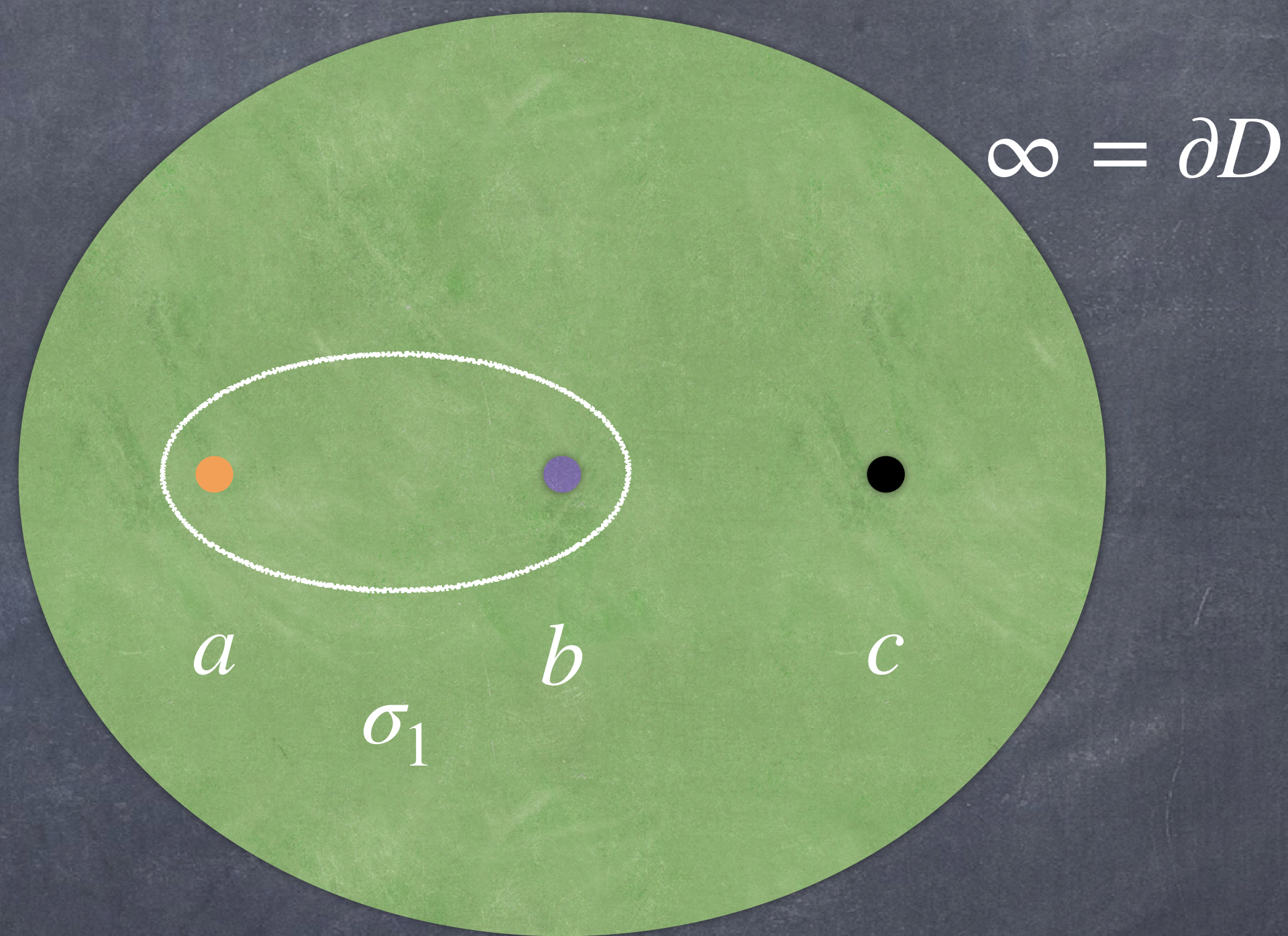
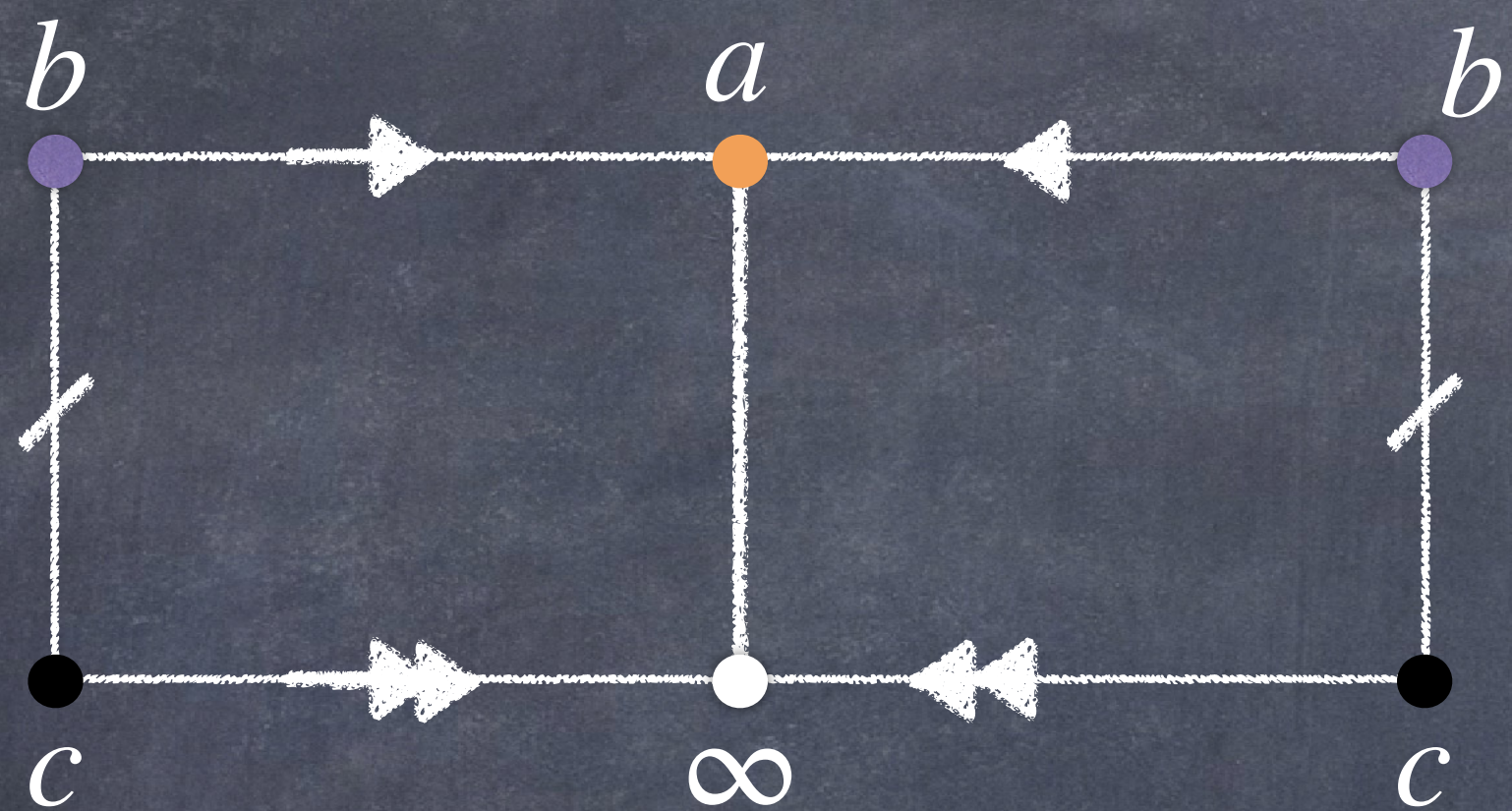
# Pseudo-Anosov homeomorphisms



# Pseudo-Anosov homeomorphisms

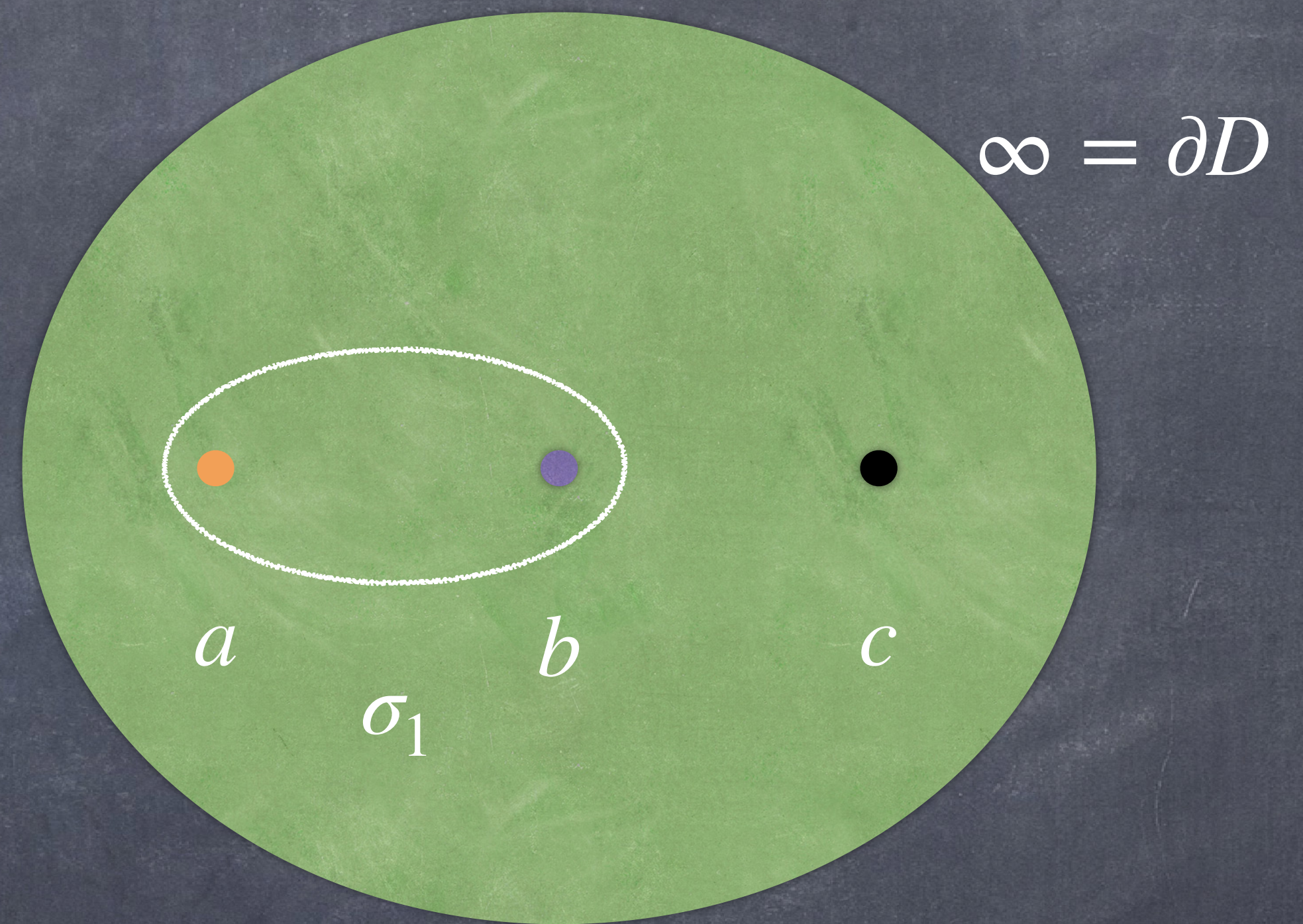
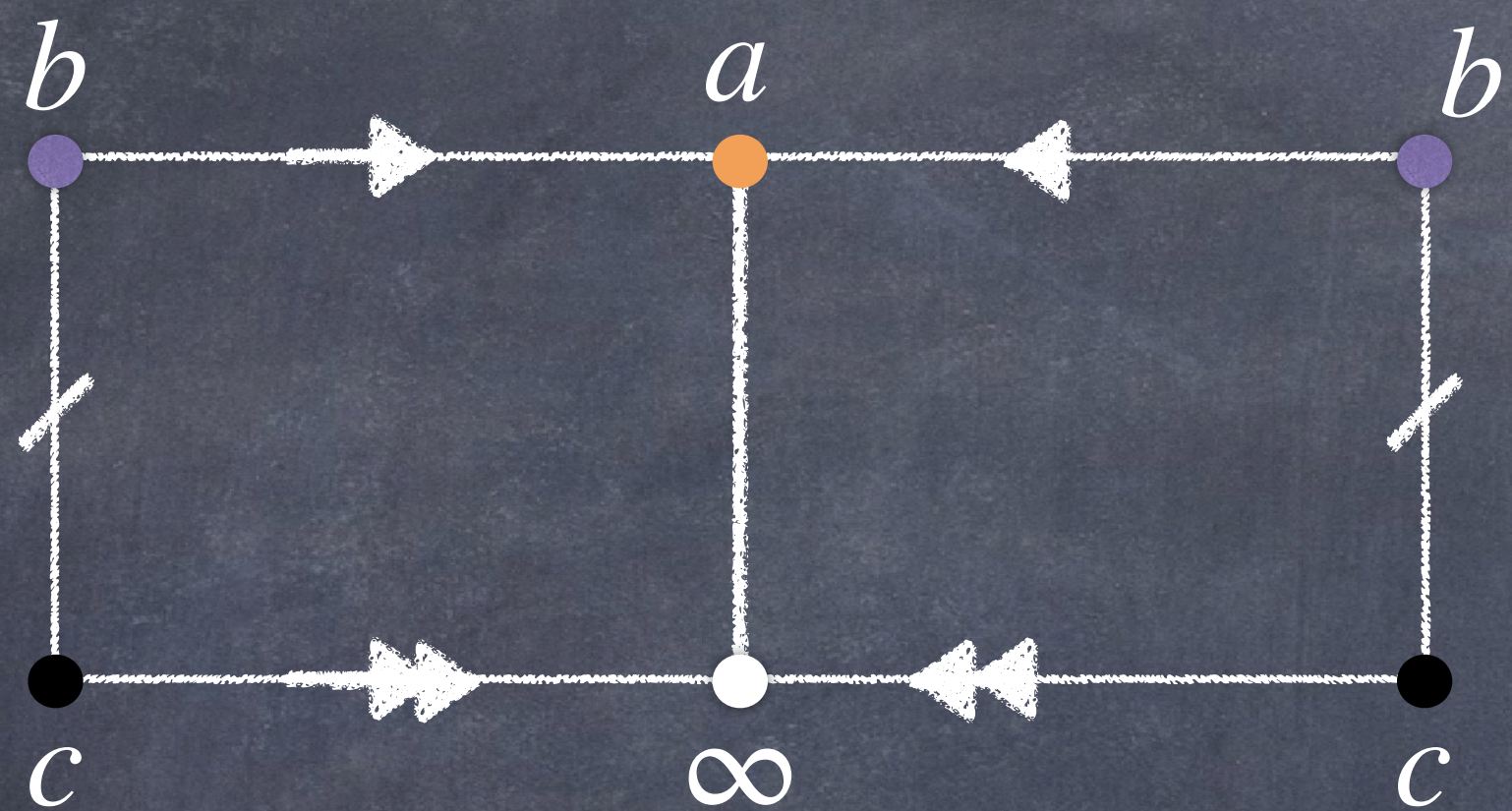


# Pseudo-Anosov homeomorphisms



half-Dehn twist  $\sigma_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \pmod{2} \\ y \end{pmatrix}$  is affine,  $D\sigma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

# Pseudo-Anosov homeomorphisms



half-Dehn twist  $\sigma_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \pmod{2} \\ y \end{pmatrix}$  is affine,  $D\sigma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

half-Dehn twist  $\sigma_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -x + y \pmod{2} \end{pmatrix}$  is affine,  $D\sigma_2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$

# Thurston-Veech construction

# Thurston-Veech construction



Dinner conference  
CIRM 2003

# Thurston-Veech construction



Dinner conference  
CIRM 2003



# Thurston-Veech construction



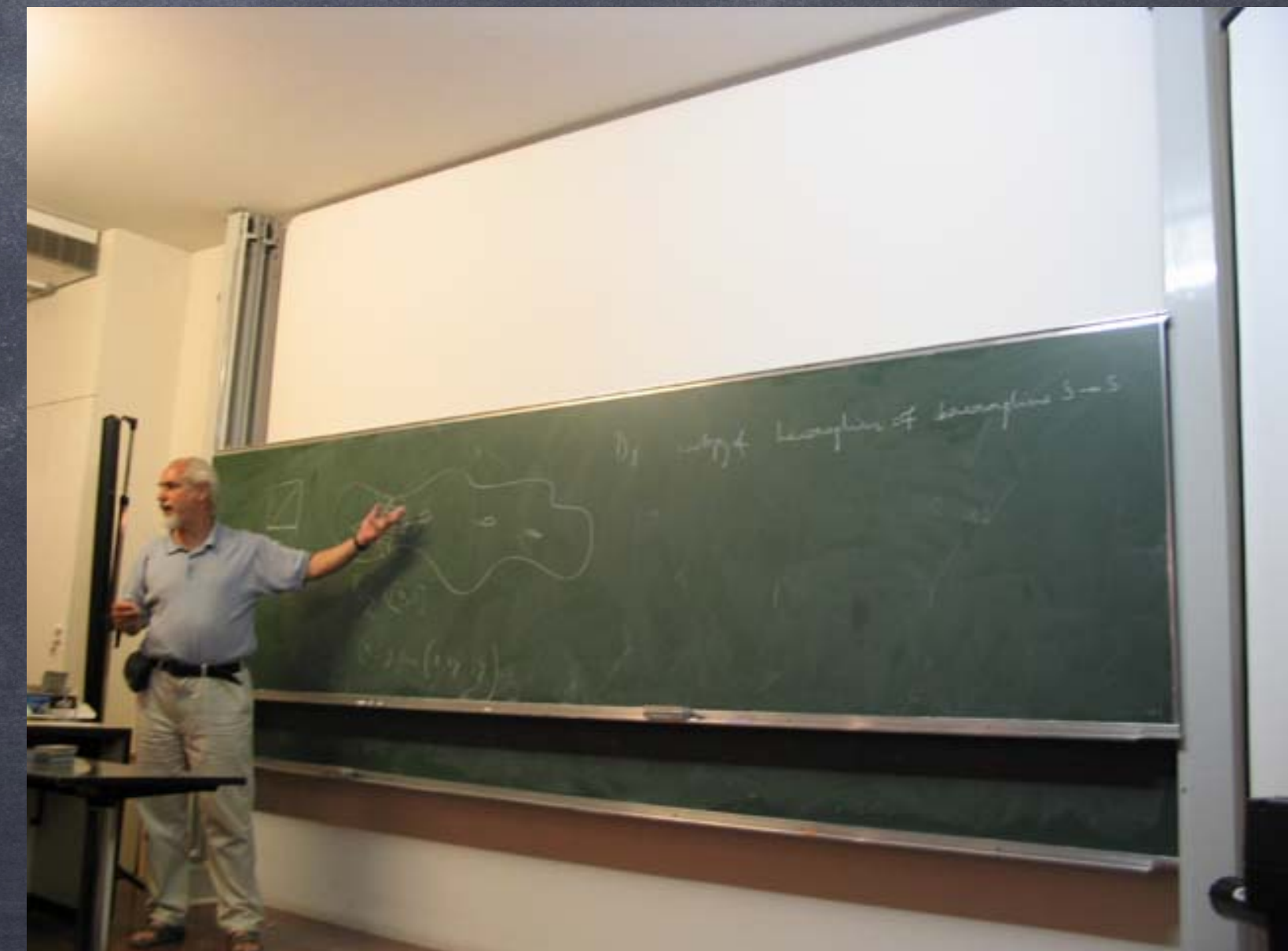
Dinner conference  
CIRM 2003



# Thurston-Veech construction



Dinner conference  
CIRM 2003



# Thurston-Veech construction

~~Thurston-Veech construction~~

~~Thurston-Veech construction~~

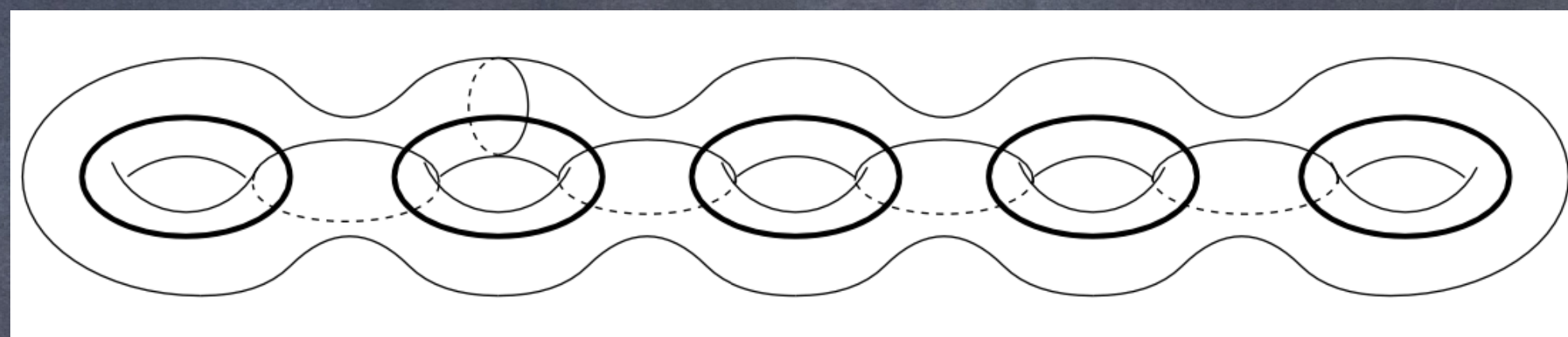
Bouillabaisse construction

# ~~Thurston-Veech construction~~ Bouillabaisse construction



# Bouillabaisse construction

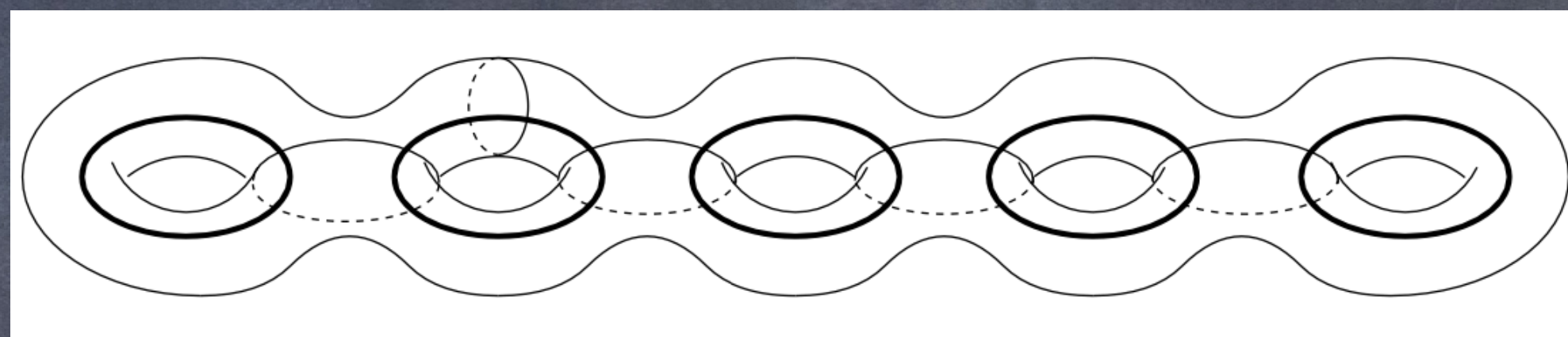
Set-up:  $(\alpha, \beta)$  is a pair of multicurves (union of disjoint essential simple closed curves) that intersect minimally and fill the surface (complement is a collection of simply polygonal regions). Let  $\Omega$  be the geometric intersection matrix of  $\alpha \cup \beta$ .



$$\alpha = \{\alpha_1, \alpha_2, \dots, \alpha_r\} \quad \beta = \{\beta_1, \beta_2, \dots, \beta_s\}$$

# Bouillabaisse construction

Set-up:  $(\alpha, \beta)$  is a pair of multicurves (union of disjoint essential simple closed curves) that intersect minimally and fill the surface (complement is a collection of simply polygonal regions). Let  $\Omega$  be the geometric intersection matrix of  $\alpha \cup \beta$ .



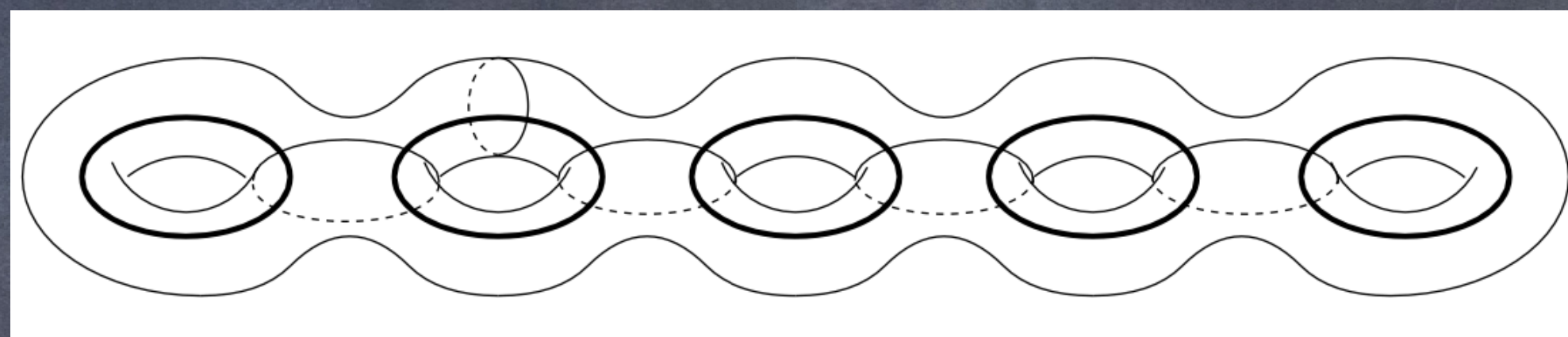
$$\alpha = \{\alpha_1, \alpha_2, \dots, \alpha_r\} \quad \beta = \{\beta_1, \beta_2, \dots, \beta_s\}$$

$$\Omega = \begin{pmatrix} 0 & X \\ X^\top & 0 \end{pmatrix} \text{ square matrix}$$

$$X_{ij} = \alpha_i \cap \beta_j$$

# Bouillabaisse construction

Set-up:  $(\alpha, \beta)$  is a pair of multicurves (union of disjoint essential simple closed curves) that intersect minimally and fill the surface (complement is a collection of simply polygonal regions). Let  $\Omega$  be the geometric intersection matrix of  $\alpha \cup \beta$ .



$$\alpha = \{\alpha_1, \alpha_2, \dots, \alpha_r\} \quad \beta = \{\beta_1, \beta_2, \dots, \beta_s\}$$

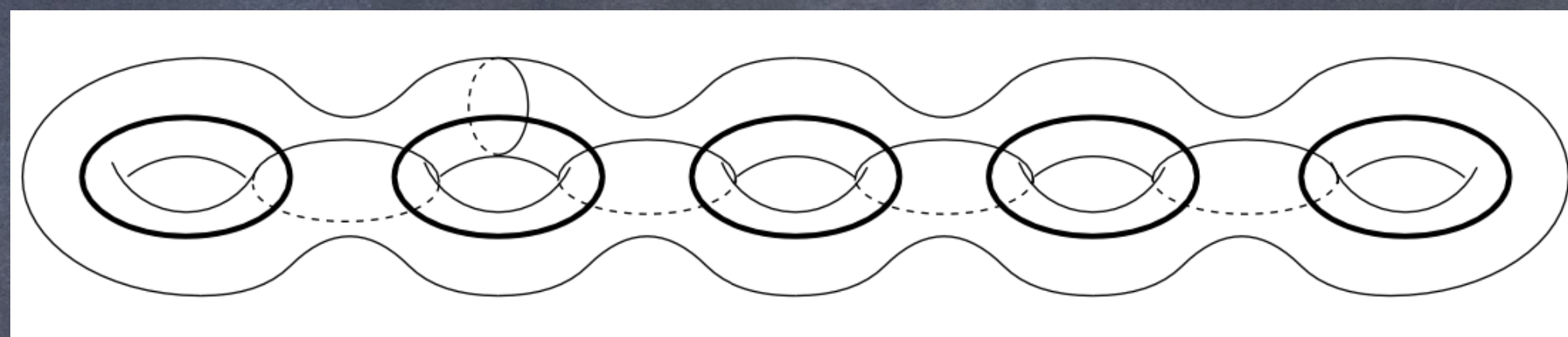
$$\Omega = \begin{pmatrix} 0 & X \\ X^\top & 0 \end{pmatrix} \text{ square matrix}$$

$$X_{ij} = \alpha_i \cap \beta_j$$

Filling is equivalent to  $\Omega$  primitive

# Bouillabaisse construction

Set-up:  $(\alpha, \beta)$  is a pair of multicurves (union of disjoint essential simple closed curves) that intersect minimally and fill the surface (complement is a collection of simply polygonal regions). Let  $\Omega$  be the geometric intersection matrix of  $\alpha \cup \beta$ .



$$\alpha = \{\alpha_1, \alpha_2, \dots, \alpha_r\} \quad \beta = \{\beta_1, \beta_2, \dots, \beta_s\}$$

$$\Omega = \begin{pmatrix} 0 & X \\ X^\top & 0 \end{pmatrix} \text{ square matrix}$$

$$X_{ij} = \alpha_i \cap \beta_j$$

Filling is equivalent to  $\Omega$  primitive

$\mu$  is the PF eigenvalue of  $\Omega$

# Bouillabaisse construction

Theorem (Thurston-Veech, 1988-1989)

$\exists D : \langle T_\alpha, T_\beta \rangle \longrightarrow \mathrm{SL}(2, \mathbb{R})$  such that

$$\bullet T_\alpha \xrightarrow{D} \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \text{ and } T_\beta \xrightarrow{D} \begin{pmatrix} 1 & 0 \\ -\mu & 1 \end{pmatrix}$$

$\bullet \phi$  is pA if and only if  $D(\phi)$  is hyperbolic.

In this case, the dilatation satisfies  $\lambda(\phi) + \lambda(\phi)^{-1} = |\mathrm{tr}(D(\phi))|$

# Bouillabaisse construction

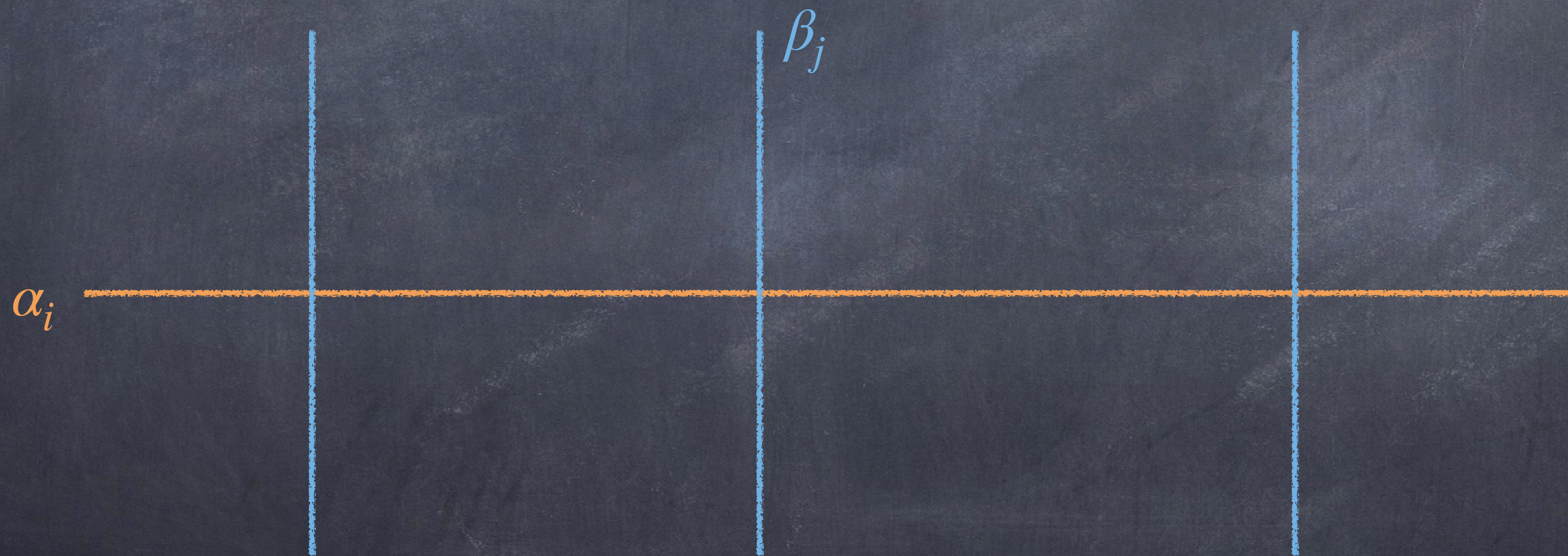
Proof: Construct a flat metric so that the multi-twists are affine. We glue a flat surface  $X$  from rectangles.

$\alpha_i$

---

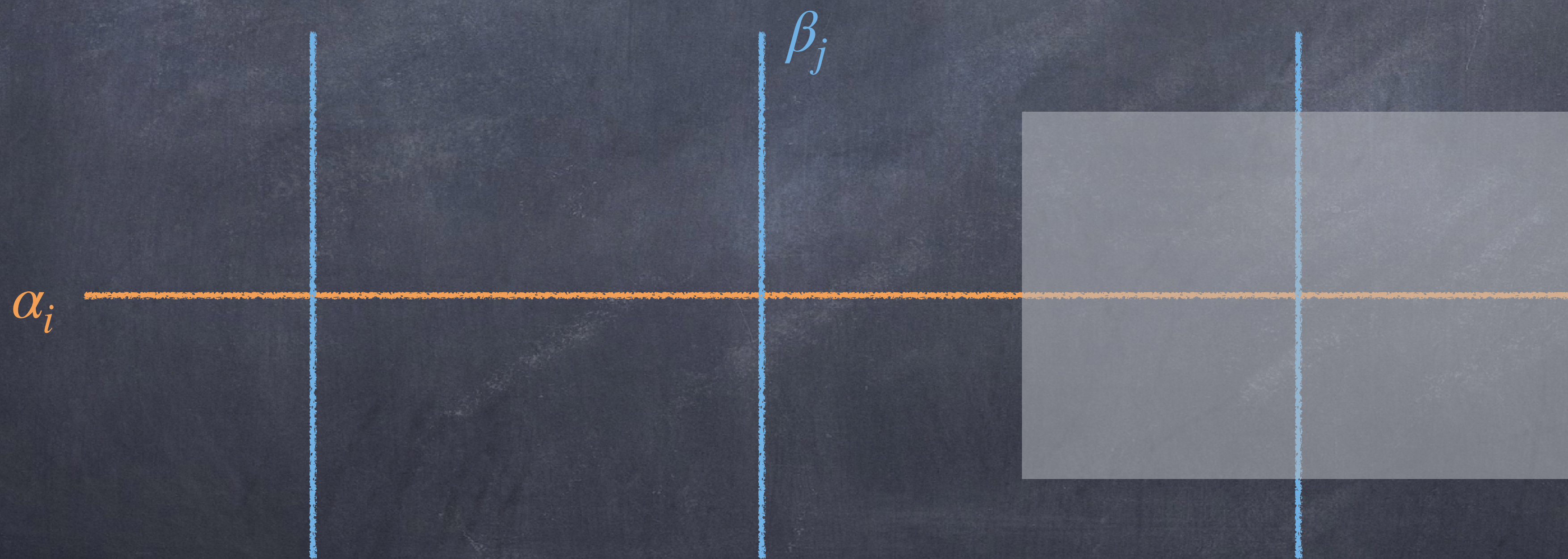
# Bouillabaisse construction

Proof: Construct a flat metric so that the multi-twists are affine. We glue a flat surface  $X$  from rectangles.



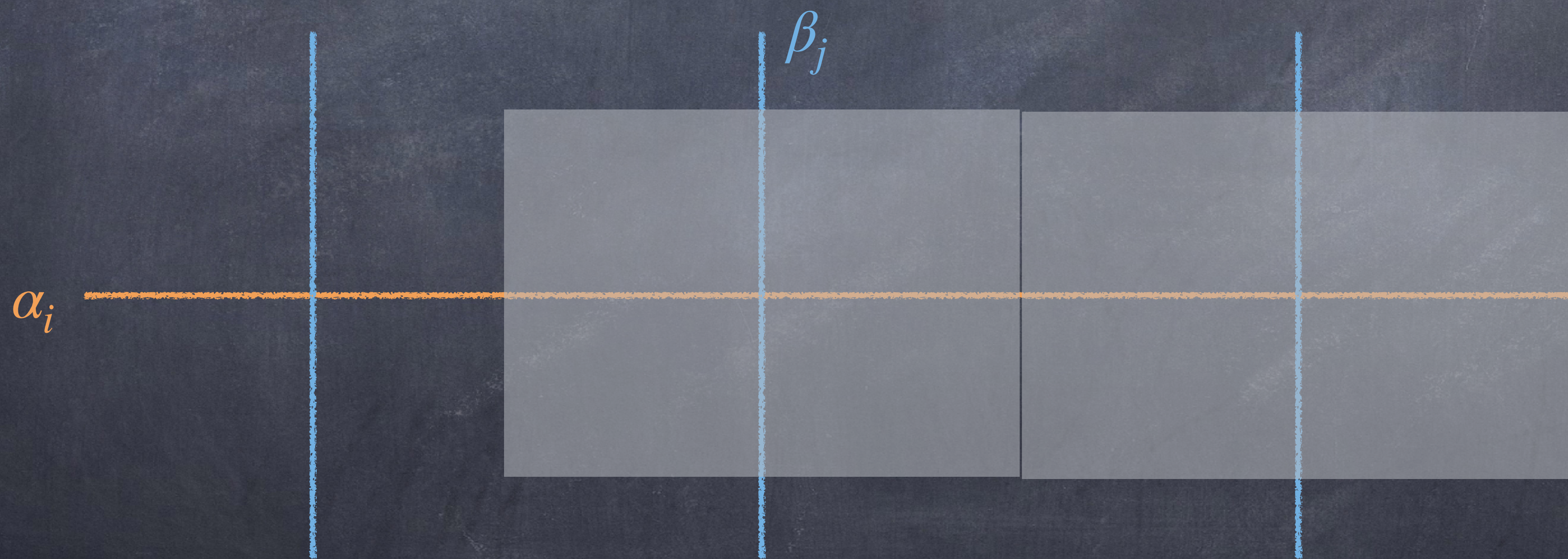
# Bouillabaisse construction

Proof: Construct a flat metric so that the multi-twists are affine. We glue a flat surface  $X$  from rectangles.



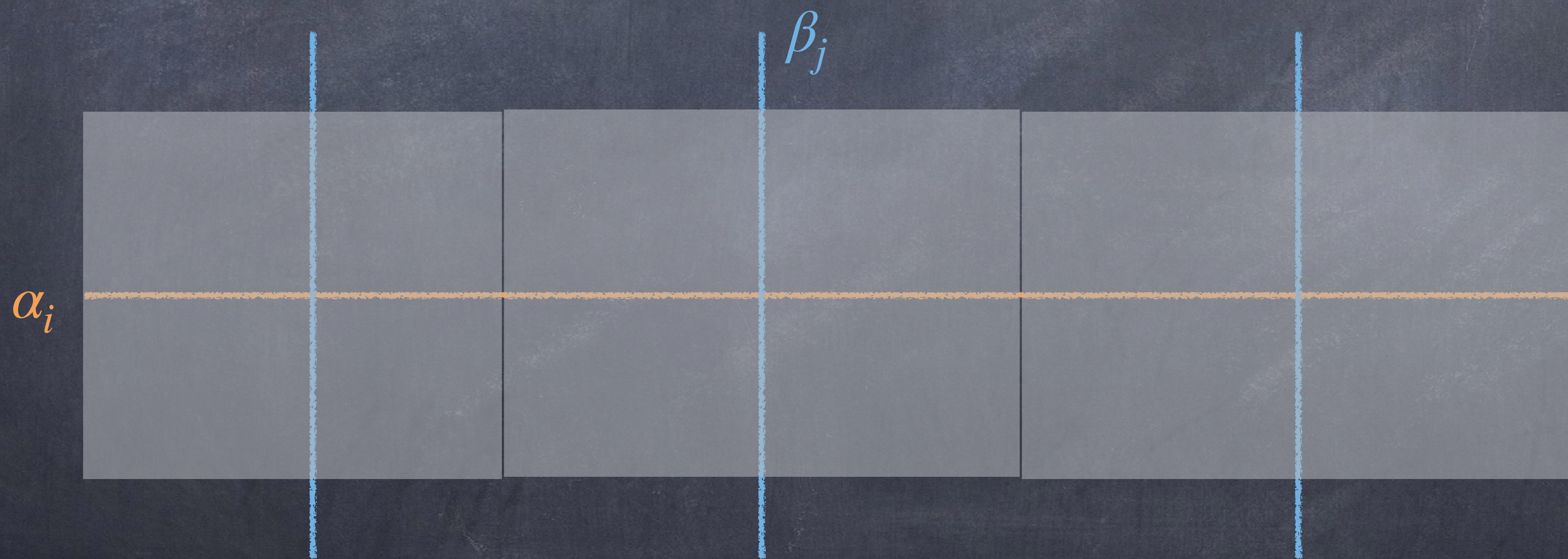
# Bouillabaisse construction

Proof: Construct a flat metric so that the multi-twists are affine. We glue a flat surface  $X$  from rectangles.



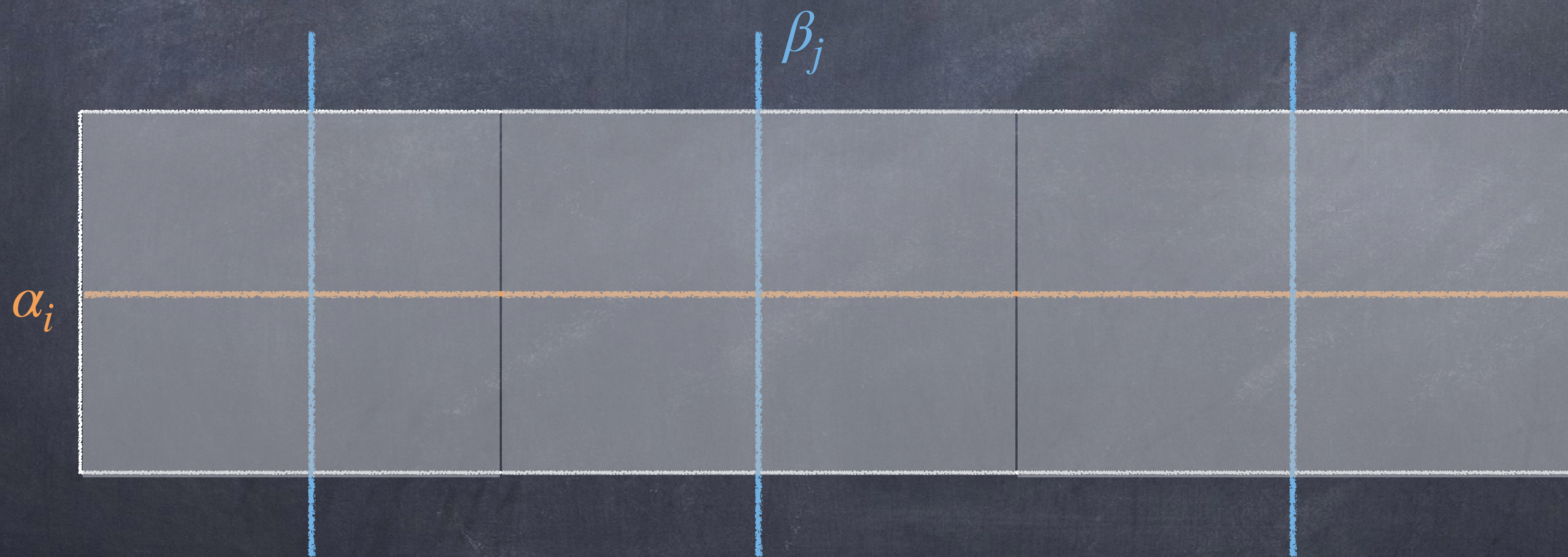
# Bouillabaisse construction

Proof: Construct a flat metric so that the multi-twists are affine. We glue a flat surface  $X$  from rectangles.



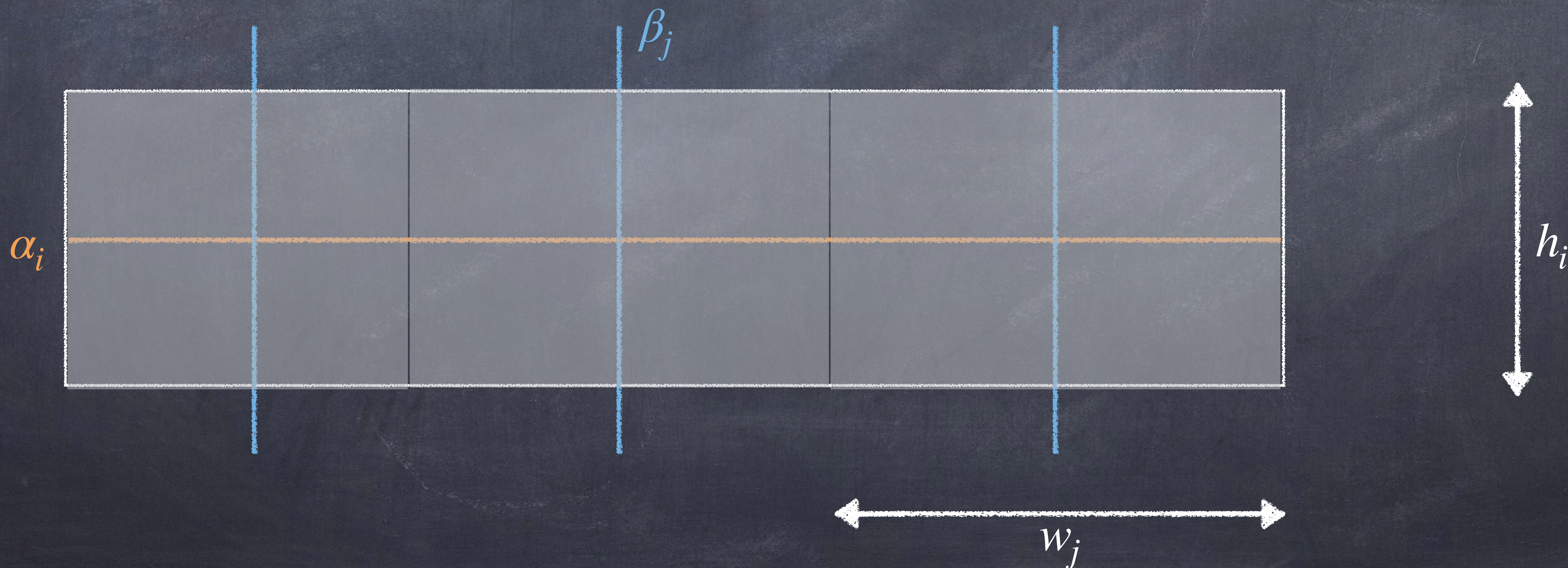
# Bouillabaisse construction

Proof: Construct a flat metric so that the multi-twists are affine. We glue a flat surface  $X$  from rectangles.



# Bouillabaisse construction

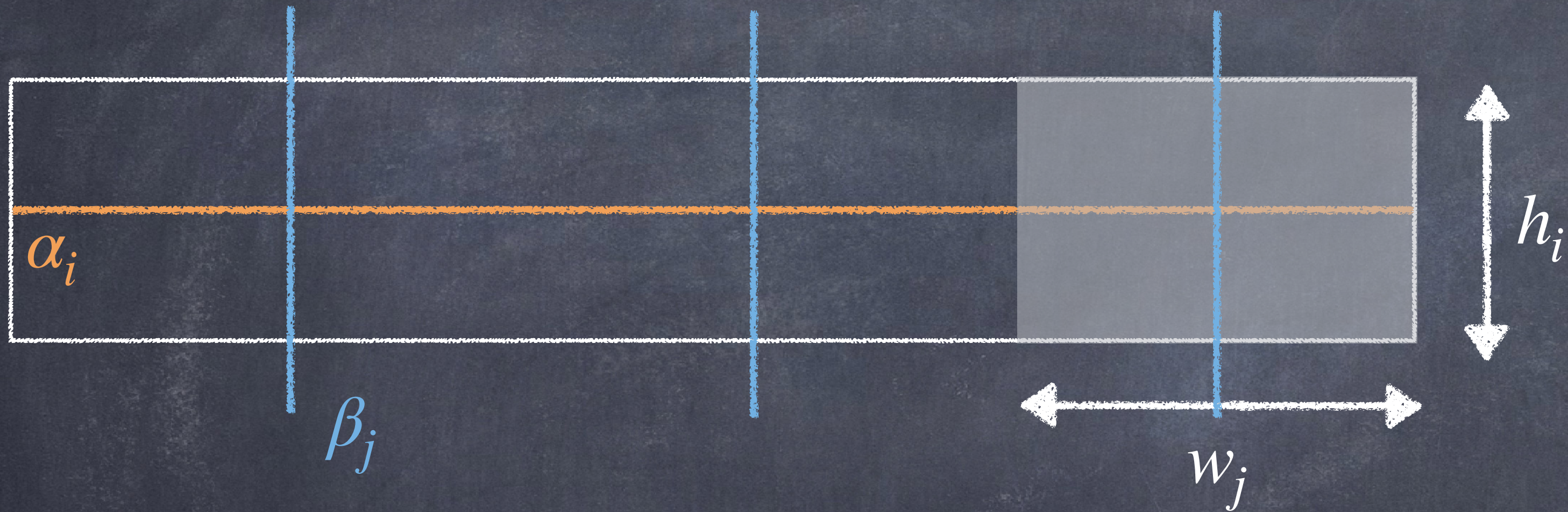
Proof: Construct a flat metric so that the multi-twists are affine. We glue a flat surface  $X$  from rectangles.



# Bouillabaisse construction

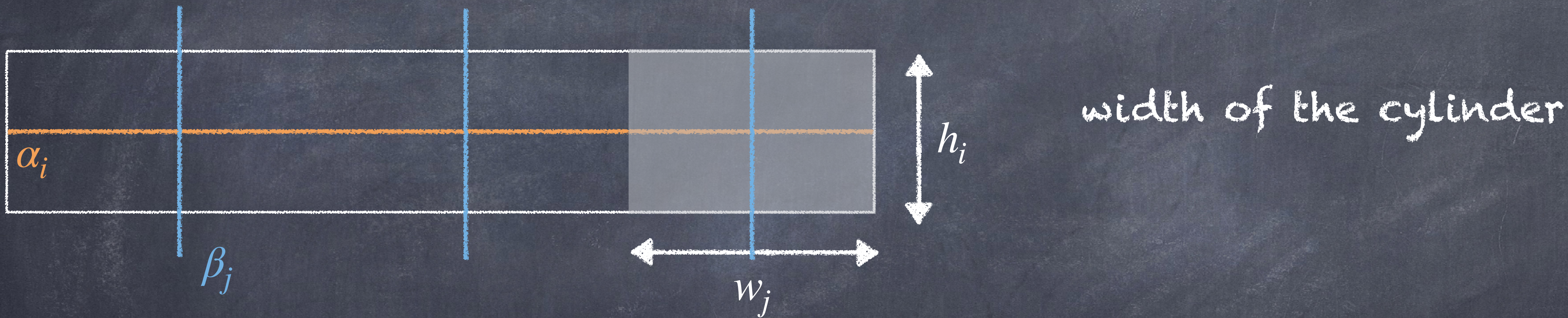
Choose a PF eigenvector  $\begin{pmatrix} h \\ w \end{pmatrix}$  of  $\Omega = \begin{pmatrix} 0 & X \\ X^\top & 0 \end{pmatrix}$  for  $\mu$

$$\begin{cases} Xw = \mu h \\ X^\top h = \mu w \end{cases}$$



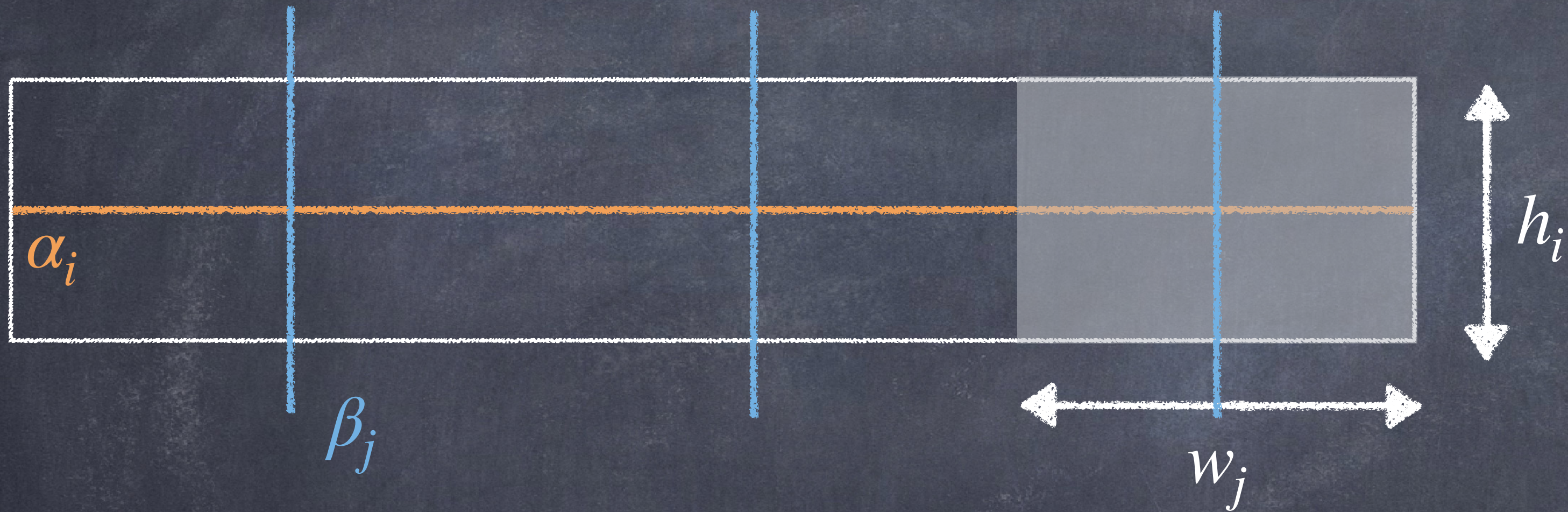
# Bouillabaisse construction

Choose a PF eigenvector  $\begin{pmatrix} h \\ w \end{pmatrix}$  of  $\Omega = \begin{pmatrix} 0 & X \\ X^T & 0 \end{pmatrix}$  for  $\mu$   $\begin{cases} Xw = \mu h \\ X^T h = \mu w \end{cases}$



# Bouillabaisse construction

Choose a PF eigenvector  $\begin{pmatrix} h \\ w \end{pmatrix}$  of  $\Omega = \begin{pmatrix} 0 & X \\ X^T & 0 \end{pmatrix}$  for  $\mu$   $\begin{cases} Xw = \mu h \\ X^T h = \mu w \end{cases}$

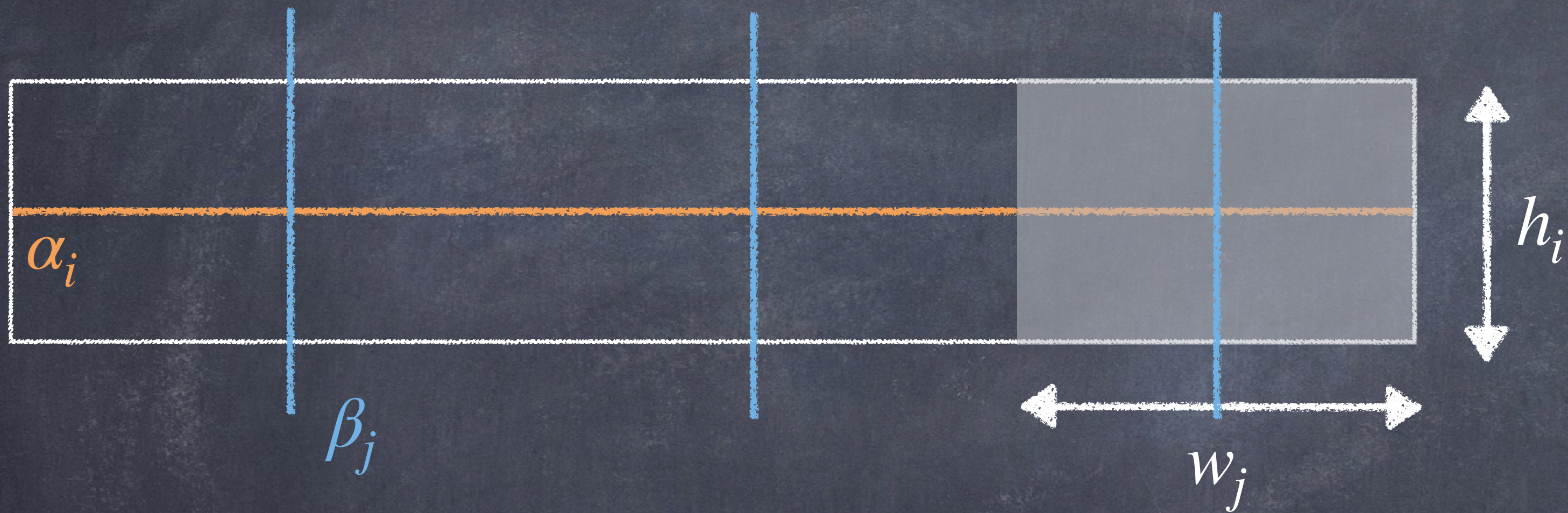


width of the cylinder

$$\sum_j \alpha_i \cap \beta_j \cdot w_j = (Xw)_i = \mu h_i$$

# Bouillabaisse construction

Choose a PF eigenvector  $\begin{pmatrix} h \\ w \end{pmatrix}$  of  $\Omega = \begin{pmatrix} 0 & X \\ X^\top & 0 \end{pmatrix}$  for  $\mu$   $\begin{cases} Xw = \mu h \\ X^\top h = \mu w \end{cases}$



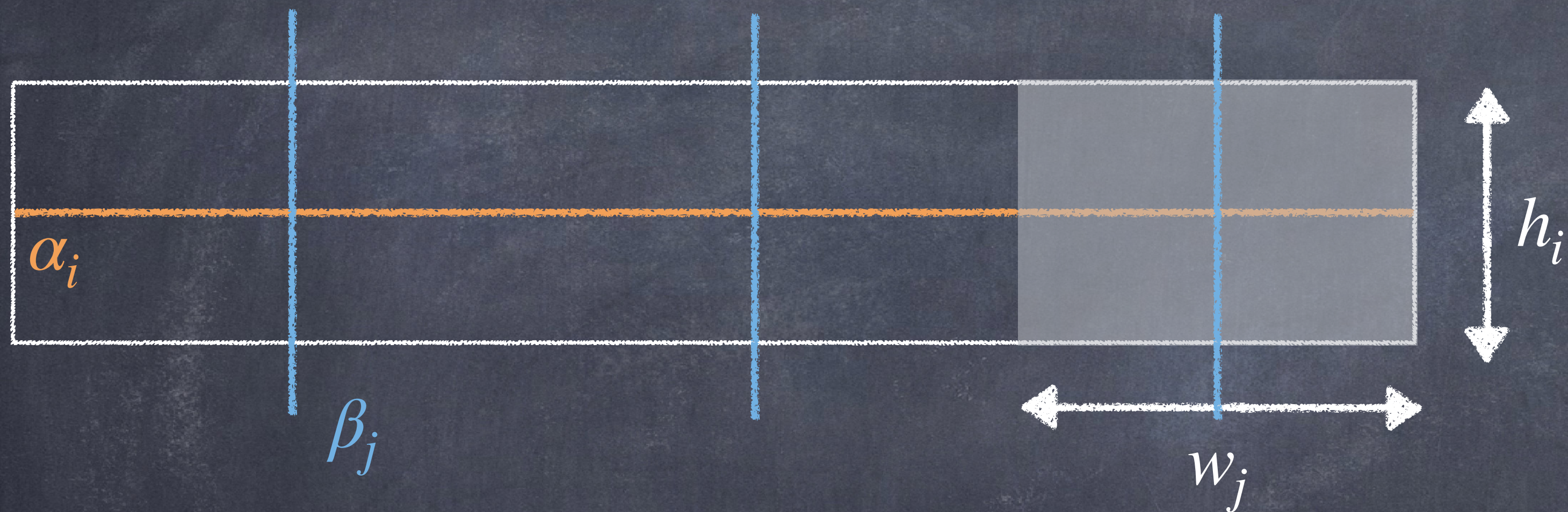
width of the cylinder

$$\sum_j \alpha_i \cap \beta_j \cdot w_j = (Xw)_i = \mu h_i$$

$$T_{\alpha_i} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + \mu y \pmod{\mu} \\ y \end{pmatrix}$$

# Bouillabaisse construction

Choose a PF eigenvector  $\begin{pmatrix} h \\ w \end{pmatrix}$  of  $\Omega = \begin{pmatrix} 0 & X \\ X^\top & 0 \end{pmatrix}$  for  $\mu$   $\begin{cases} Xw = \mu h \\ X^\top h = \mu w \end{cases}$



width of the cylinder

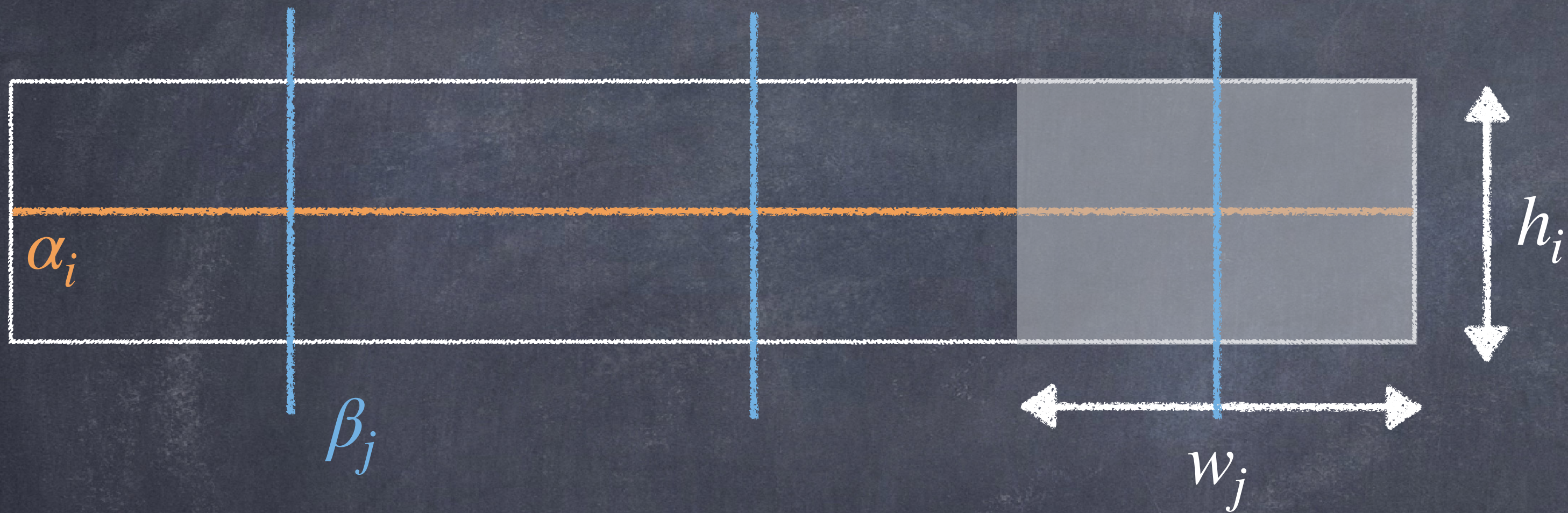
$$\sum_j \alpha_i \cap \beta_j \cdot w_j = (Xw)_i = \mu h_i$$

$$T_{\alpha_i} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + \mu y \pmod{\mu} \\ y \end{pmatrix}$$

$$DT_{\alpha_i} = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$$

# Bouillabaisse construction

Choose a PF eigenvector  $\begin{pmatrix} h \\ w \end{pmatrix}$  of  $\Omega = \begin{pmatrix} 0 & X \\ X^\top & 0 \end{pmatrix}$  for  $\mu$   $\begin{cases} Xw = \mu h \\ X^\top h = \mu w \end{cases}$



width of the cylinder

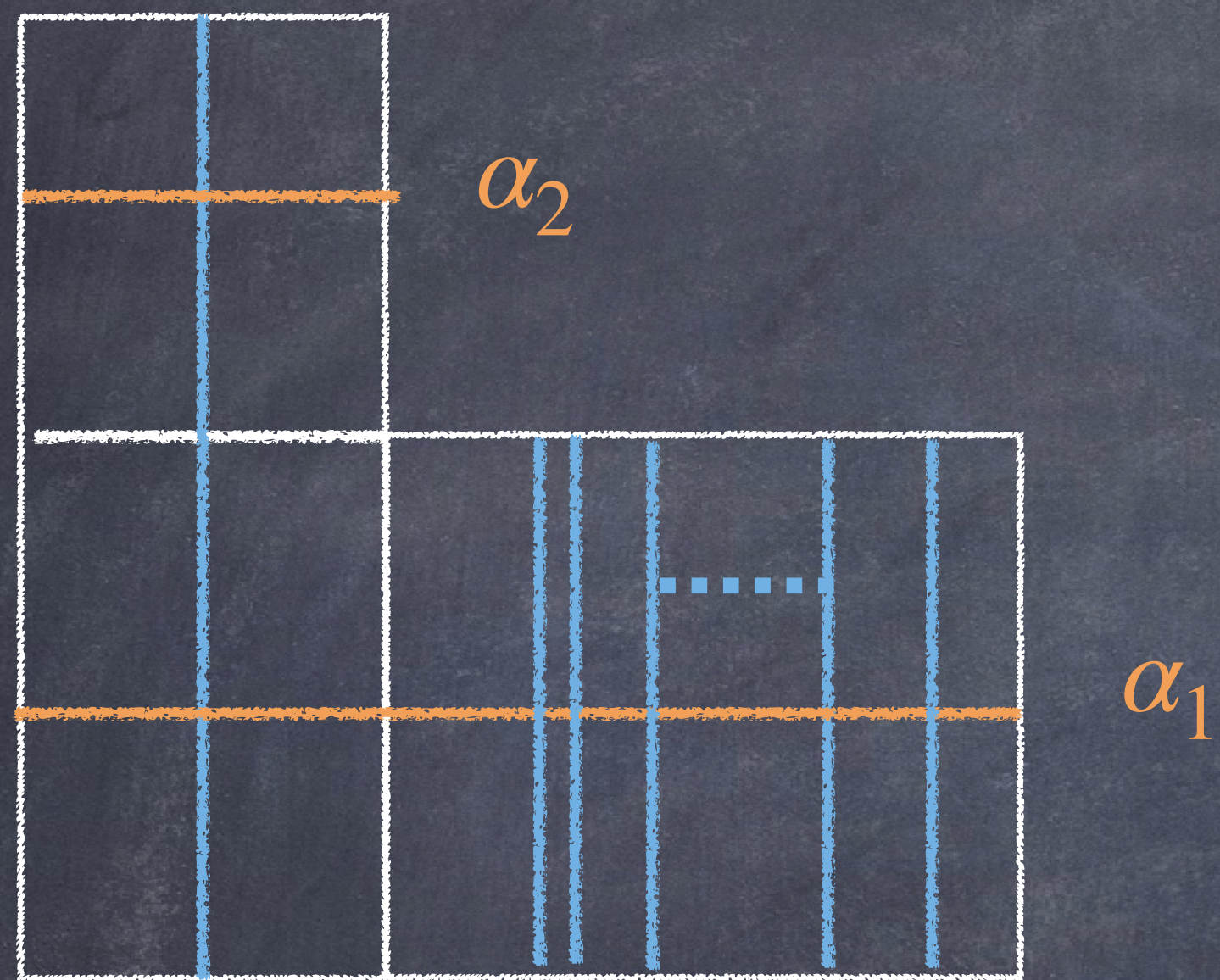
$$\sum_j \alpha_i \cap \beta_j \cdot w_j = (Xw)_i = \mu h_i$$

$$T_{\alpha_i} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + \mu y \pmod{\mu} \\ y \end{pmatrix}$$

$$DT_{\alpha_i} = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$$

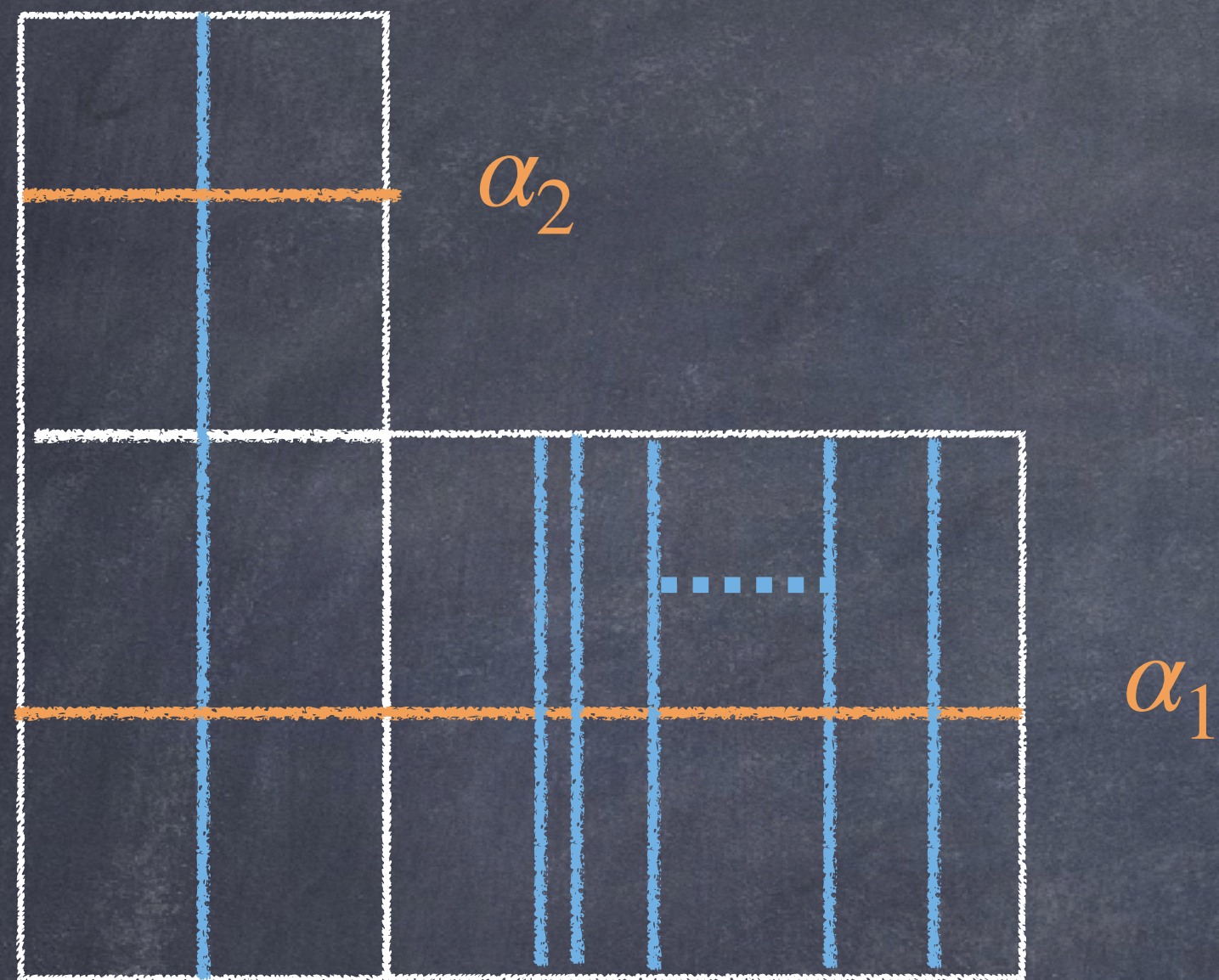
So the multi-twist  $T_\alpha$  is affine!

# An example



$\beta_1$      $\beta_j, j = 2, \dots, u$

# An example

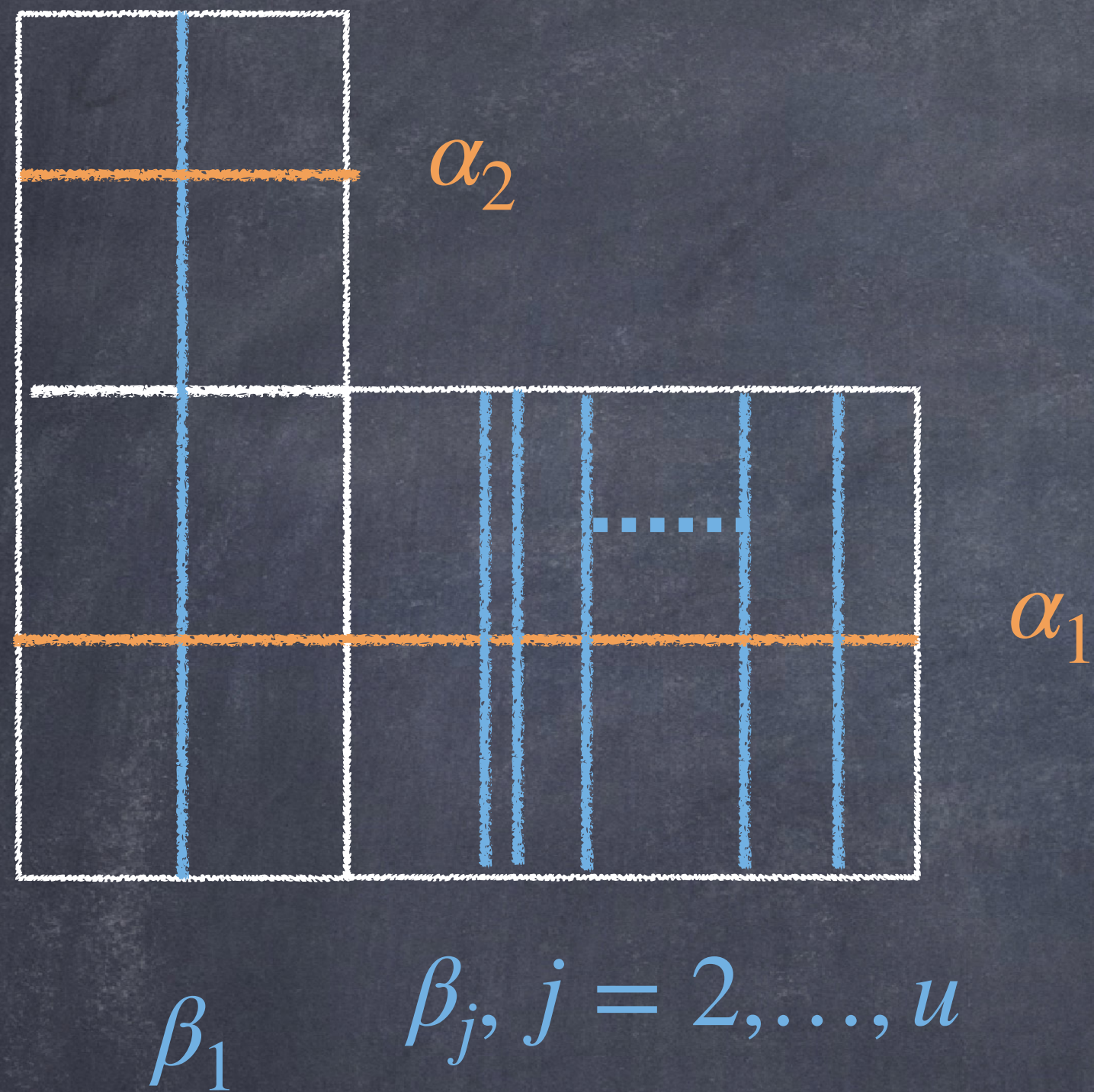


$$\Omega = \begin{pmatrix} 0 & X \\ X^\top & 0 \end{pmatrix}$$

$$X = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

$\beta_1$      $\beta_j, j = 2, \dots, u$

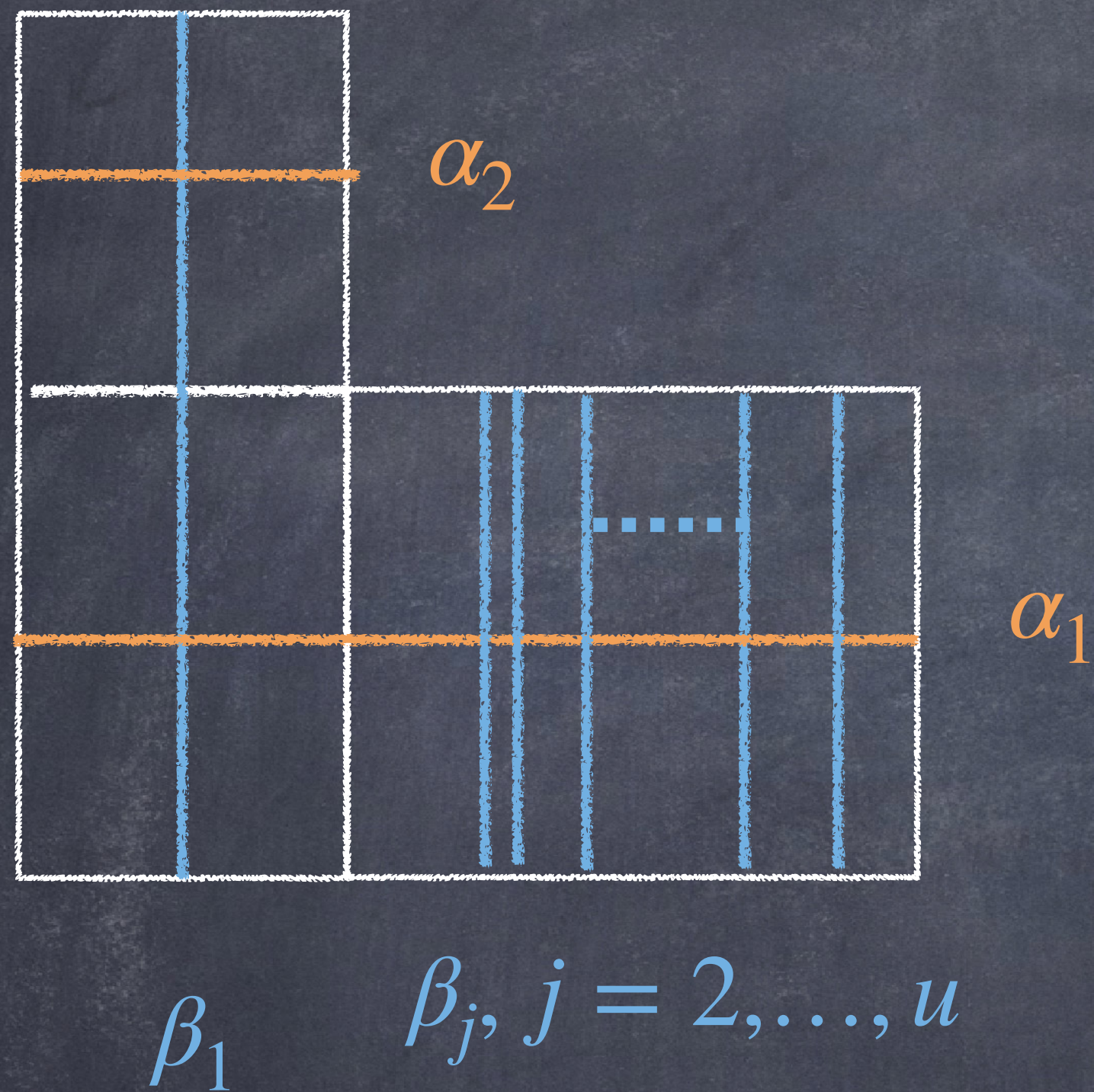
# An example



$$\Omega = \begin{pmatrix} 0 & X \\ X^\top & 0 \end{pmatrix} \quad X = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

$$\mu^2 \text{ eigenvalue of } \Omega^2 = \begin{pmatrix} XX^\top & 0 \\ 0 & X^\top X \end{pmatrix}$$

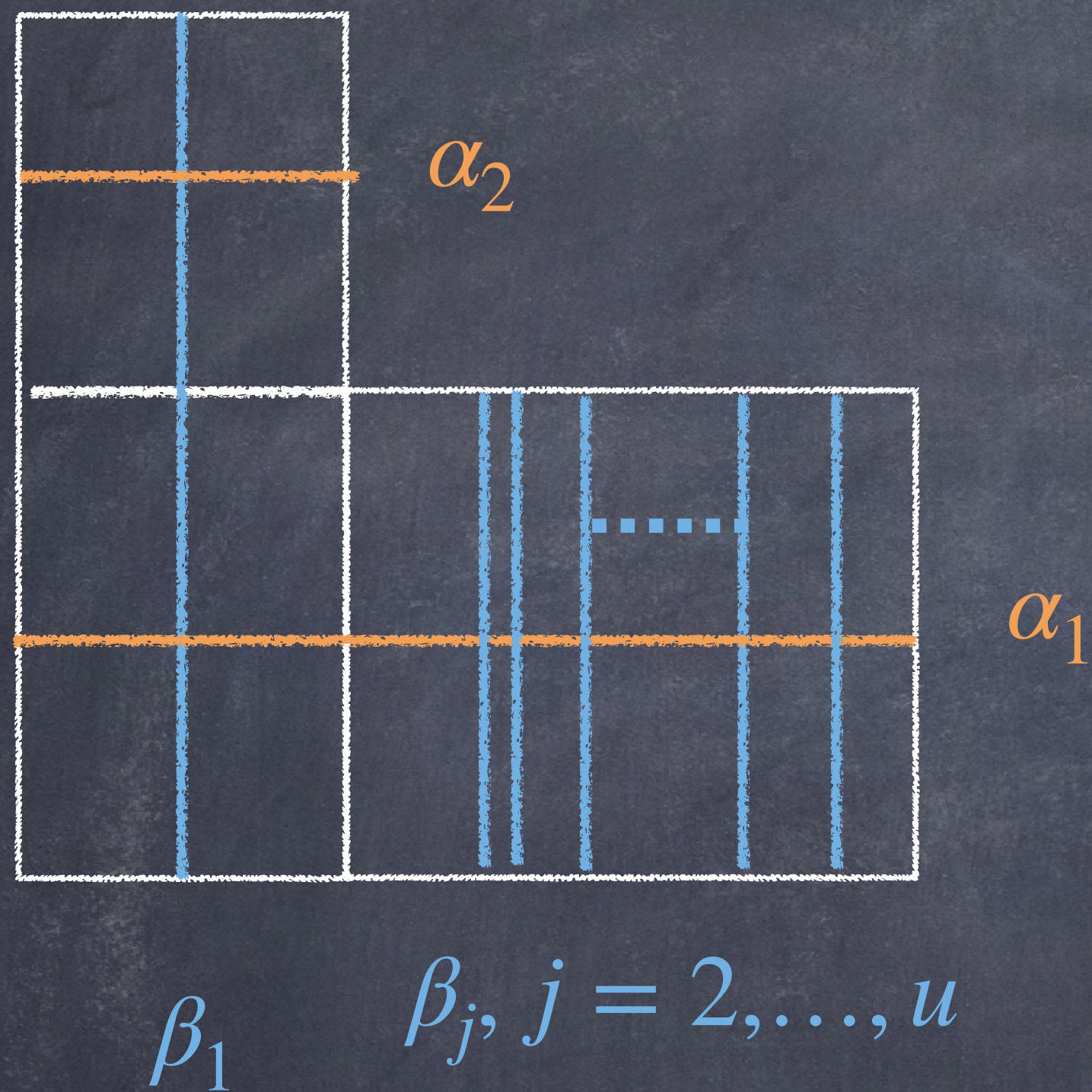
# An example



$$\Omega = \begin{pmatrix} 0 & X \\ X^\top & 0 \end{pmatrix} \quad X = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

$$\mu^2 \text{ eigenvalue of } \Omega^2 = \begin{pmatrix} XX^\top & 0 \\ 0 & X^\top X \end{pmatrix} \quad XX^\top = \begin{pmatrix} 1 & 1 \\ 1 & u \end{pmatrix}$$

# An example



$$\Omega = \begin{pmatrix} 0 & X \\ X^\top & 0 \end{pmatrix} \quad X = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

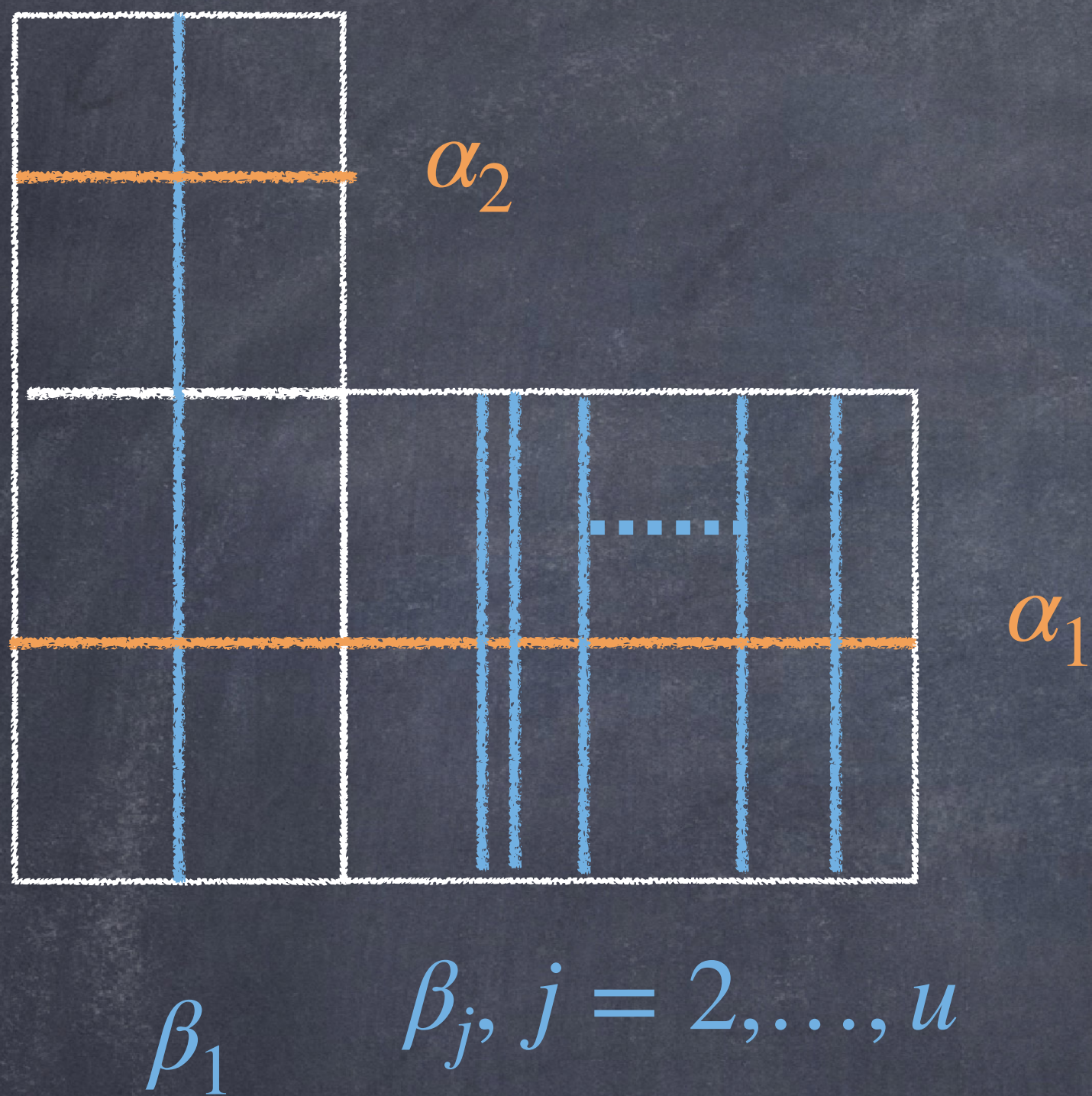
$$\mu^2 \text{ eigenvalue of } \Omega^2 = \begin{pmatrix} XX^\top & 0 \\ 0 & X^\top X \end{pmatrix} \quad XX^\top = \begin{pmatrix} 1 & 1 \\ 1 & u \end{pmatrix}$$

$$\mu^2 - (u+1)\mu + u - 1 = 0$$

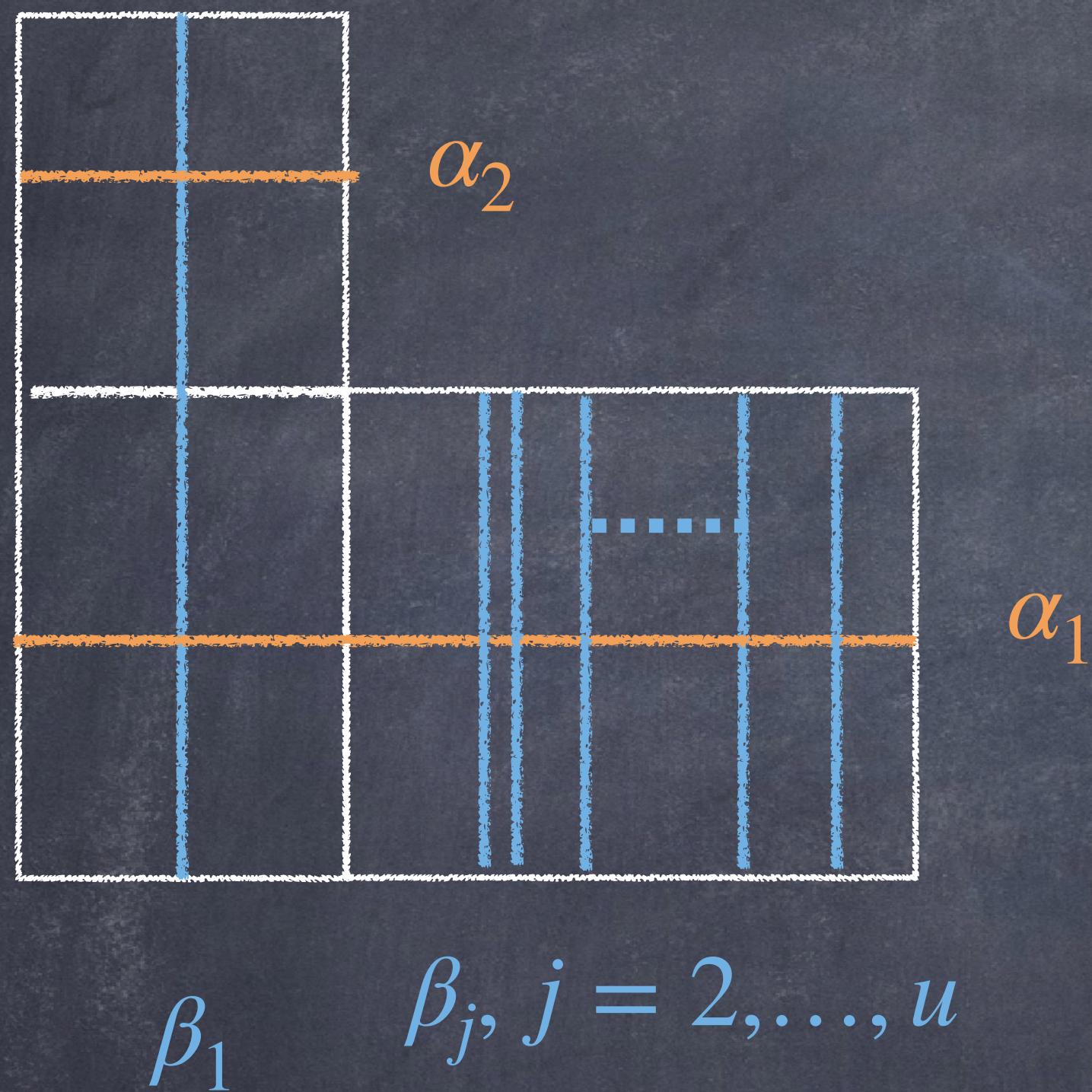
# An example

$$\mu^2 - (u + 1)\mu + u - 1 = 0$$

$$\text{For } u \geq 4, \operatorname{tr} T_\alpha T_\beta = 2 - \mu^2 < -2$$



# An example

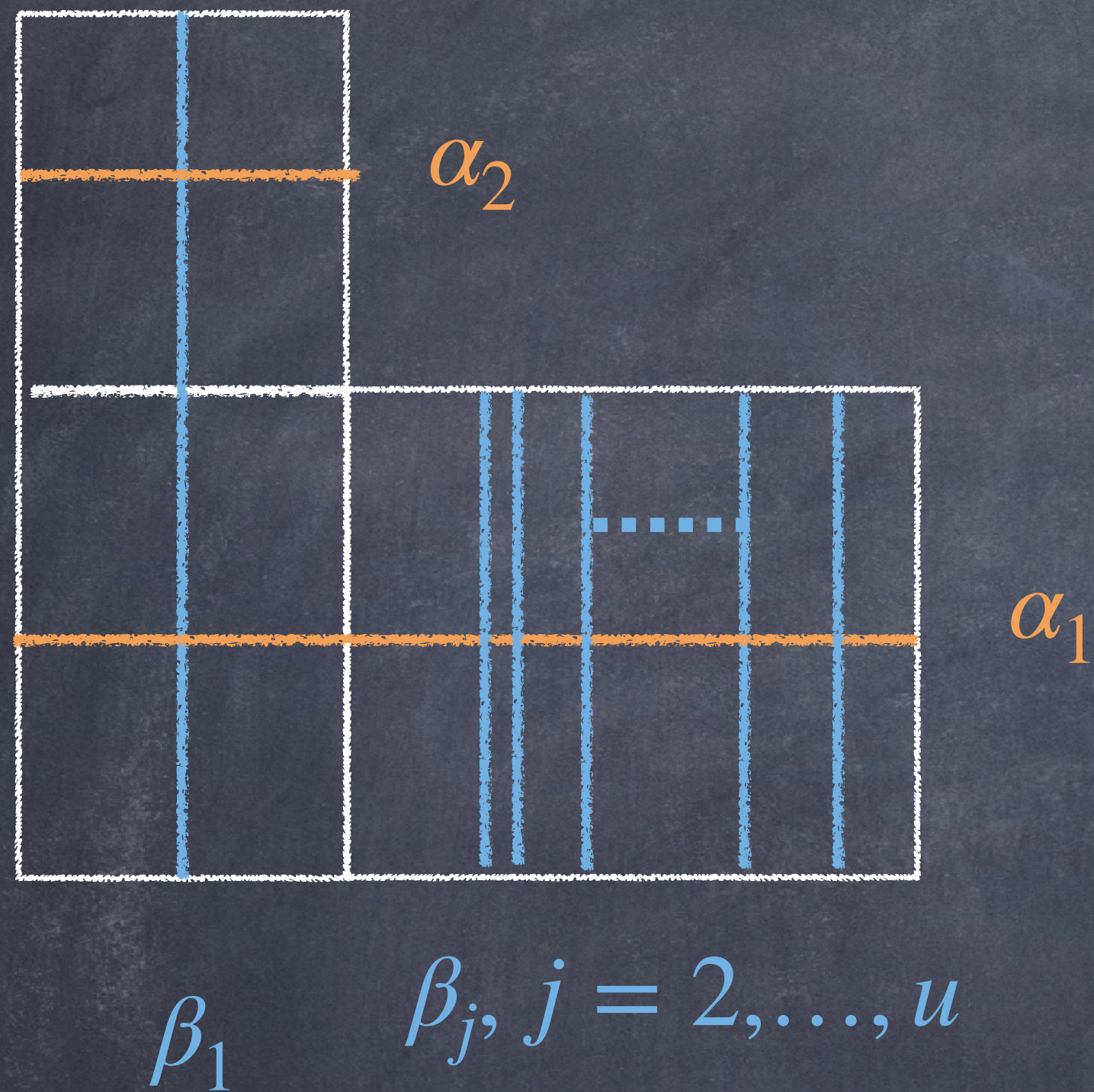


$$\mu^2 - (u + 1)\mu + u - 1 = 0$$

$$\text{For } u \geq 4, \text{tr } T_\alpha T_\beta = 2 - \mu^2 < -2$$

$$T_\alpha T_\beta \text{ is pA and } \lambda + \lambda^{-1} = \mu^2 - 2$$

# An example



$$\mu^2 - (u + 1)\mu + u - 1 = 0$$

$$\text{For } u \geq 4, \text{tr } T_\alpha T_\beta = 2 - \mu^2 < -2$$

$$T_\alpha T_\beta \text{ is pA and } \lambda + \lambda^{-1} = \mu^2 - 2$$

The smallest  $\lambda$  is at  $u = 4$ , where  $\lambda^4 - \lambda^3 - \lambda^2 - \lambda + 1 = 0$

# Bouillabaisse construction

Not all pseudo-Anosovs are obtained this way!

There are several dynamical and algebraic constraints

# Bouillabaisse construction

Not all pseudo-Anosovs are obtained this way!

There are several dynamical and algebraic constraints

We may ask about several possible restrictions:

1. What algebraic numbers can be realized?
2. What degrees are possible, for a given genus?

# Bouillabaisse construction

Not all pseudo-Anosovs are obtained this way!

There are several dynamical and algebraic constraints

We may ask about several possible restrictions:

1. What algebraic numbers can be realized?
2. What degrees are possible, for a given genus?

Hooper-Thurston-Veech construction for big mapping class groups (2015)

Bouillabaisse construction

# Bouillabaisse construction

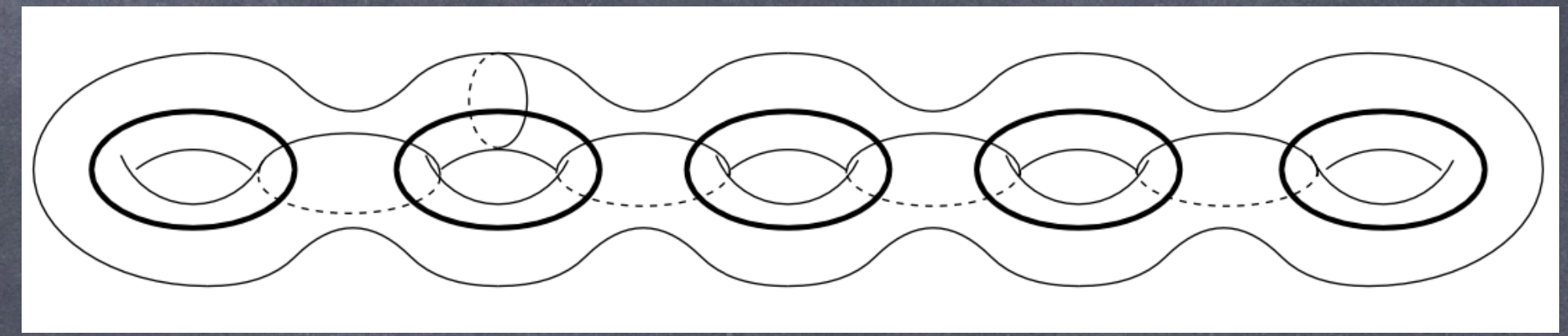
Theorem (Leininger, 2003)

$$\lambda(\phi) \geq \text{Lehmer's number} \simeq 1.17628 > 1$$

# Bouillabaisse construction

Theorem (Leininger, 2003)

$\lambda(\phi) \geq$  Lehmer's number  $\simeq 1.17628 > 1$

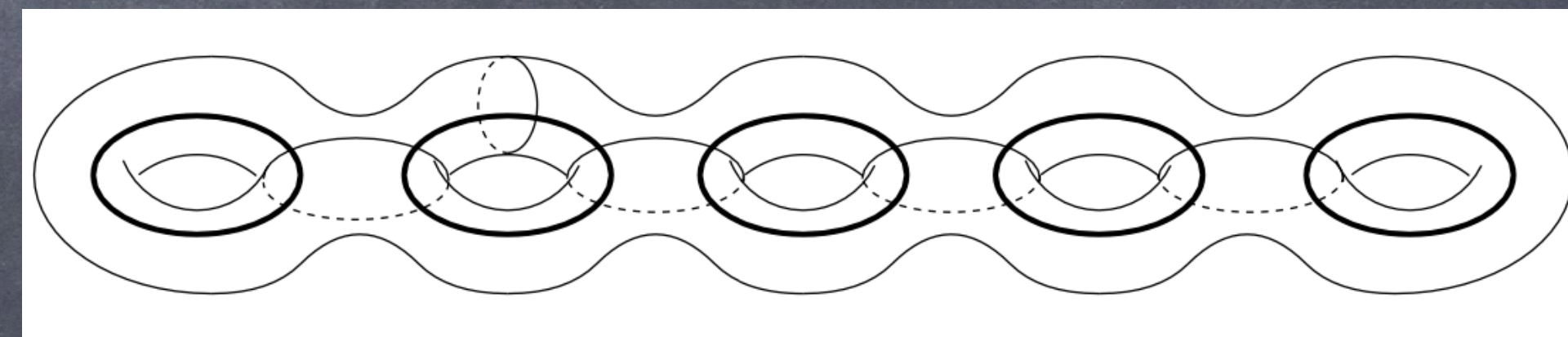


$$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1 = 0$$

# Bouillabaisse construction

Theorem (Leininger, 2003)

$\lambda(\phi) \geq$  Lehmer's number  $\simeq 1.17628 > 1$



$$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1 = 0$$

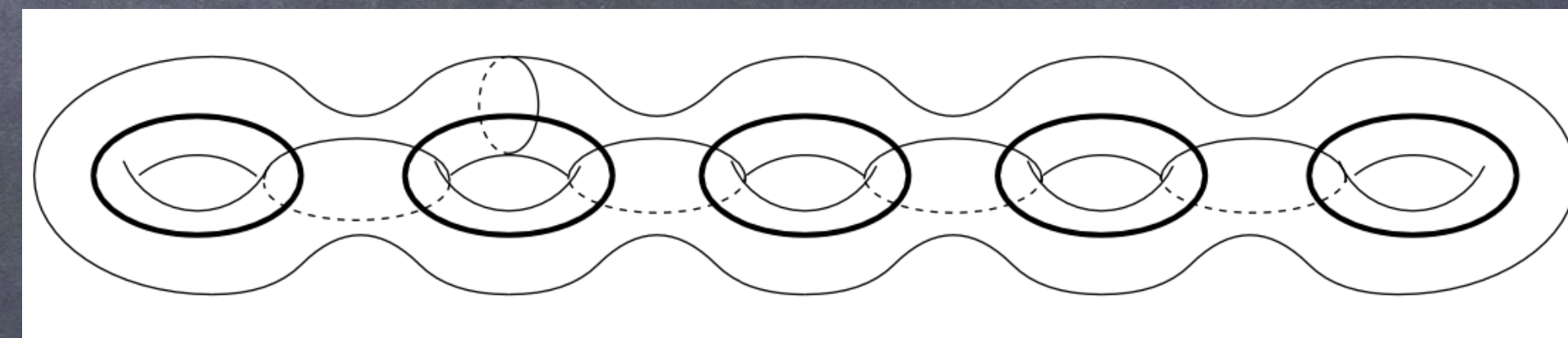
Theorem (Hubert-L, 2007)

$\lambda^\sigma \in \mathbb{R} \cup S^1$  (Galois conjugates are unimodular/real)

# Bouillabaisse construction

Theorem (Leininger, 2003)

$\lambda(\phi) \geq$  Lehmer's number  $\simeq 1.17628 > 1$



$$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1 = 0$$

Theorem (Hubert-L, 2007)

$\lambda^\sigma \in \mathbb{R} \cup S^1$  (Galois conjugates are unimodular/real)

Theorem (Liechti-Pankau, 2022)

$\lambda$  bi-Perron and  $\lambda^\sigma \in \mathbb{R} \cup S^1$  then there exists a bouillabaisse pA  $\phi$  and  $k \in \mathbb{N}$  such that  $\lambda(\phi) = \lambda^k$

# Degrees realization

It is well known that the upper bound on the algebraic degree of stretch factor is  $6g-6$

# Degrees realization

It is well known that the upper bound on the algebraic degree of stretch factor is  $6g-6$

This was first established in Thurston's seminal 1988 bulletin paper, and claims, without proof, that

« the examples of Theorem 7 show that this bound is sharp »

The referenced examples are exactly the bouillabaisse examples

# Degrees realization

It is well known that the upper bound on the algebraic degree of stretch factor is  $6g-6$

This was first established in Thurston's seminal 1988 bulletin paper, and claims, without proof, that

« the examples of Theorem 7 show that this bound is sharp »

The referenced examples are exactly the bouillabaisse examples

Margalit remarked in 2011 that no proof of Thurston's claim has ever been published!

# Degrees realization

It is well known that the upper bound on the algebraic degree of stretch factor is  $6g-6$

This was first established in Thurston's seminal 1988 bulletin paper, and claims, without proof, that

« the examples of Theorem 7 show that this bound is sharp »

The referenced examples are exactly the bouillabaisse examples

Margalit remarked in 2011 that no proof of Thurston's claim has ever been published!

Strenner (2018) determined the set of possible degrees (by using Penner's construction)

# Degrees realization

Theorem (L-Liechti, 2025)

For  $g \geq 2$ , every even integer between 2 and  $6g - 6$  arises as the stretch factor degree of a bouillabaisse mapping class in the Torelli group  $T(S_g)$

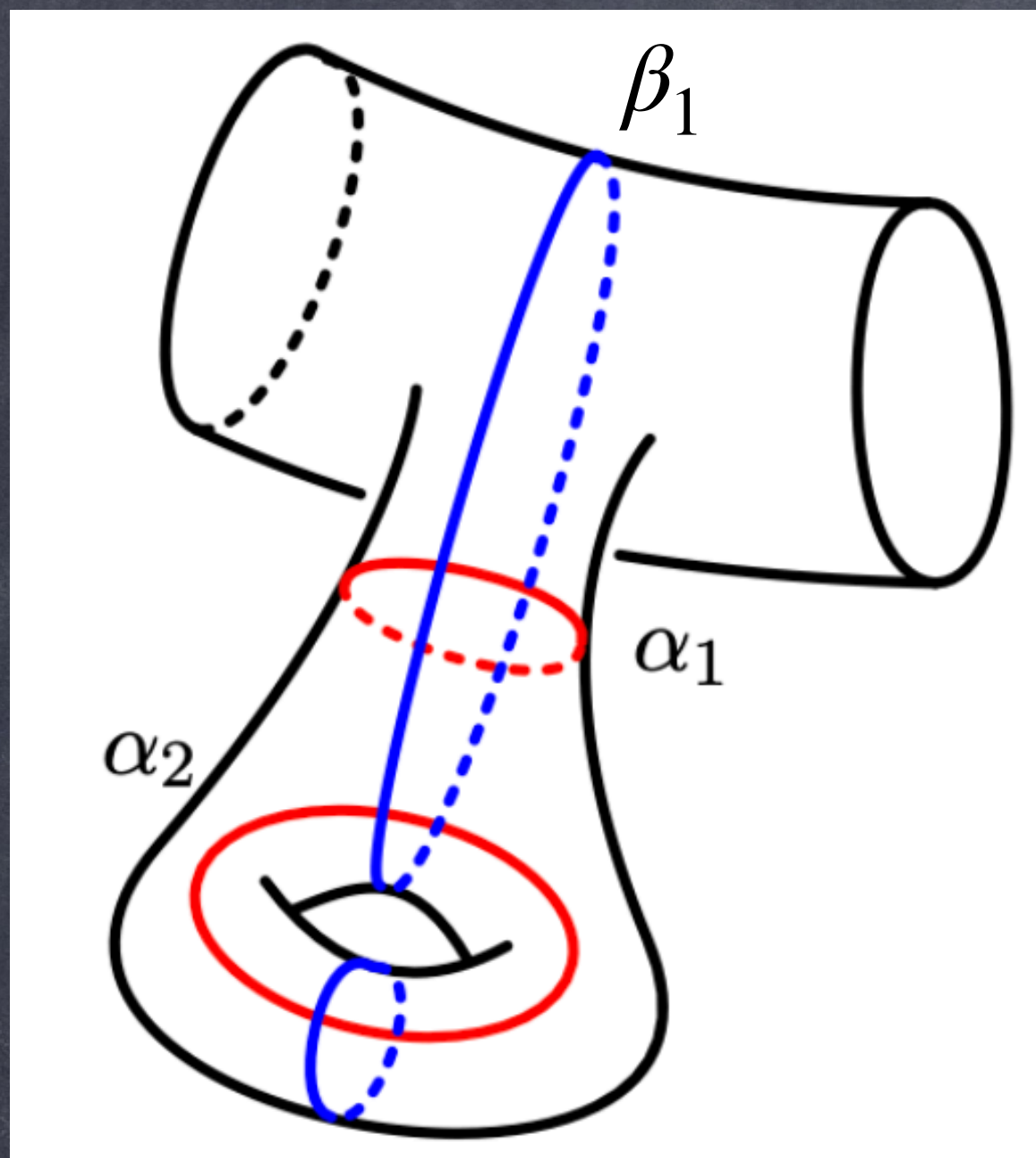
# Degrees realization

Theorem (L-Liechti, 2025)

For  $g \geq 2$ , every even integer between 2 and  $6g - 6$  arises as the stretch factor degree of a bouillabaisse mapping class in the Torelli group  $T(S_g)$

We proved a version for connected components of strata (2024)

# Warm-up / Wake-up



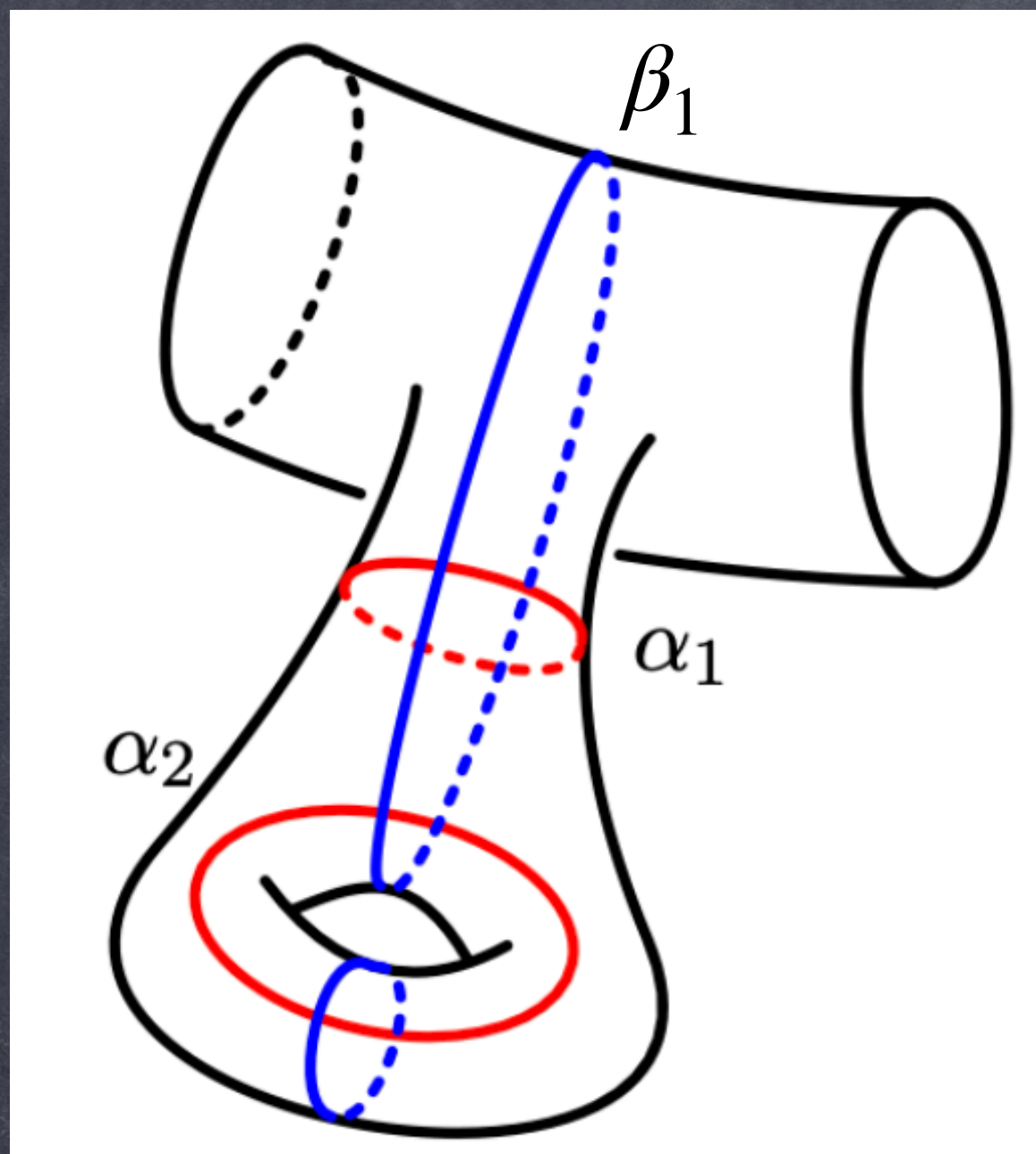
$u - 1$  copies of  $\beta_j$

genus  $g = 1$  with  
2 boundary components

$$XX^T = \begin{pmatrix} 4 & 2 \\ 2 & u \end{pmatrix}$$

$$X = (|\alpha_i \cap \beta_j|)_{i,j}$$

# Warm-up / Wake-up



$u - 1$  copies of  $\beta_j$

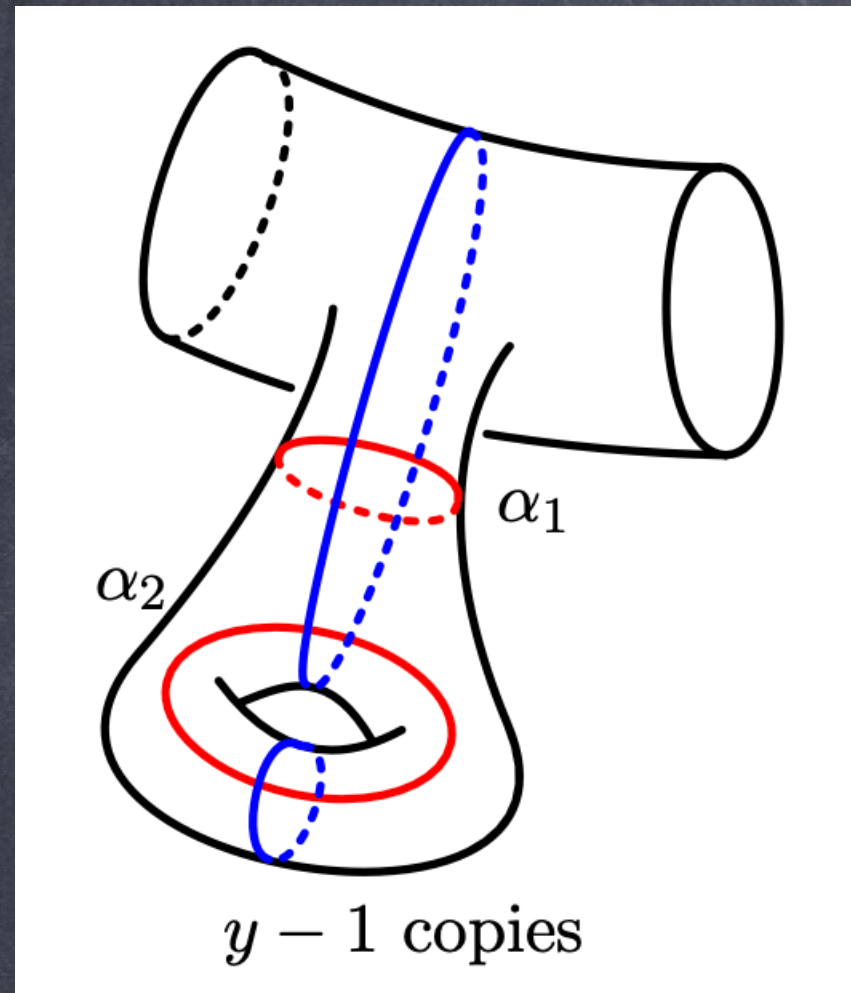
genus  $g = 1$  with  
2 boundary components

$$XX^T = \begin{pmatrix} 4 & 2 \\ 2 & u \end{pmatrix}$$

$$X = (|\alpha_i \cap \beta_j|)_{i,j}$$

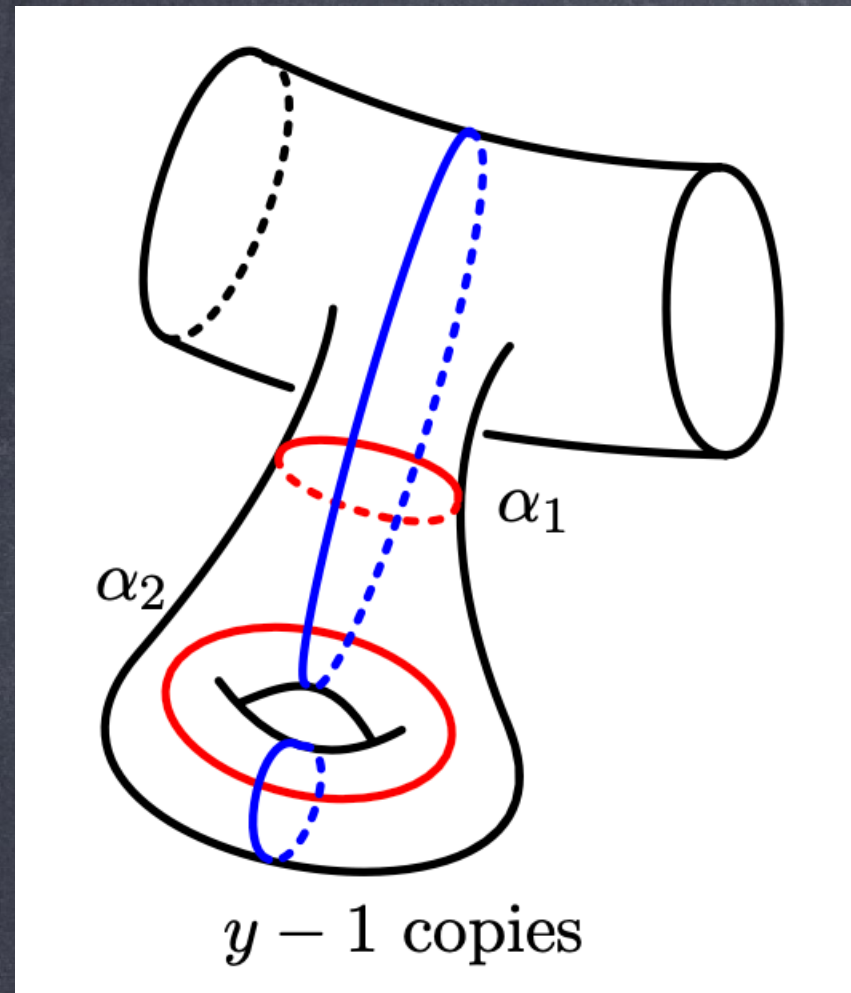
$(\alpha, \beta)$  pair of filling multicurves such that  $\chi_{XX^T} \in \mathbb{Z}[t]$  is irreducible and of degree  $2 = 3g - 1$

genus  $g > 1$

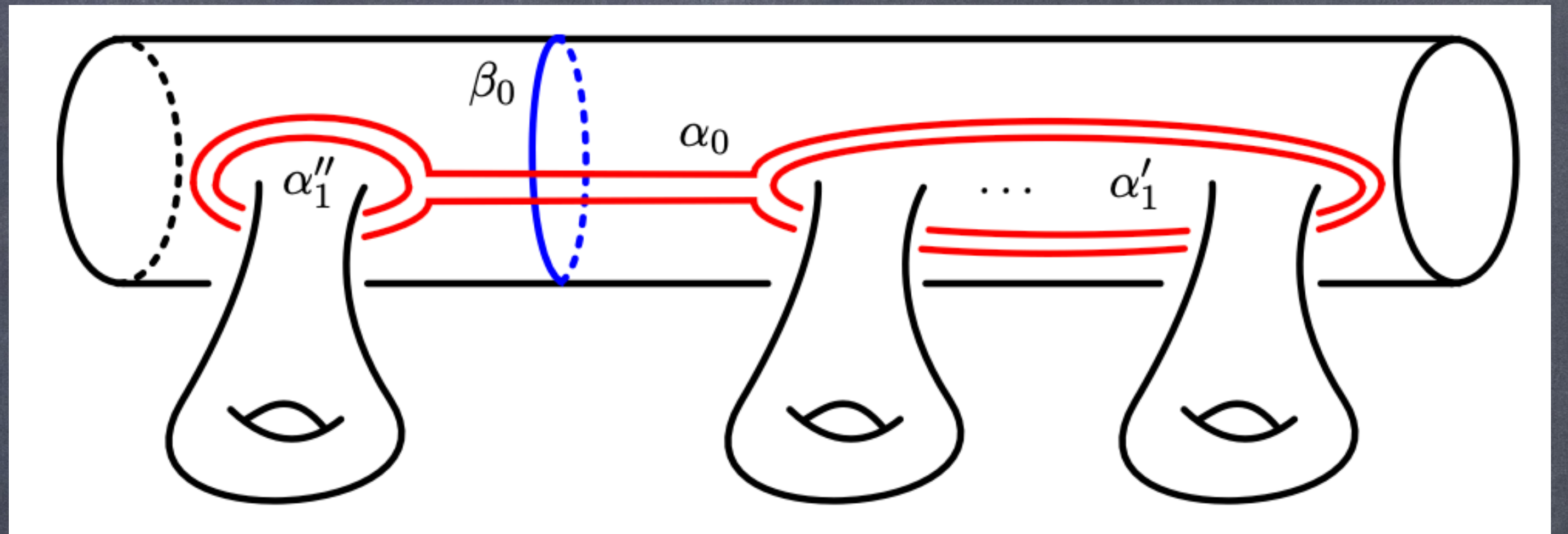


Initial case

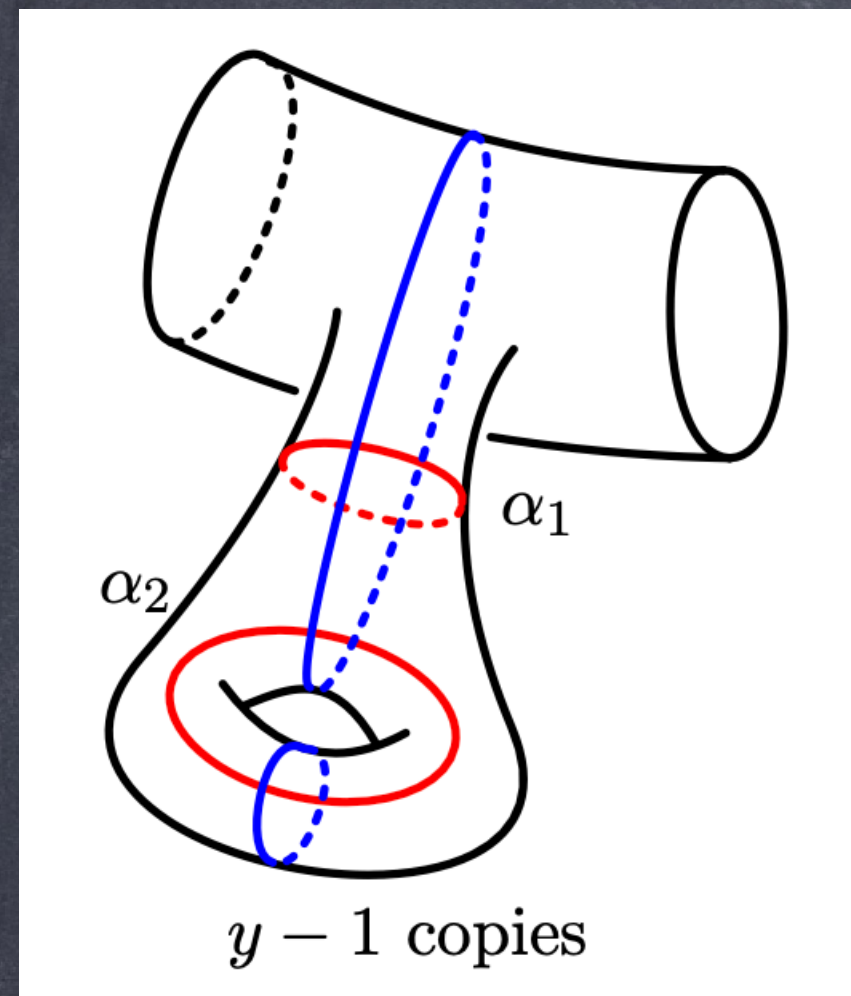
genus  $g > 1$



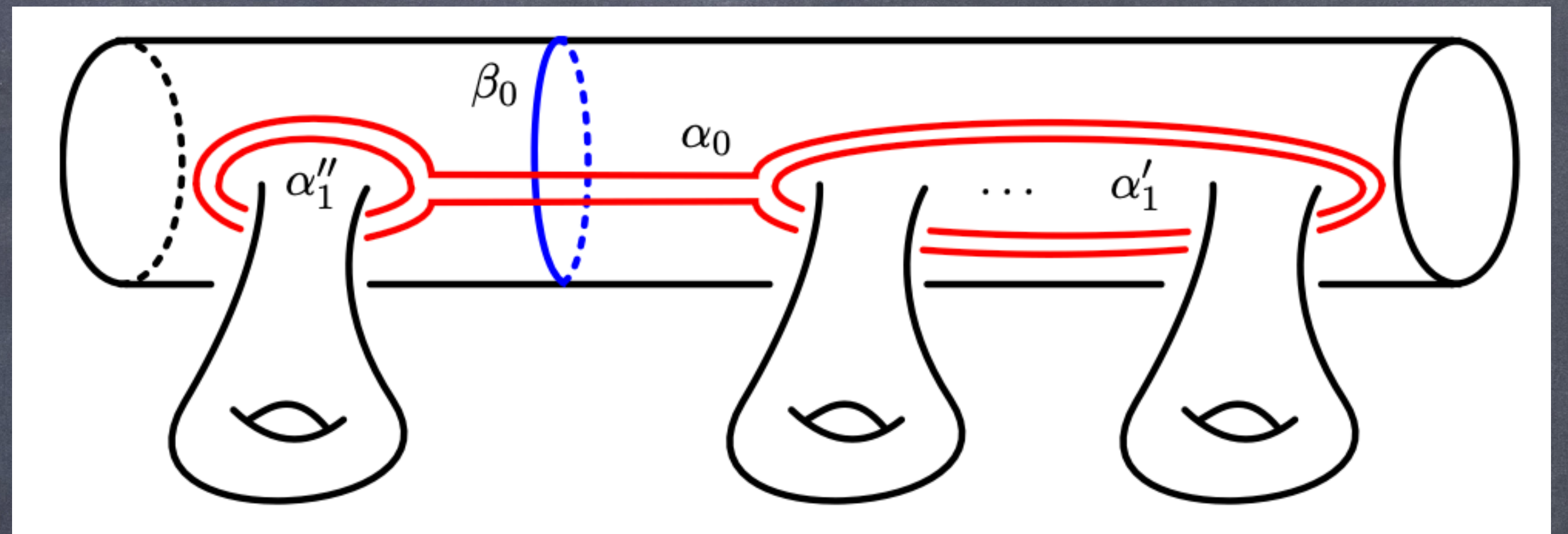
Initial case



genus  $g > 1$

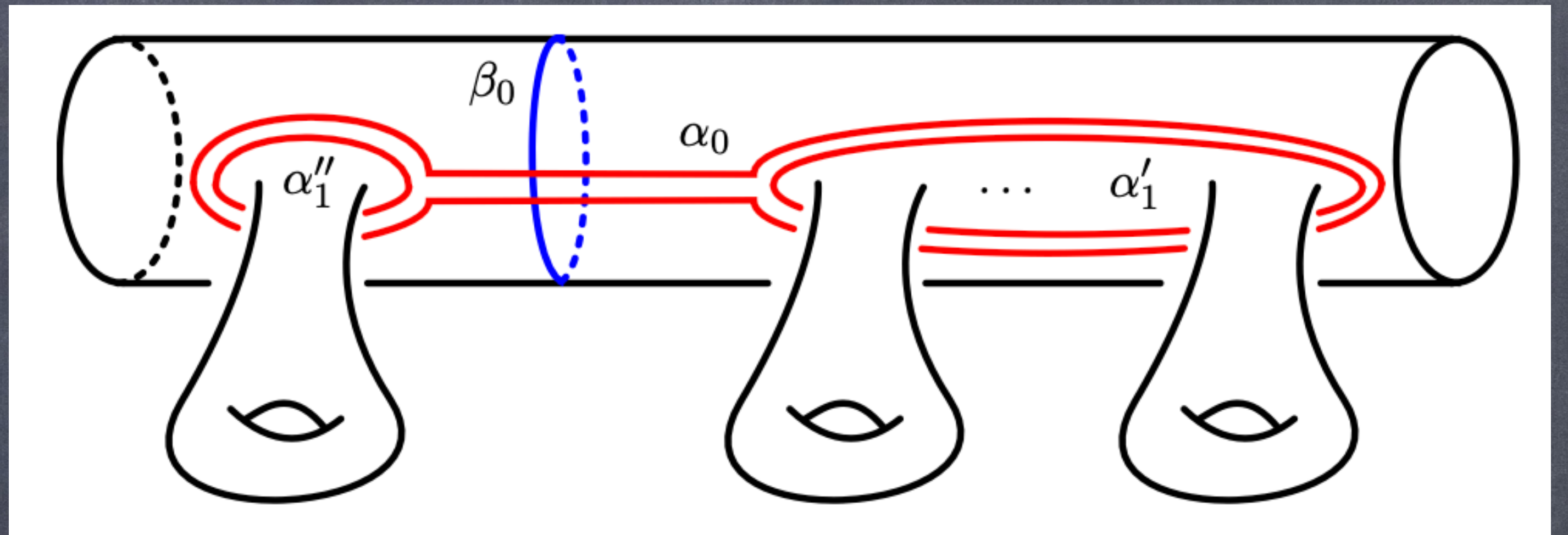


Initial case



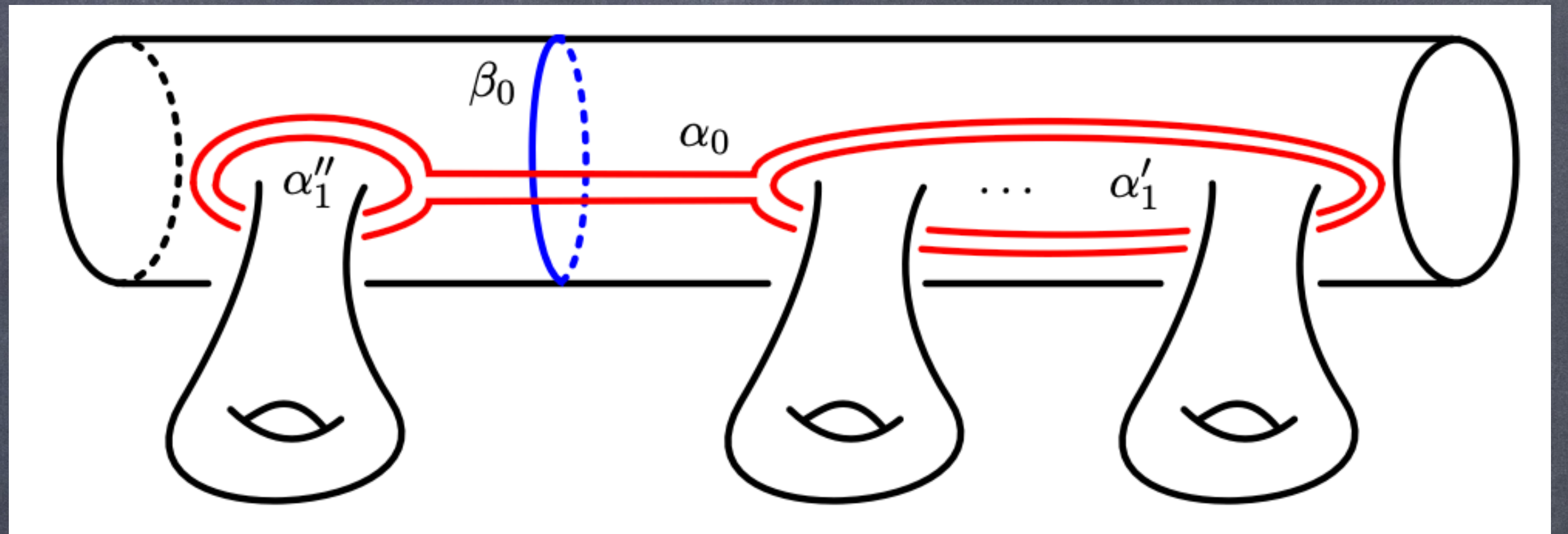
- $\alpha_0$  curve encircle all the handles of the newly formed surface
- $y$  copies of  $\beta_0$

genus  $g > 1$



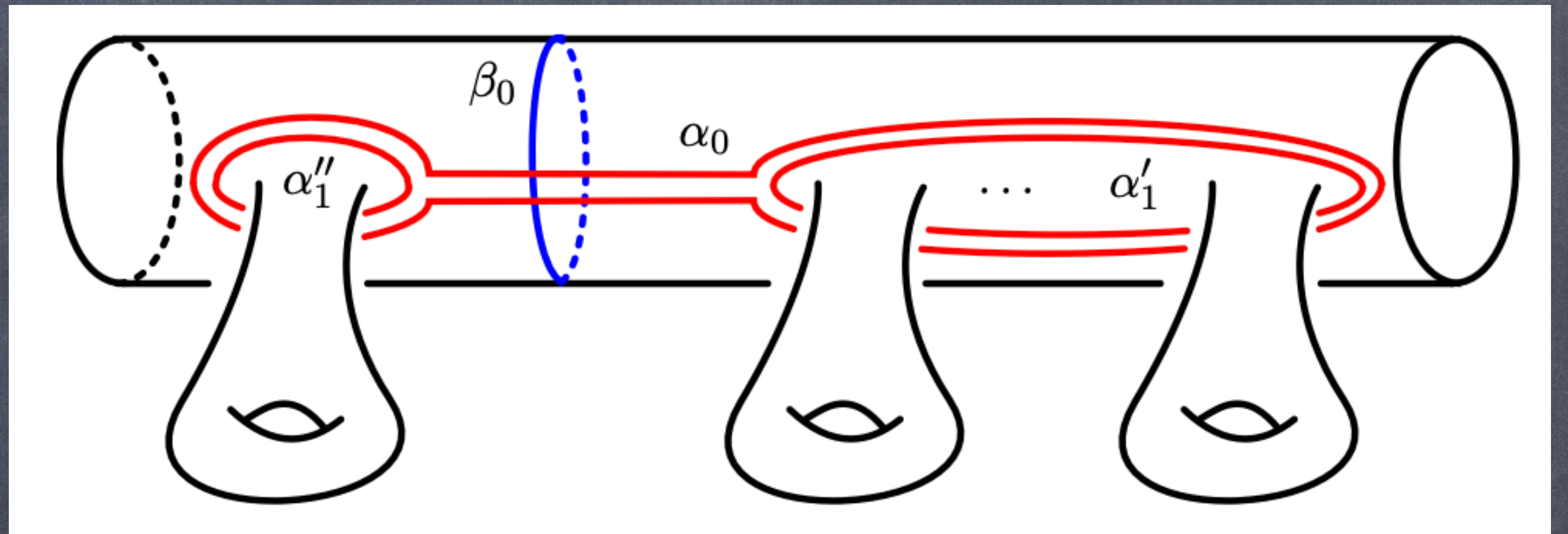
genus  $g > 1$

$$XX^T = \left( \begin{array}{c|ccc|cc} 4y^2 & a_1 & \dots & a_n & 4 & 2 \\ \hline a_1 & & & & & \\ \vdots & & A & & & \\ a_n & & & & & \\ \hline 4 & & & & 4 & 2 \\ 2 & & & & 2 & b \end{array} \right)$$



genus  $g > 1$

$$XX^T = \left( \begin{array}{c|ccc|cc} 4y^2 & a_1 & \dots & a_n & 4 & 2 \\ \hline a_1 & & & & & \\ \vdots & & & & & \\ a_n & & & & & \\ \hline 4 & & & & 4 & 2 \\ 2 & & & & 2 & b \end{array} \right)$$

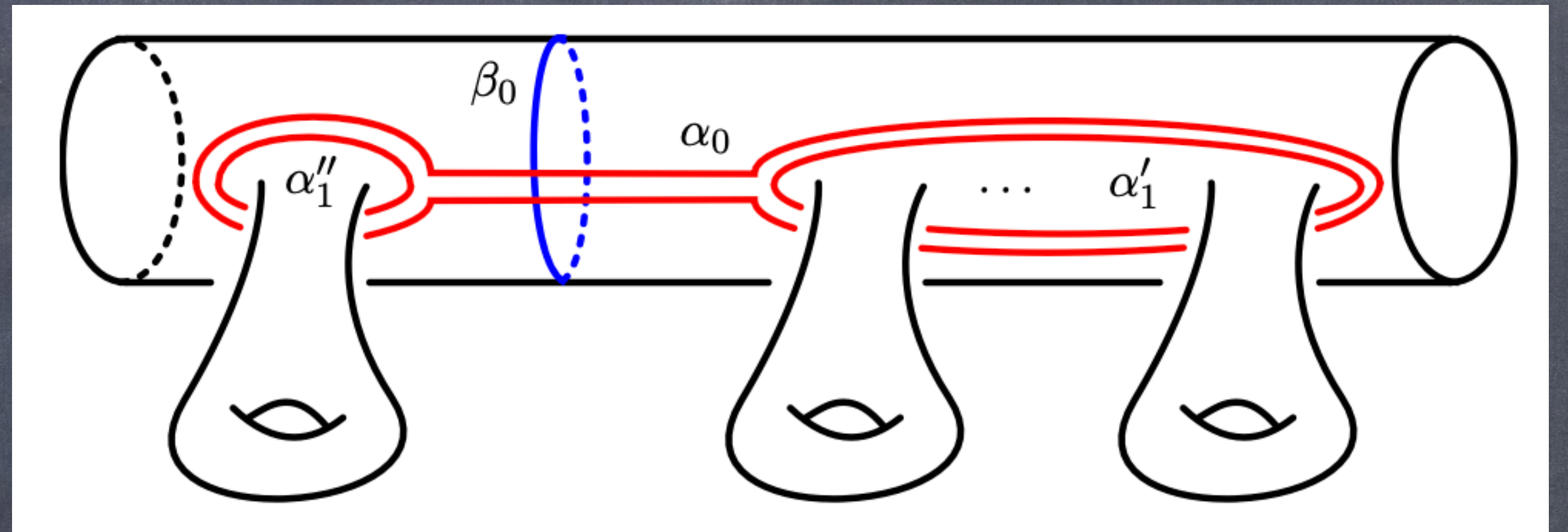


Inductively

genus  $g' = g - 1$  and  $\chi_{XX^T} \in \mathbb{Z}[t]$  has degree  $3g' - 1$

genus  $g > 1$

$$XX^T = \left( \begin{array}{c|ccc|cc} 4y^2 & a_1 & \dots & a_n & 4 & 2 \\ \hline a_1 & & & & & \\ \vdots & & & & & \\ a_n & & & & & \\ \hline 4 & & & & 4 & 2 \\ 2 & & & & 2 & b \end{array} \right)$$

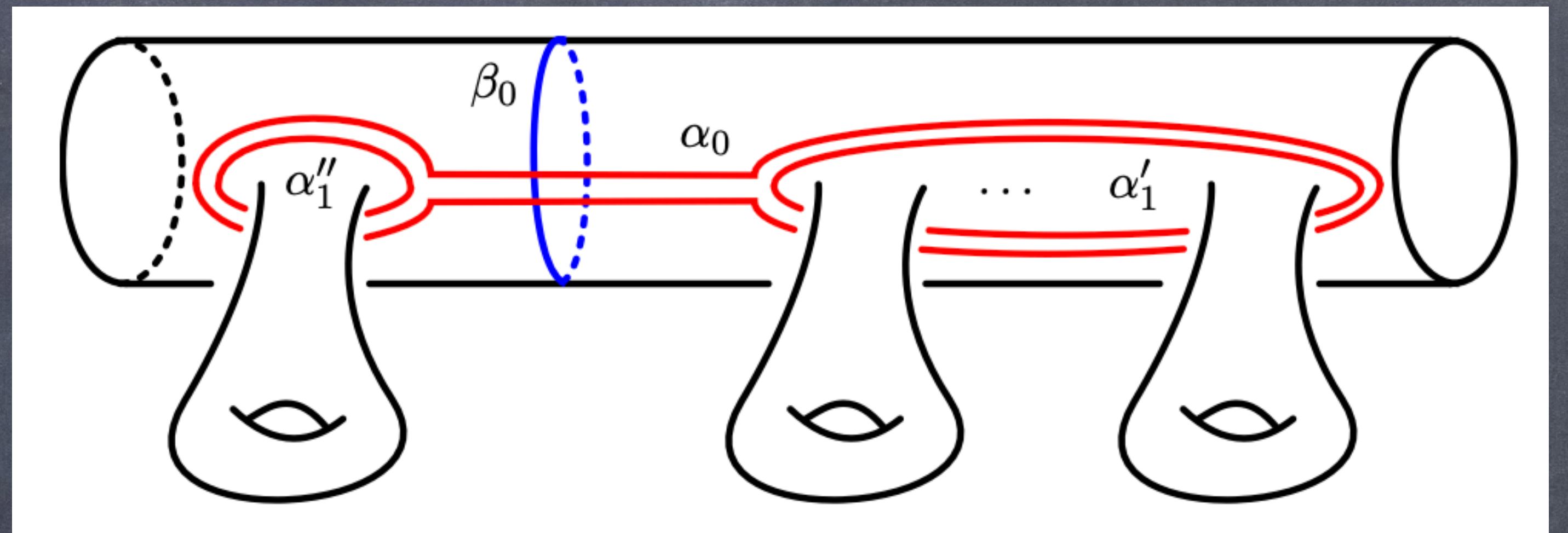


Inductively  
genus  $g' = g - 1$  and  $\chi_{XX^T} \in \mathbb{Z}[t]$  has degree  $3g' - 1$

Claim:  $\chi_{XX^T} \in \mathbb{Z}[t]$  is irreducible for some  $y$

genus  $g > 1$

$$XX^T = \left( \begin{array}{c|ccc|cc} 4y^2 & a_1 & \dots & a_n & 4 & 2 \\ \hline a_1 & & & & & \\ \vdots & & & & & \\ a_n & & & & & \\ \hline 4 & & & & 4 & 2 \\ 2 & & & & 2 & b \end{array} \right)$$



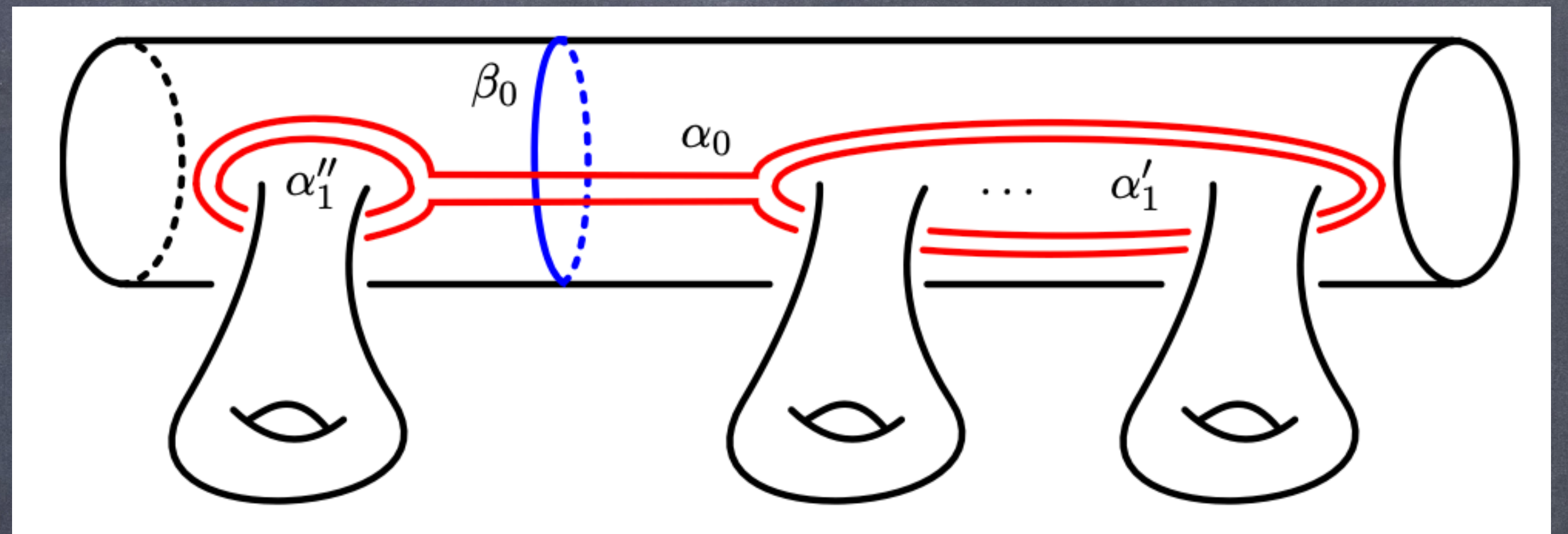
Inductively  
genus  $g' = g - 1$  and  $\chi_{XX^T} \in \mathbb{Z}[t]$  has degree  $3g' - 1$

Claim:  $\chi_{XX^T} \in \mathbb{Z}[t]$  is irreducible for some  $y$

$\chi_{XX^T} \in \mathbb{Z}[t, y] = (\mathbb{Z}[t])[y]$  is irreducible

genus  $g > 1$

$$XX^T = \left( \begin{array}{c|ccc|cc} 4y^2 & a_1 & \dots & a_n & 4 & 2 \\ \hline a_1 & & & & & \\ \vdots & & & & & \\ a_n & & & & & \\ \hline 4 & & & & 4 & 2 \\ 2 & & & & 2 & b \end{array} \right)$$



Inductively

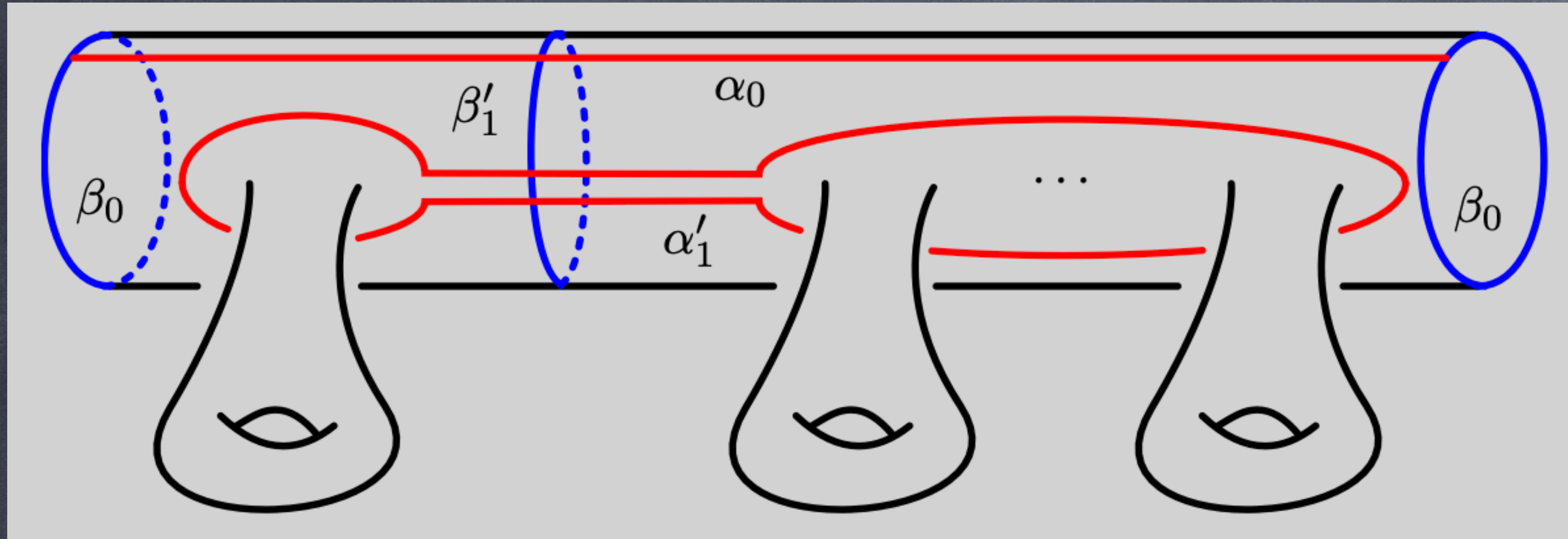
genus  $g' = g - 1$  and  $\chi_{XX^T} \in \mathbb{Z}[t]$  has degree  $3g' - 1$

**Claim:**  $\chi_{XX^T} \in \mathbb{Z}[t]$  is irreducible for some  $y$

$\chi_{XX^T} \in \mathbb{Z}[t, y] = (\mathbb{Z}[t])[y]$  is irreducible

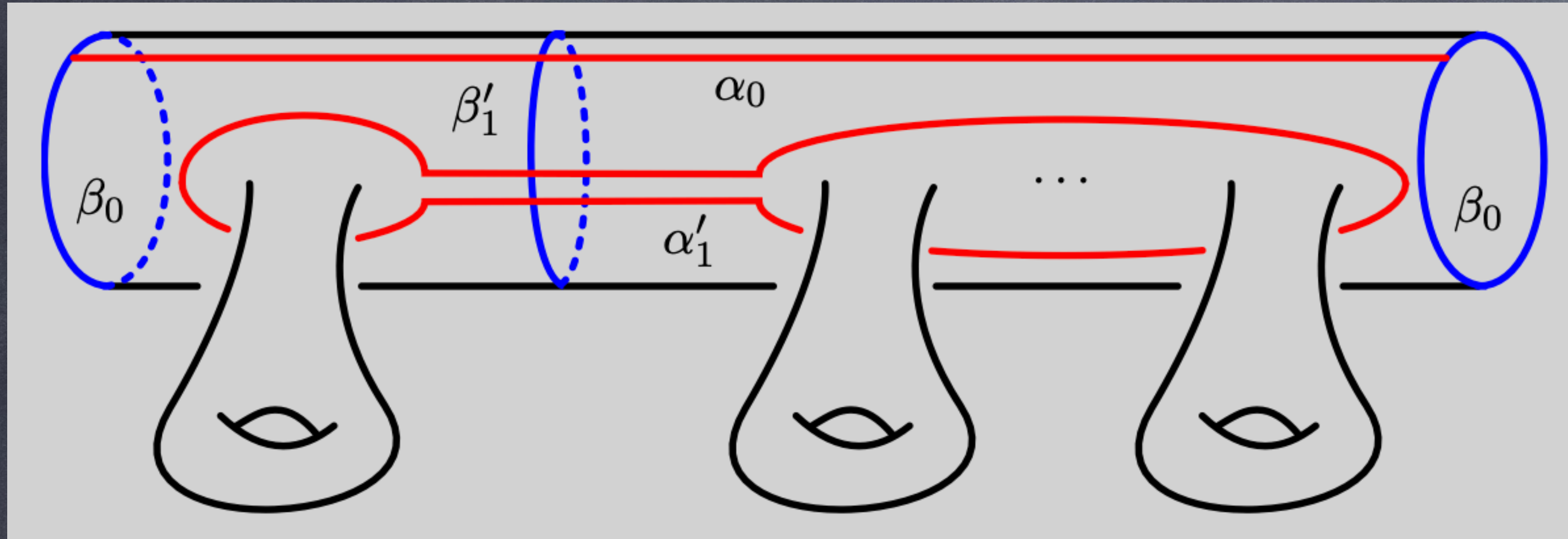
Hilbert's irreducibility theorem: there exist infinitely many specializations of  $y$  such that  $\chi_{XX^T} \in \mathbb{Z}[t]$  is irreducible

genus  $g > 1$



genus  $g' = g - 1$

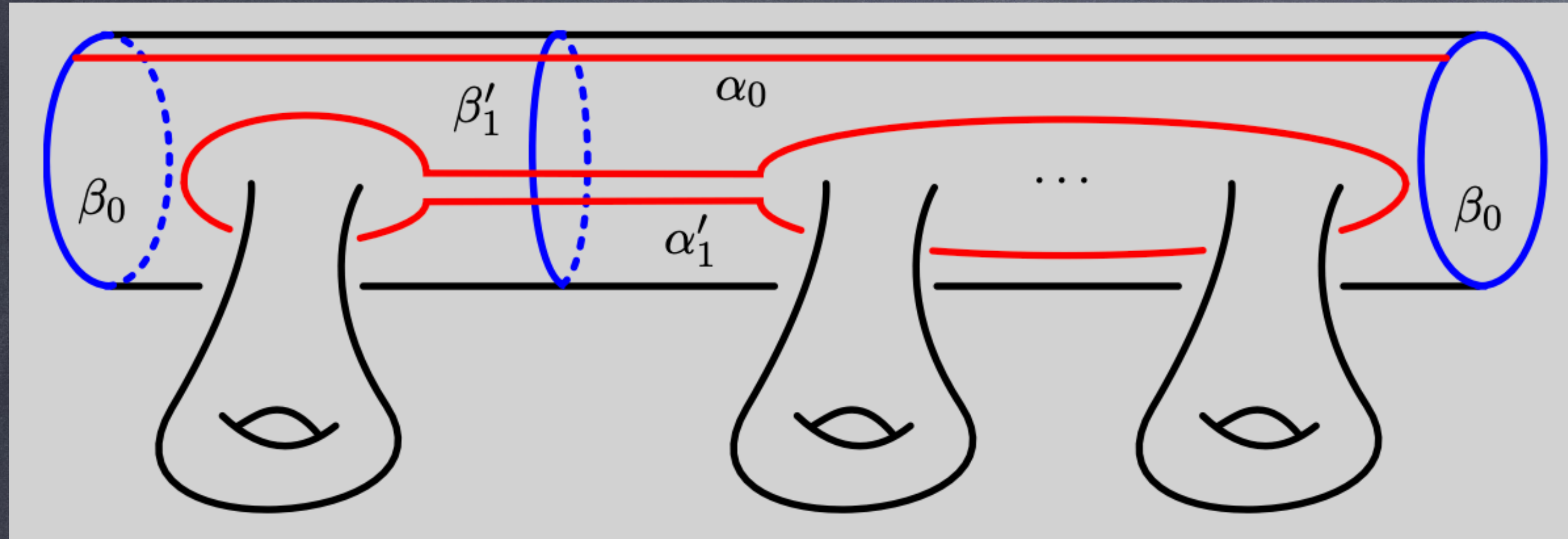
genus  $g > 1$



genus  $g' = g - 1$

We identify the two boundary components of the surface to increase the genus by one.

genus  $g > 1$

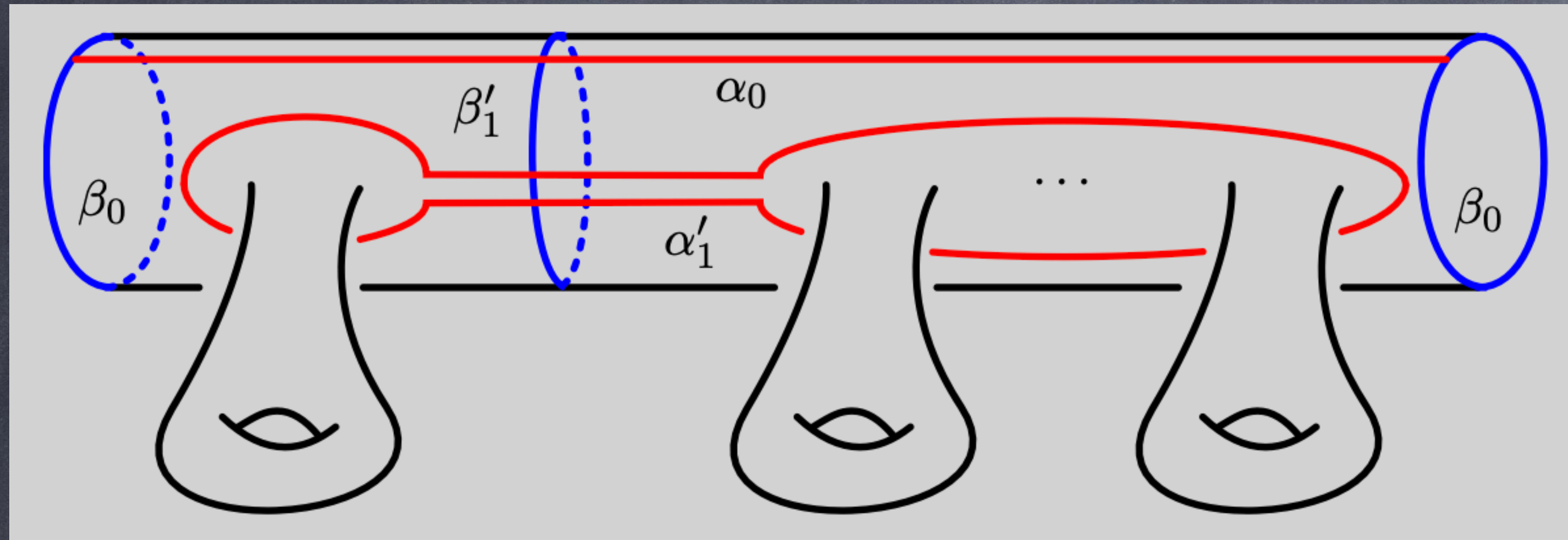


genus  $g' = g - 1$

We identify the two boundary components of the surface to increase the genus by one.

degree of  $\chi_{XX^T}$  is  $(3g' - 1) + 1 = 3g - 3$

genus  $g > 1$



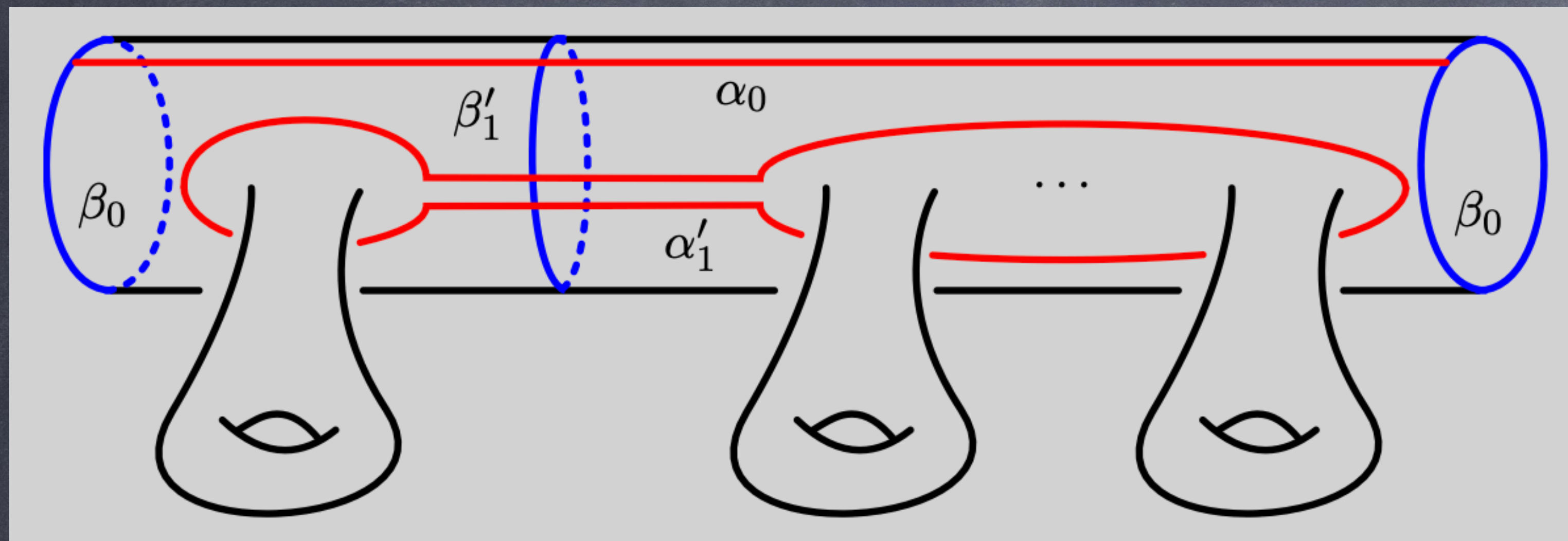
Hilbert's irreducibility theorem ensures there exist infinitely many specializations of  $y$  such that  $\chi_{XX^T} \in \mathbb{Z}[t]$  is irreducible

genus  $g' = g - 1$

We identify the two boundary components of the surface to increase the genus by one.

degree of  $\chi_{XX^T}$  is  $(3g' - 1) + 1 = 3g - 3$

genus  $g > 1$



Hilbert's irreducibility theorem ensures there exist infinitely many specializations of  $y$  such that  $\chi_{XX^T} \in \mathbb{Z}[t]$  is irreducible

genus  $g' = g - 1$

We identify the two boundary components of the surface to increase the genus by one.

degree of  $\chi_{XX^T}$  is  $(3g' - 1) + 1 = 3g - 3$

$\mathbb{Q}(\mu^2) = \mathbb{Q}(\lambda + \lambda^{-1})$   
has degree  $3g - 3$

genus  $g > 1$

Conclusion:  $\lambda$  solves the quadratic equation

$$\lambda^2 - (\lambda + \lambda^{-1}) \cdot \lambda + 1 = 0$$

Need to show that  $\lambda \notin \mathbb{Q}(\lambda + \lambda^{-1})$

genus  $g > 1$

Conclusion:  $\lambda$  solves the quadratic equation

$$\lambda^2 - (\lambda + \lambda^{-1}) \cdot \lambda + 1 = 0$$

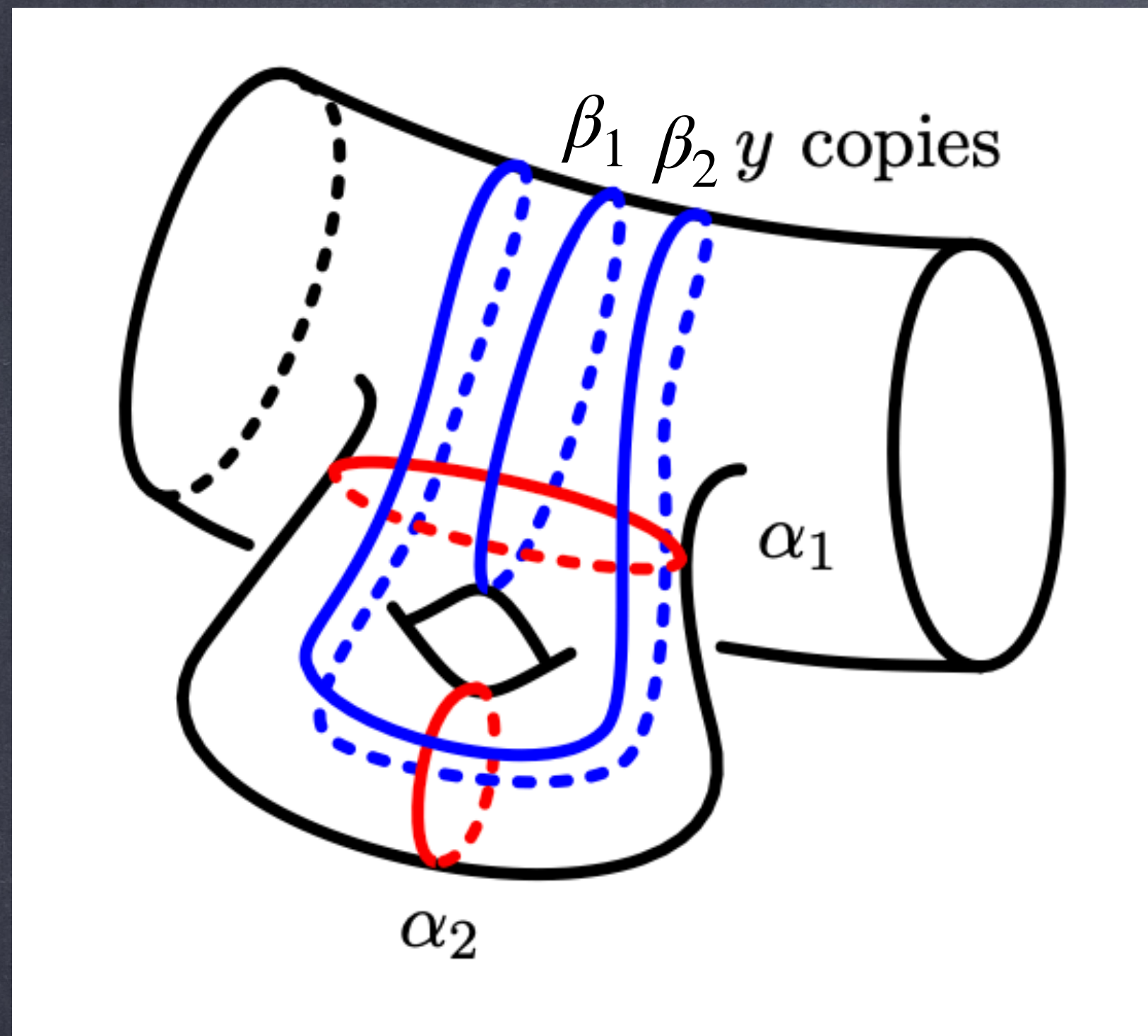
Need to show that  $\lambda \notin \mathbb{Q}(\lambda + \lambda^{-1})$

Theorem (L-Liechti, 2025)

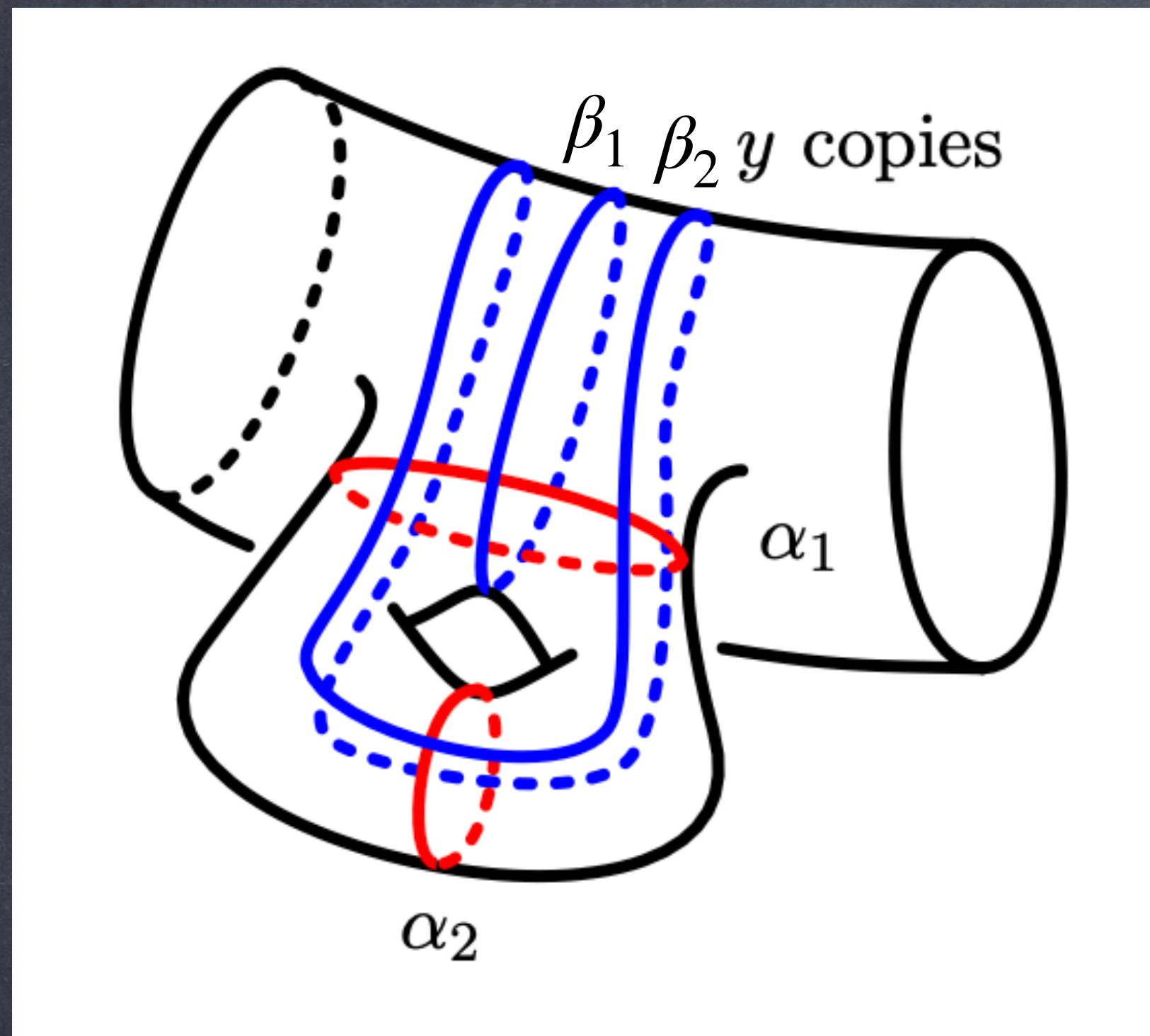
Let  $(\alpha, \beta)$  be filling multicurves. There exists  $n > 0$  s.t. the bouillabaisse mapping class  $T_\alpha^n \circ T_\beta^{\pm n}$  is pseudo-Anosov and

$\lambda \notin \mathbb{Q}(\lambda + \lambda^{-1})$

For the Torelli subgroup



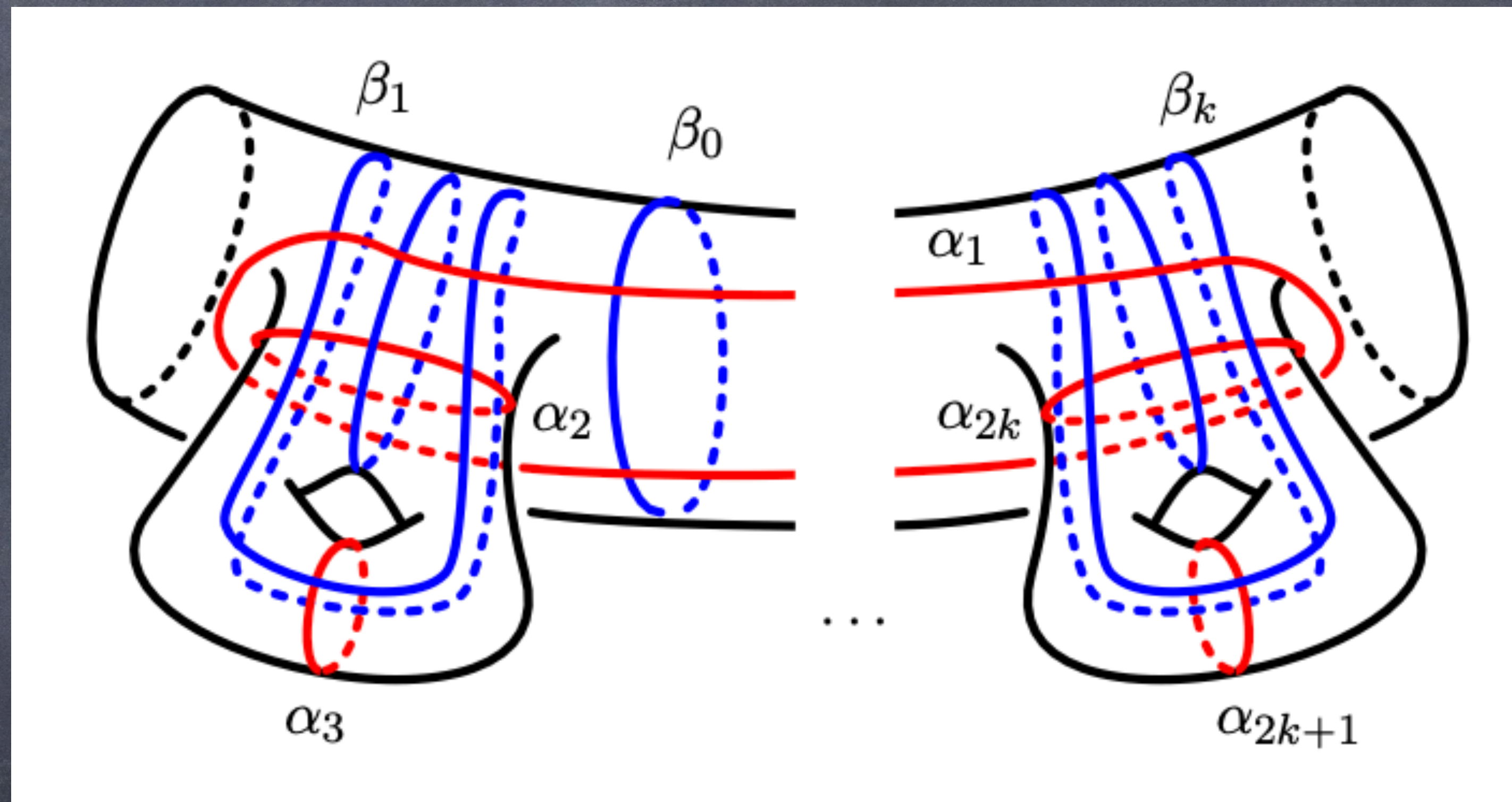
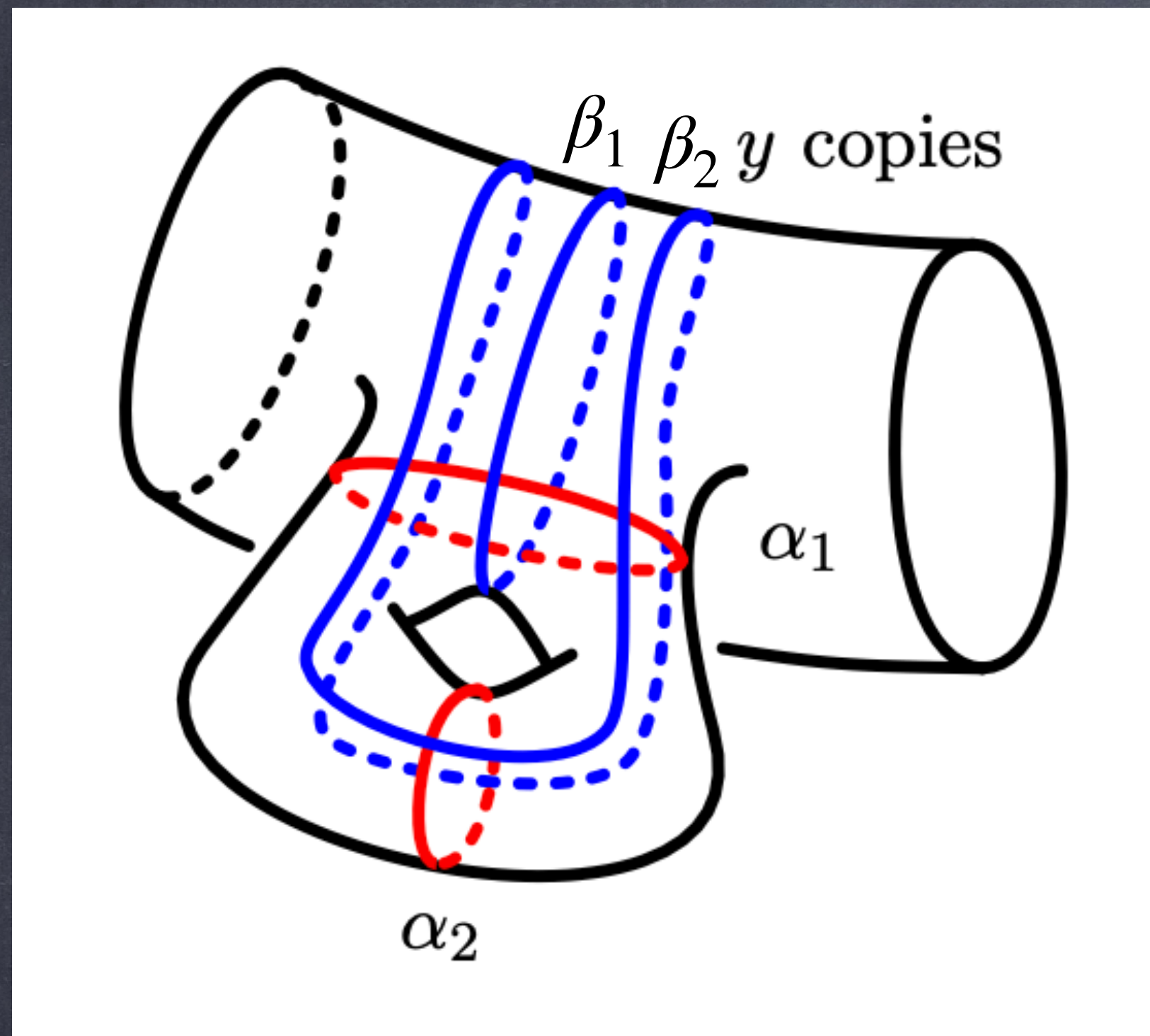
# For the Torelli subgroup



bounding pair  $\alpha_i, \beta_j$

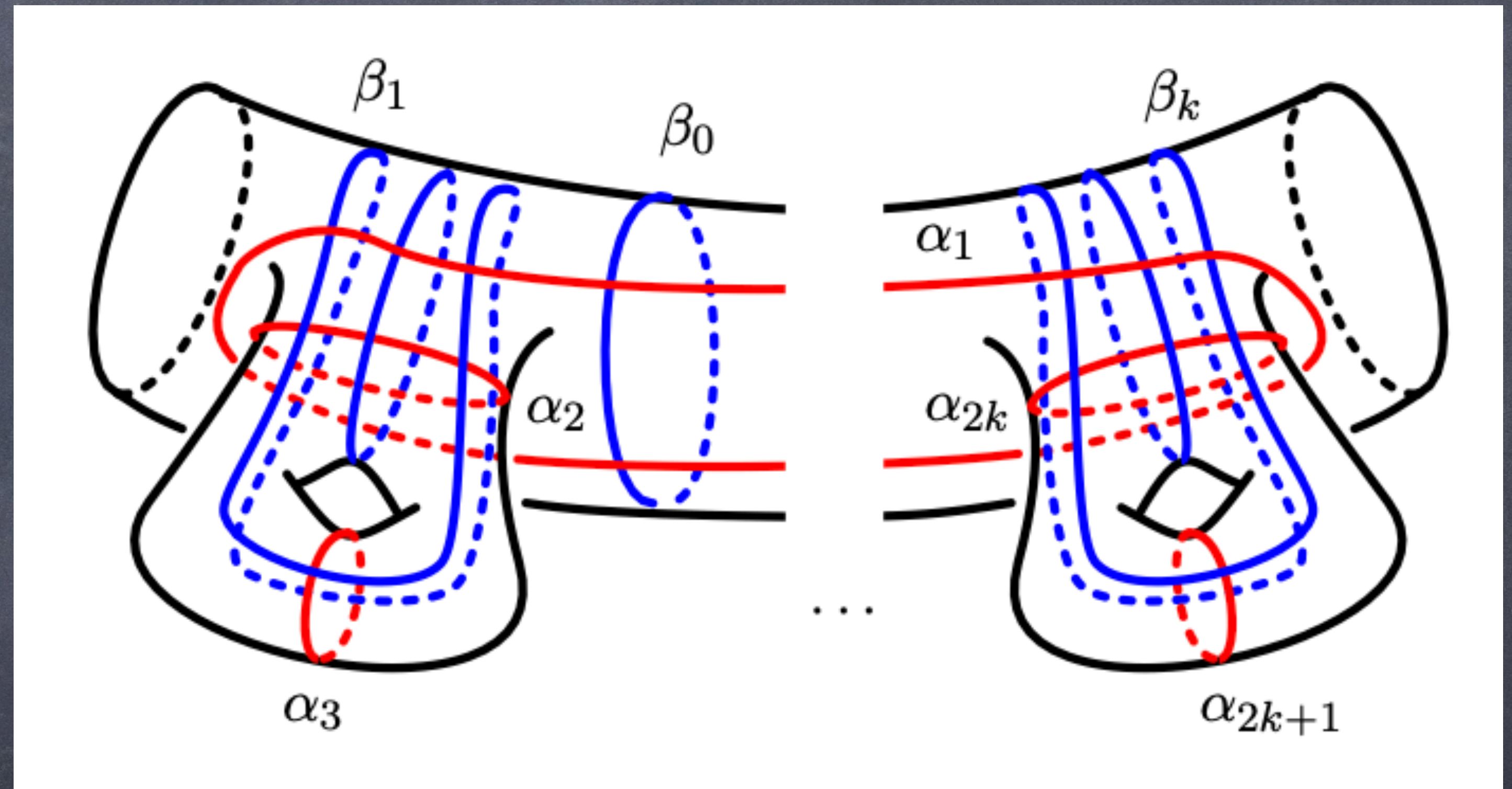
$$T_{\alpha_i} \circ T_{\beta_j}^{-1} \in T(S)$$

# For the Torelli subgroup

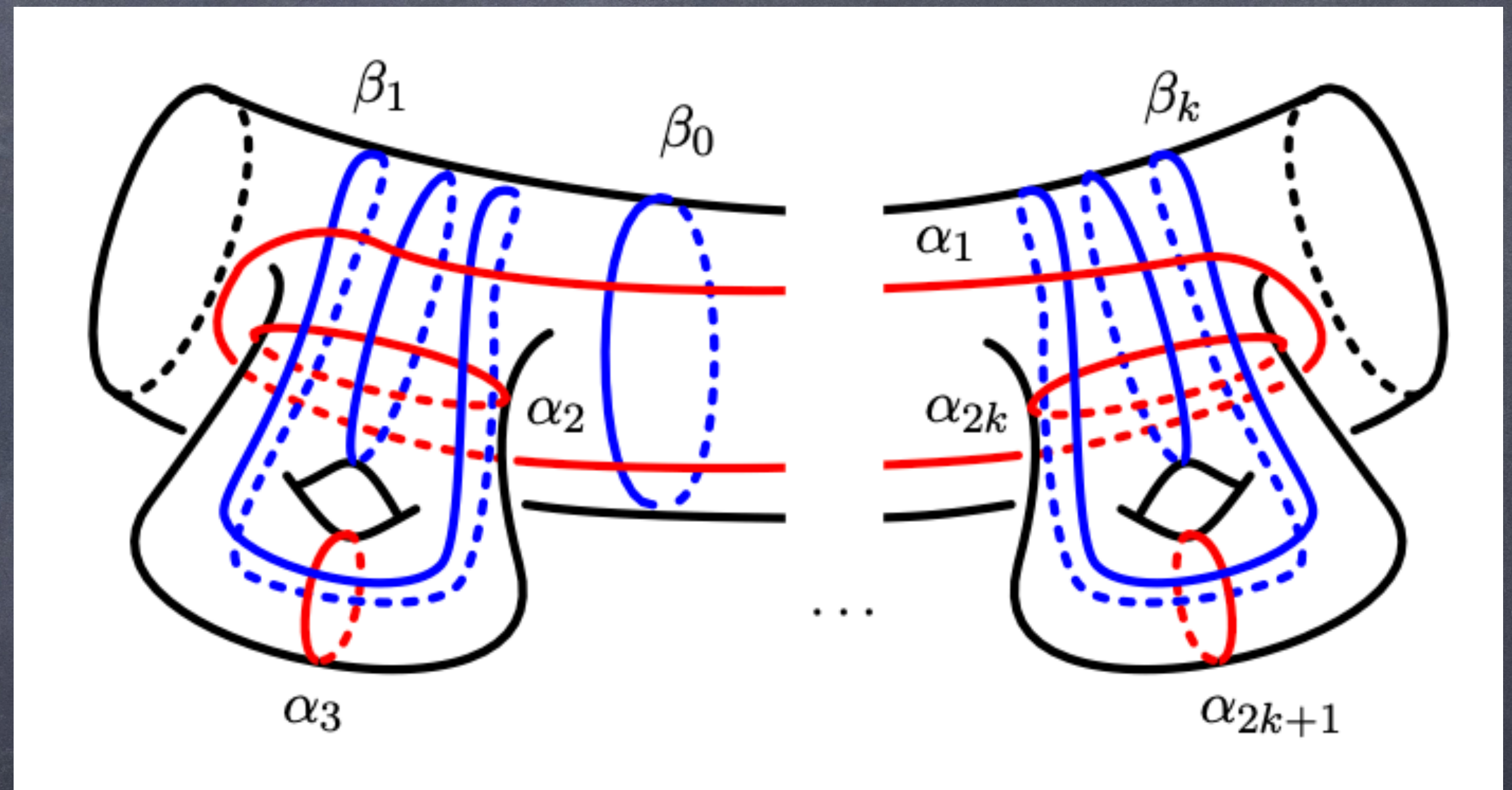
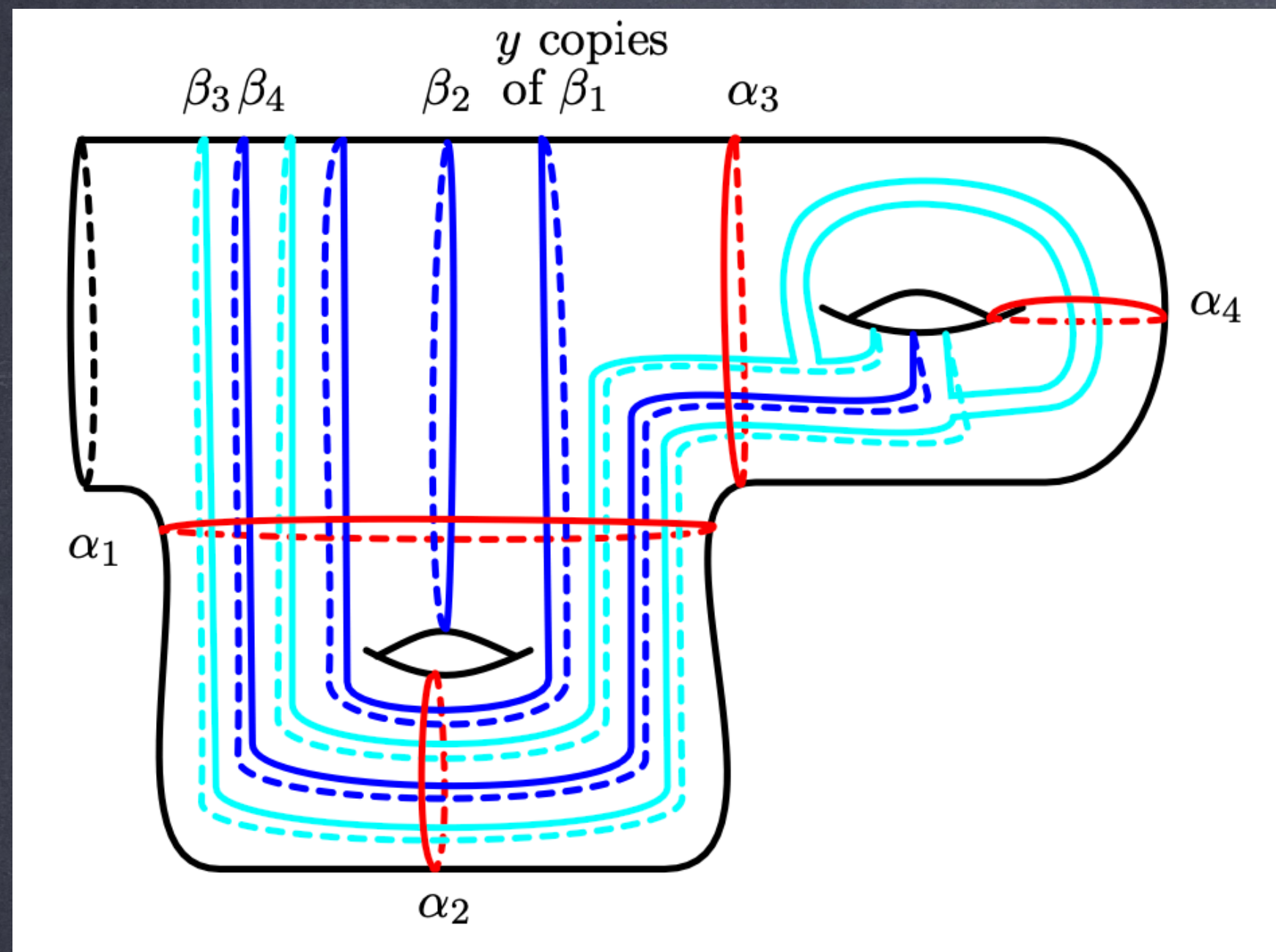


bounding pair  $\alpha_i, \beta_j$   
 $T_{\alpha_i} \circ T_{\beta_j}^{-1} \in T(S)$

For the Torelli subgroup



# For the Torelli subgroup



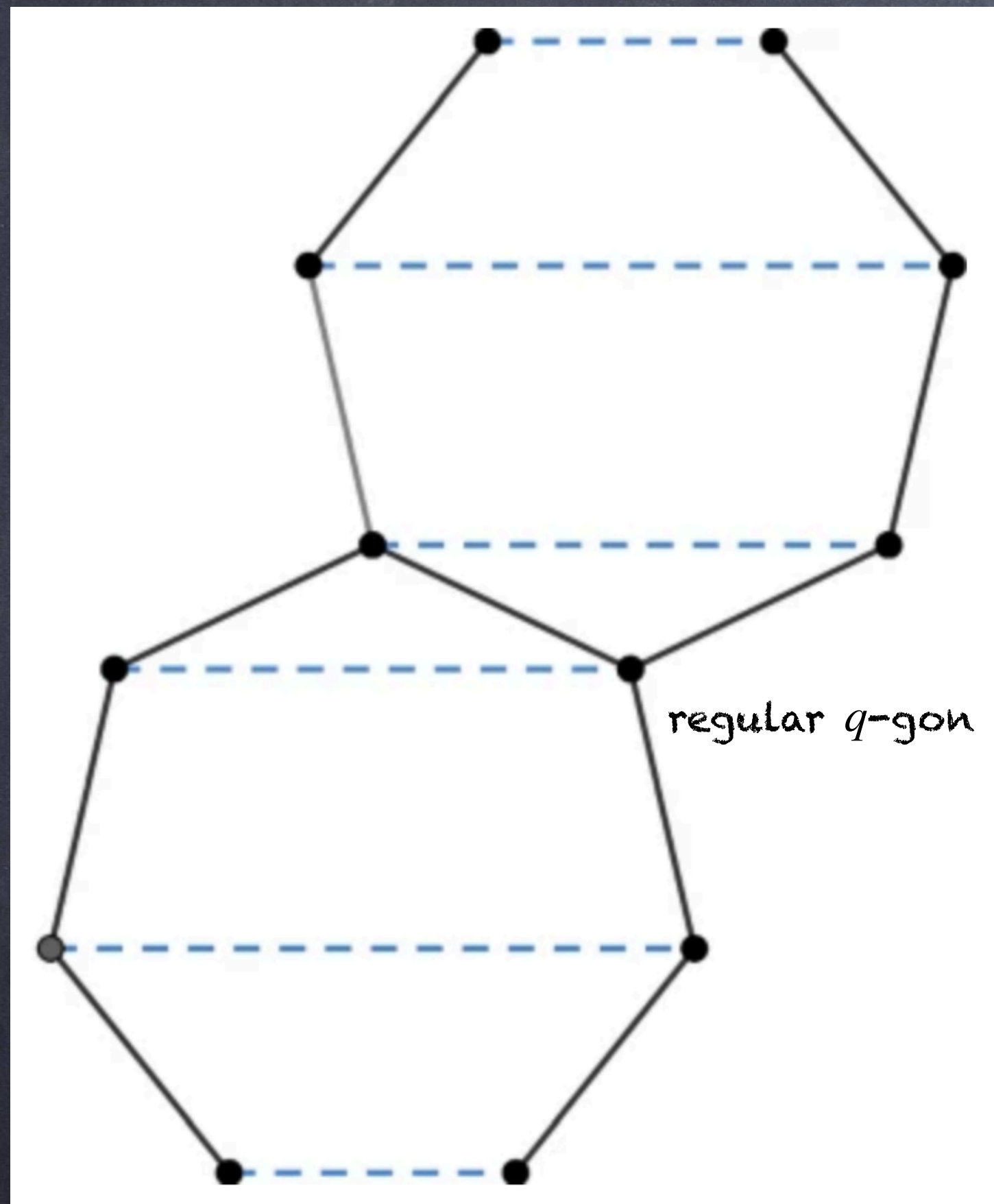
And for odd degrees?

And for odd degrees?

The realization of odd degrees seems more mysterious

And for odd degrees?

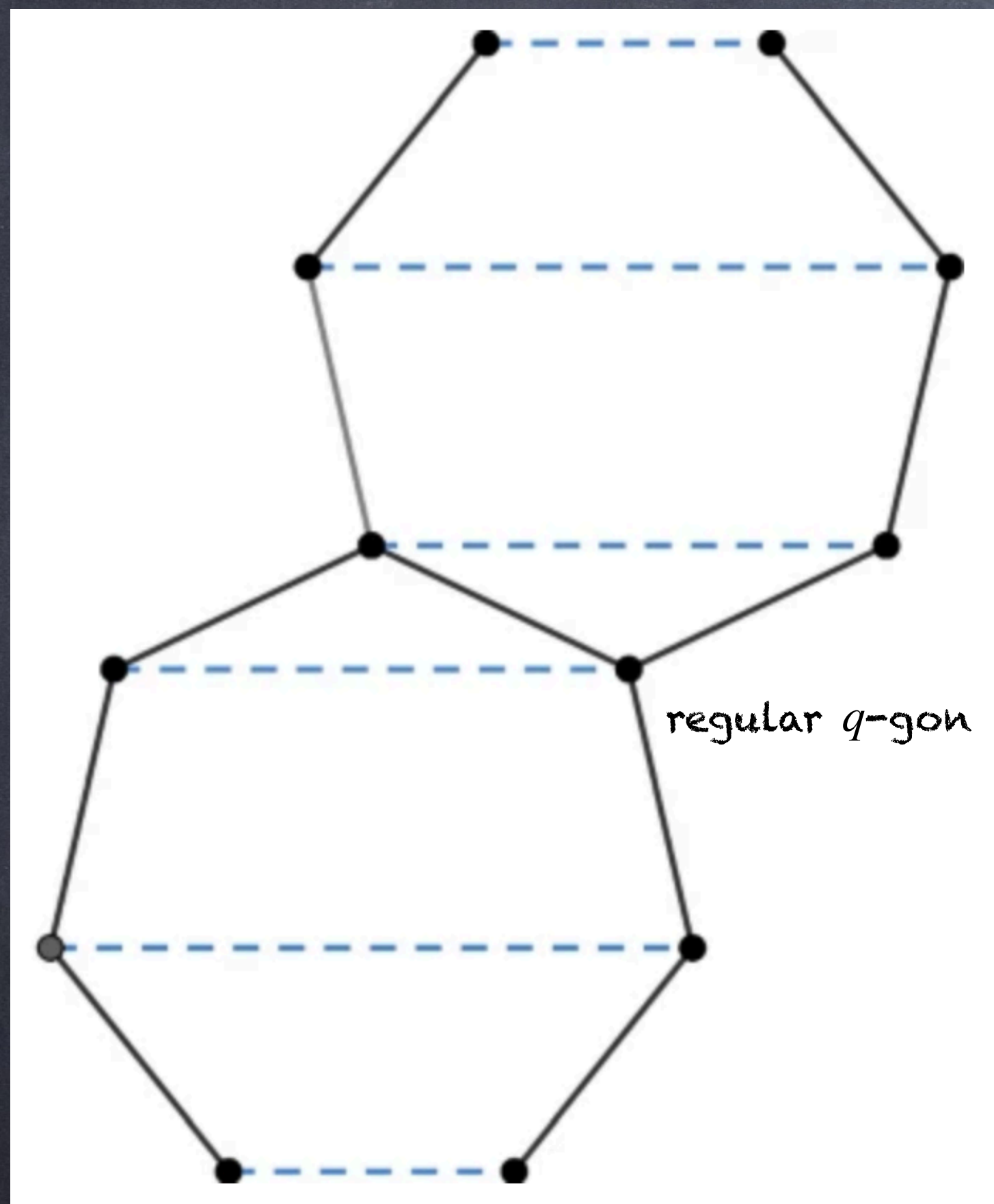
The realization of odd degrees seems more mysterious



$$g = (q - 1)/2 \text{ for odd } q$$

And for odd degrees?

The realization of odd degrees seems more mysterious



$$g = (q - 1)/2 \text{ for odd } q$$

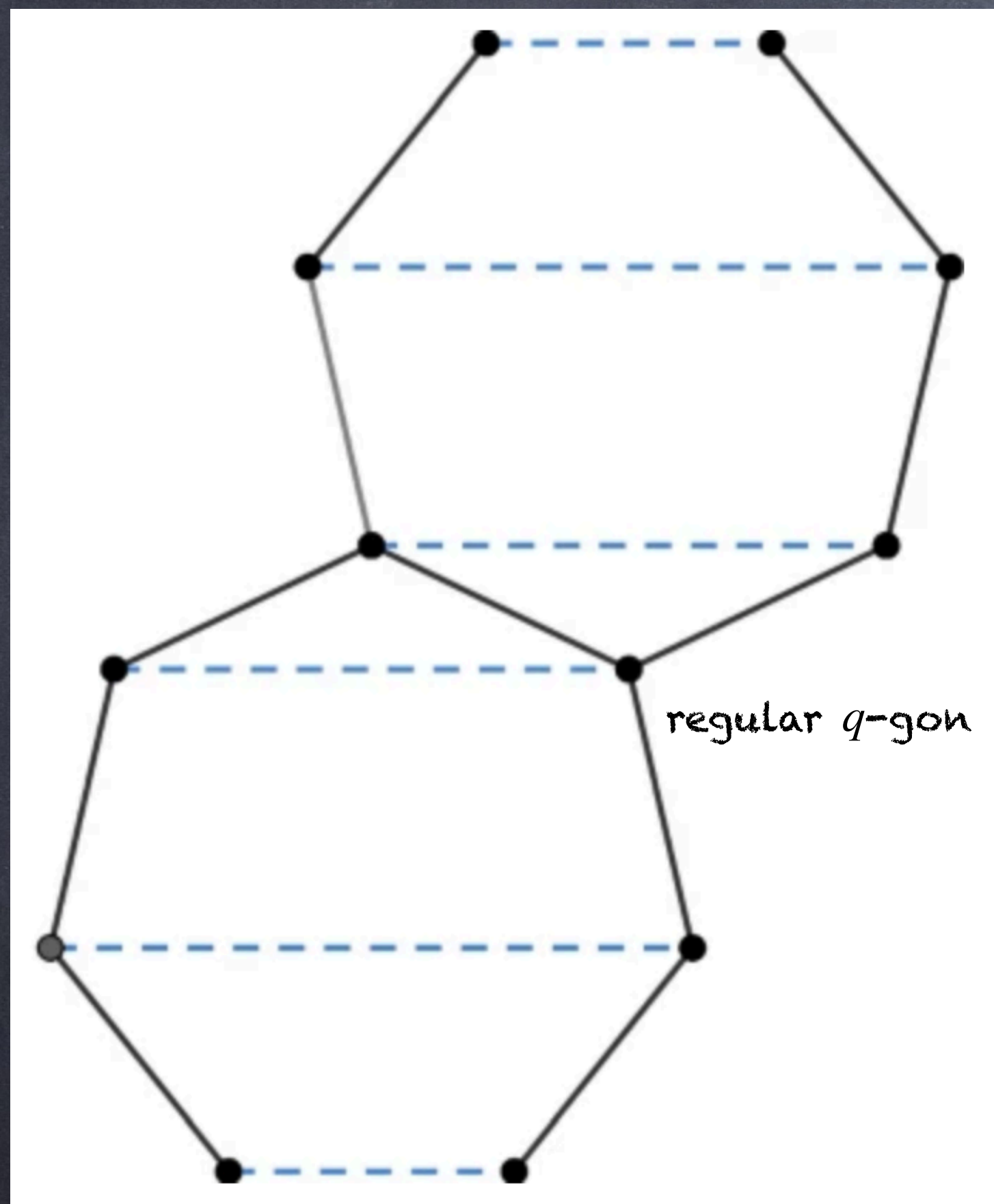
Veech's family of Hecke groups

$$\langle T_\alpha, T_\beta \rangle \simeq \left\langle \begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle$$

$$\lambda_q = 2 \cos \pi/q \text{ and } q \geq 3$$

# And for odd degrees?

The realization of odd degrees seems more mysterious



$$g = (q - 1)/2 \text{ for odd } q$$

Veech's family of Hecke groups

$$\langle T_\alpha, T_\beta \rangle \simeq \left\langle \begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle$$

$$\lambda_q = 2 \cos \pi/q \text{ and } q \geq 3$$

$$q = 3: \text{PSL}(2, \mathbb{Z})$$

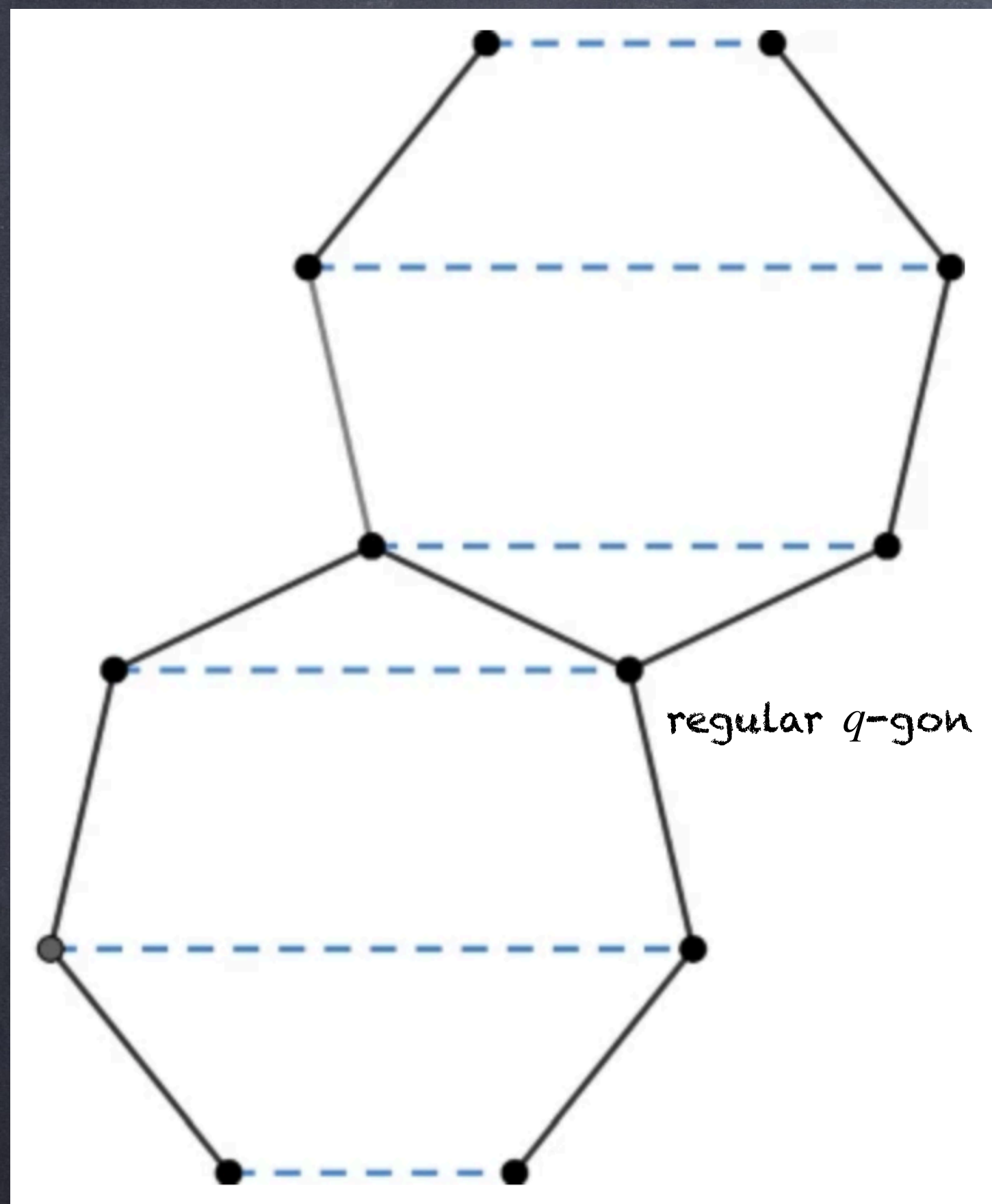
$$q = 5: \text{quadratic field (genus 2)}$$

$$q = 7, 9: \text{cubic field (genus 3 and 4)}$$

$$q \geq 11: \text{degree } > 3 \text{ (genus } > 4)$$

And for odd degrees?

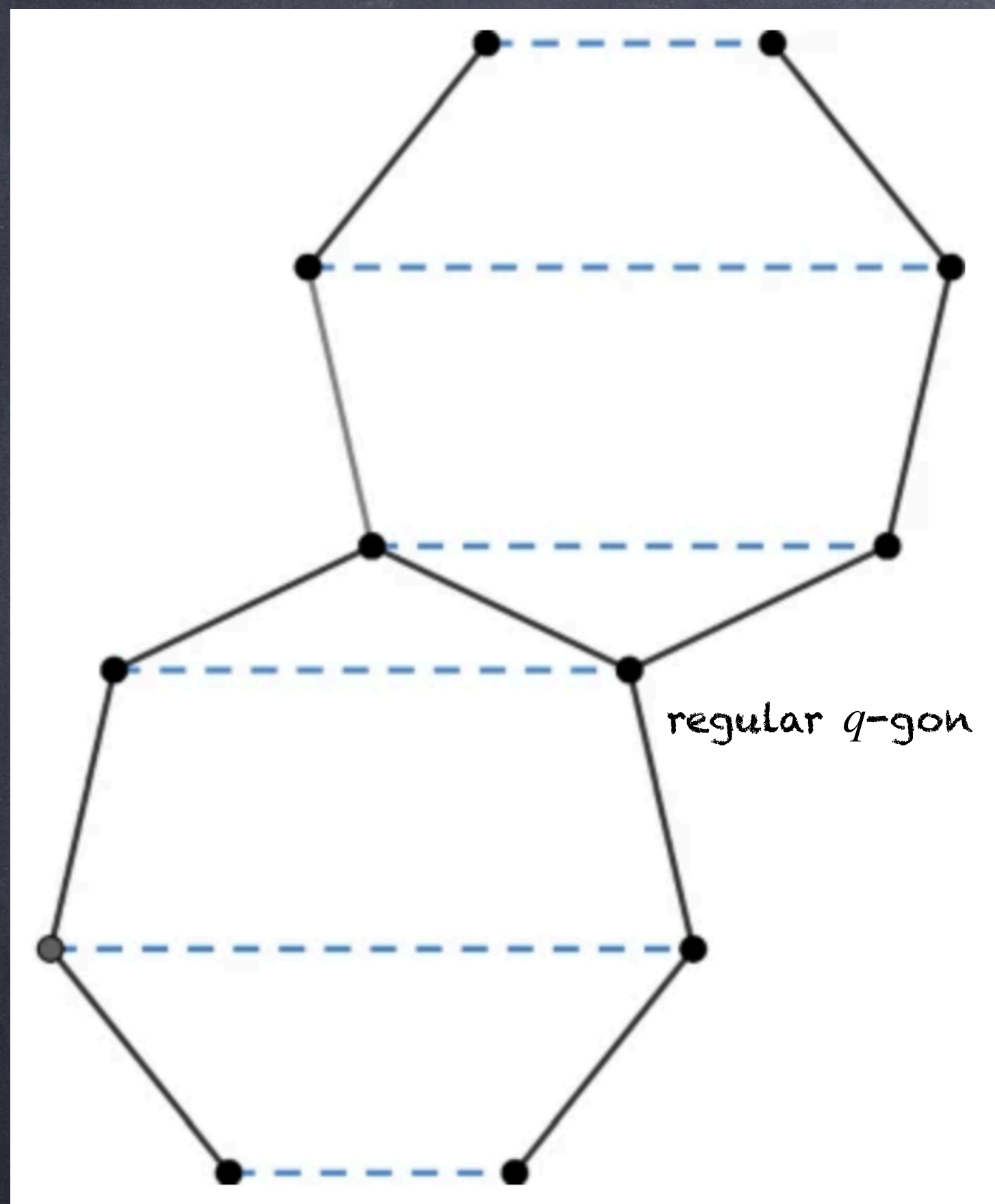
The realization of odd degrees seems more mysterious



$$g = (q - 1)/2 \text{ for odd } q$$

# And for odd degrees?

The realization of odd degrees seems more mysterious

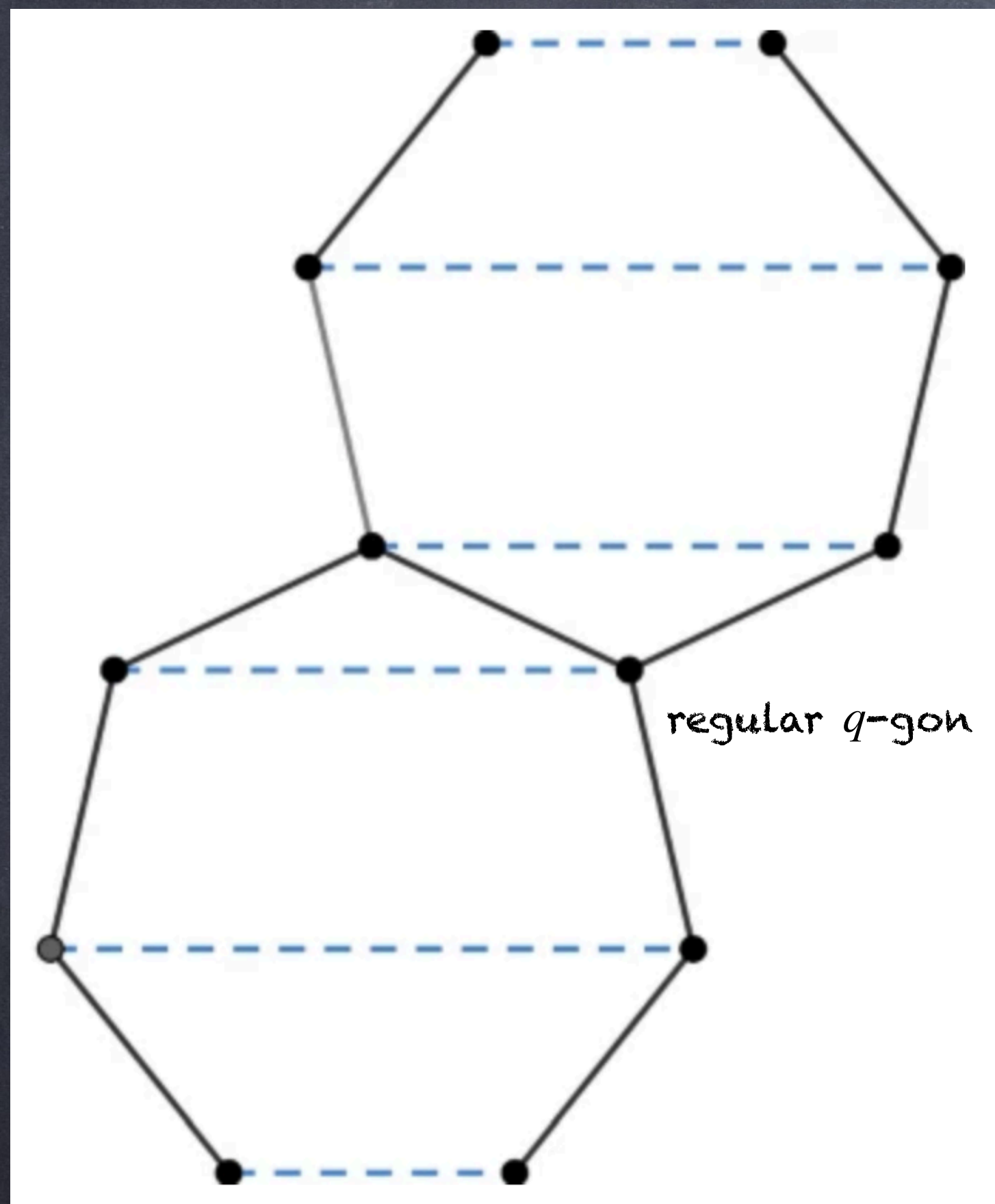


For  $q = 7$ , there exists bouillabaisse pA with cubic stretch factor e.g.  $T_\alpha \circ T_\beta^{-1}$

$$g = (q - 1)/2 \text{ for odd } q$$

# And for odd degrees?

The realization of odd degrees seems more mysterious



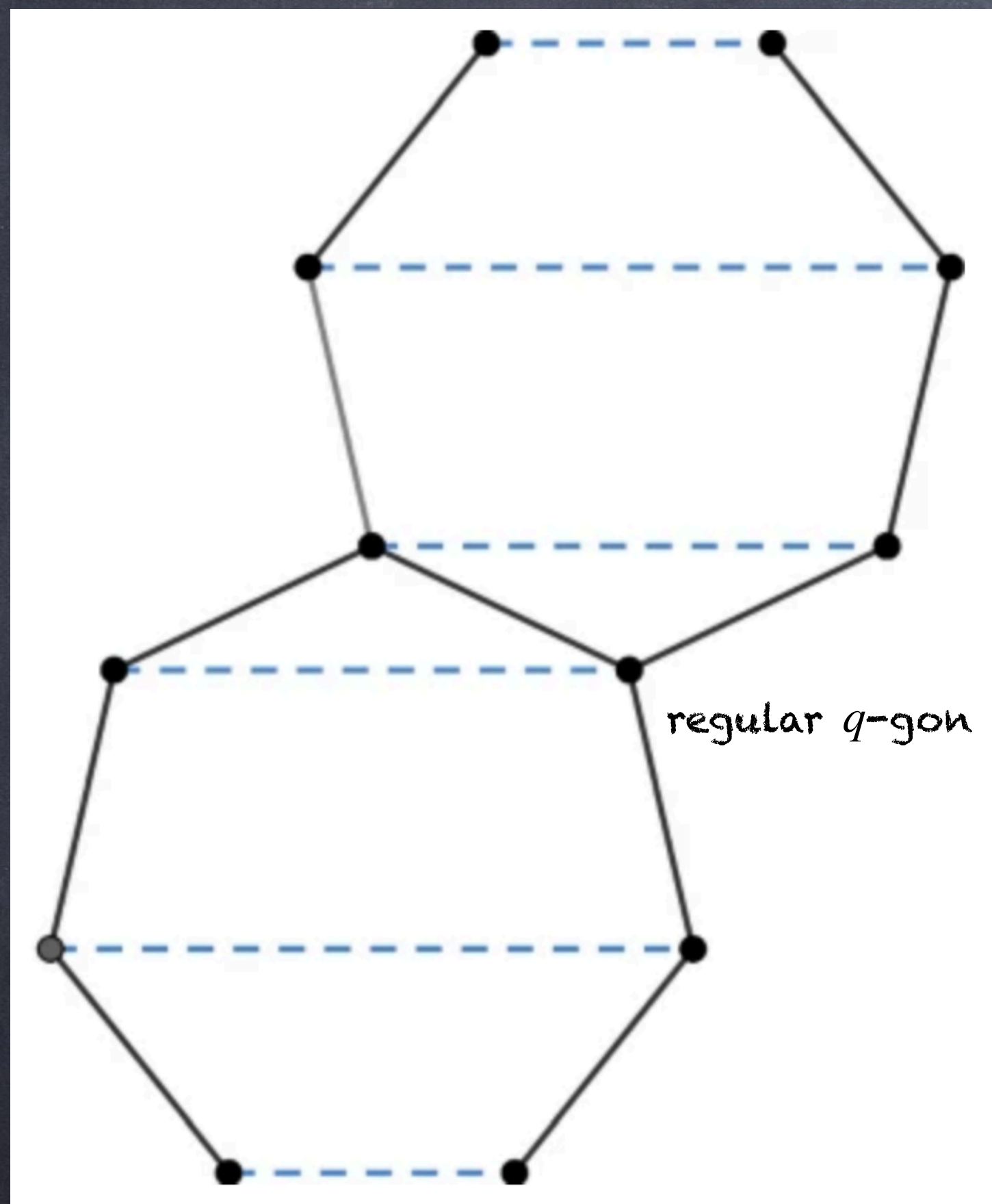
For  $q = 7$ , there exists bouillabaisse  $pA$  with cubic stretch factor e.g.  $T_\alpha \circ T_\beta^{-1}$

For  $q = 9$ , there exists bouillabaisse  $pA$  with cubic stretch factor (Julien Boulanger)

$$g = (q - 1)/2 \text{ for odd } q$$

# And for odd degrees?

The realization of odd degrees seems more mysterious



$$g = (q - 1)/2 \text{ for odd } q$$

For  $q = 7$ , there exists bouillabaisse  $pA$  with cubic stretch factor e.g.  $T_\alpha \circ T_\beta^{-1}$

For  $q = 9$ , there exists bouillabaisse  $pA$  with cubic stretch factor (Julien Boulanger)

However, it is conjectured that no such element exists for odd  $q \geq 11$

# Minimum dilatation problem

$\text{Spec}(\mathcal{M}_g) = \{\log \lambda(\phi), \phi \in \text{Mod}(S_g)\}$  is discrete (Ivanov ~1985)

What is the smallest element?

$$\text{sys}(\mathcal{M}_{g,n}) = \min \text{Spec}(\mathcal{M}_{g,n})$$

# Minimum dilatation problem

$\text{Spec}(\mathcal{M}_g) = \{\log \lambda(\phi), \phi \in \text{Mod}(S_g)\}$  is discrete (Ivanov ~1985)

What is the smallest element?

$$\text{sys}(\mathcal{M}_{g,n}) = \min \text{Spec}(\mathcal{M}_{g,n})$$

$$\xrightarrow{g \rightarrow \infty} 0$$

# Minimum dilatation problem

$\text{Spec}(\mathcal{M}_g) = \{\log \lambda(\phi), \phi \in \text{Mod}(S_g)\}$  is discrete (Ivanov ~1985)

What is the smallest element?

$$\text{sys}(\mathcal{M}_{g,n}) = \min \text{Spec}(\mathcal{M}_{g,n})$$

$$\xrightarrow[g \rightarrow \infty]{} 0$$

Conjecture (Hironaka; McMullen 2000)

$$\lim_{g \rightarrow \infty} g \cdot \text{sys}(\mathcal{M}_g) = \log \varphi^2, \text{ where } \varphi \text{ is the golden ratio.}$$

Proof of Hironaka's conjecture  
for matrices

# Proof of Hironaka's conjecture for matrices

Theorem (McMullen 2015)

The minimum value of the spectral radius  $\lambda(A)$  over all reciprocal Perron-Frobenius matrices  $A \in M(2g, \mathbb{Z}_{\geq 0})$  satisfies

# Proof of Hironaka's conjecture for matrices

Theorem (McMullen 2015)

The minimum value of the spectral radius  $\lambda(A)$  over all reciprocal Perron-Frobenius matrices  $A \in M(2g, \mathbb{Z}_{\geq 0})$  satisfies

$$\lambda^g \geq \varphi^2$$

# Proof of Hironaka's conjecture for matrices

Theorem (McMullen 2015)

The minimum value of the spectral radius  $\lambda(A)$  over all reciprocal Perron-Frobenius matrices  $A \in M(2g, \mathbb{Z}_{\geq 0})$  satisfies

$$\lambda^g \geq \varphi^2$$

Connection with surface dynamics:  $\lambda(f)$  is the PF eigenvalue of  $A \in \text{Sp}(2d, \mathbb{Z})$

# Proof of Hironaka's conjecture for matrices

Theorem (McMullen 2015)

The minimum value of the spectral radius  $\lambda(A)$  over all reciprocal Perron-Frobenius matrices  $A \in M(2g, \mathbb{Z}_{\geq 0})$  satisfies

$$\lambda^g \geq \varphi^2$$

Connection with surface dynamics:  $\lambda(f)$  is the PF eigenvalue of  $A \in \text{Sp}(2d, \mathbb{Z})$

$$\lambda(f)^d \geq \varphi^2$$

# Proof of Hironaka's conjecture for matrices

Theorem (McMullen 2015)

The minimum value of the spectral radius  $\lambda(A)$  over all reciprocal Perron-Frobenius matrices  $A \in M(2g, \mathbb{Z}_{\geq 0})$  satisfies

$$\lambda^g \geq \varphi^2$$

Connection with surface dynamics:  $\lambda(f)$  is the PF eigenvalue of  $A \in \text{Sp}(2d, \mathbb{Z})$

$$\lambda(f)^d \geq \varphi^2$$

Since  $d \leq 6g - 6$ , we get  $\lambda^g \geq \varphi^{2/3}$

# Minimum dilatation problem

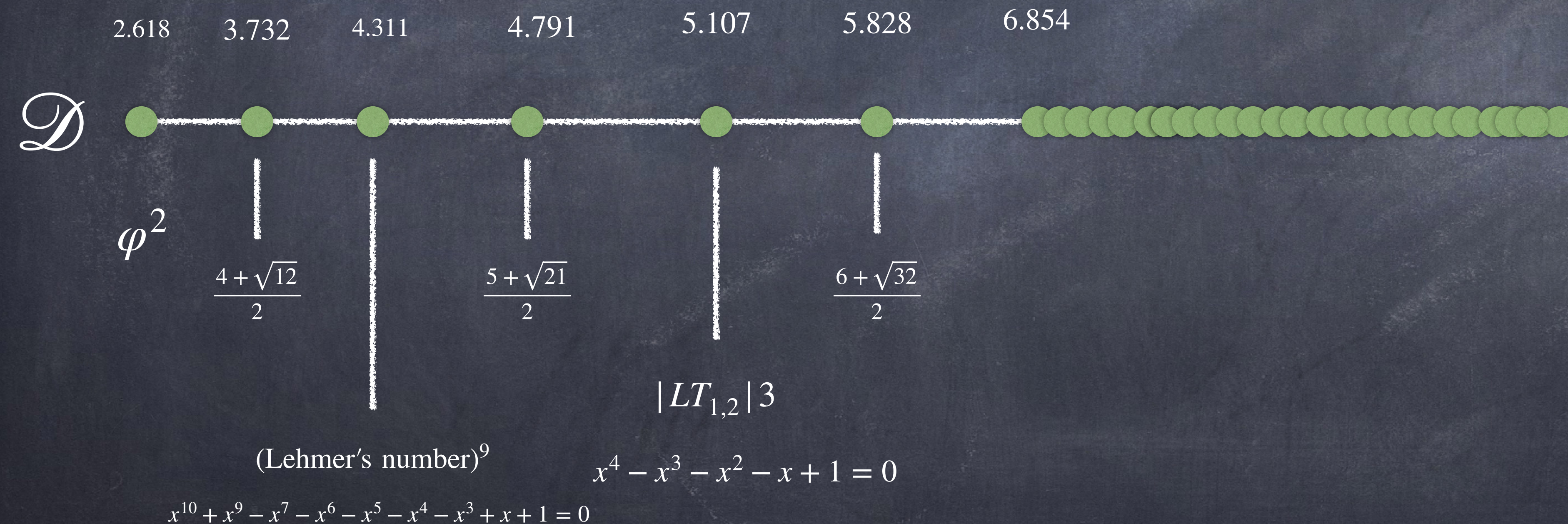
Theorem (Hironaka-Tsang; Tsang 2024)

The set  $\mathcal{D} = \{\lambda(\phi)^{2g-2+n(\phi)}; \phi \in \text{Mod}(g)\}$  of normalized dilatations of « fully-punctured » pseudo-Anosov map is:

# Minimum dilatation problem

Theorem (Hironaka-Tsang; Tsang 2024)

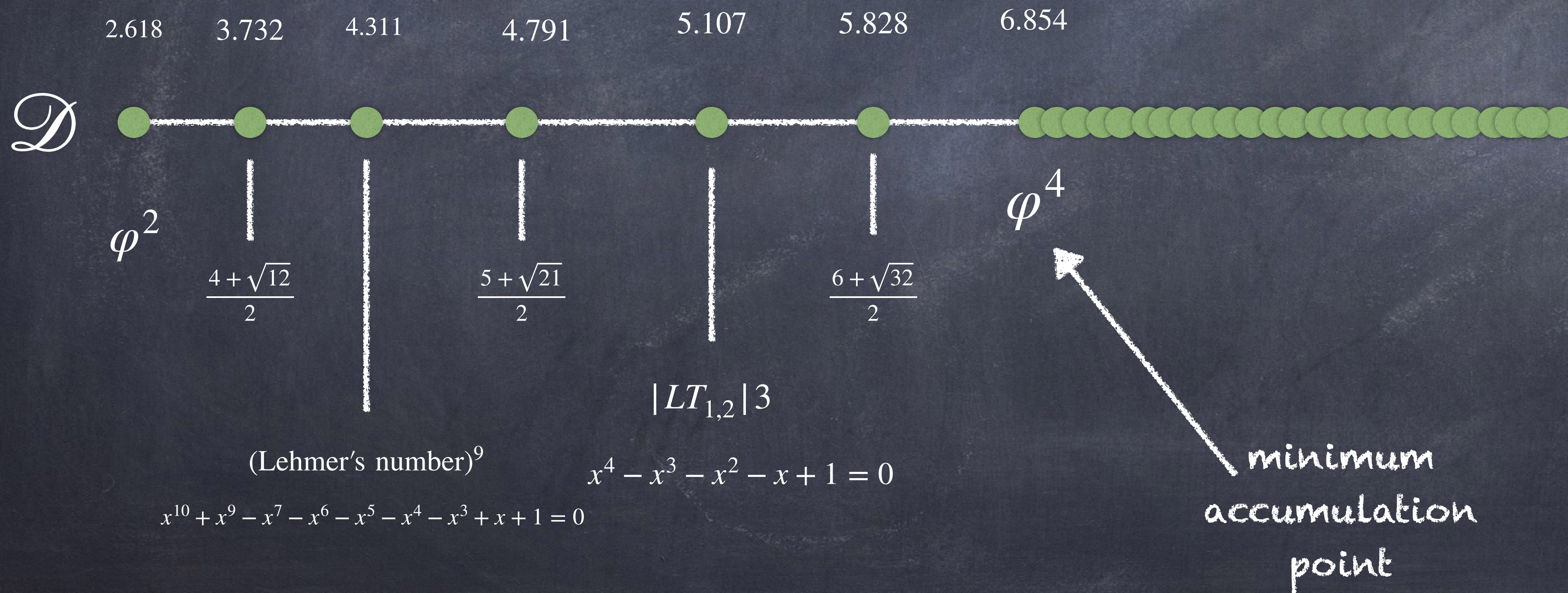
The set  $\mathcal{D} = \{\lambda(\phi)^{2g-2+n(\phi)}; \phi \in \text{Mod}(g)\}$  of normalized dilatations of « fully-punctured » pseudo-Anosov map is:



# Minimum dilatation problem

Theorem (Hironaka-Tsang; Tsang 2024)

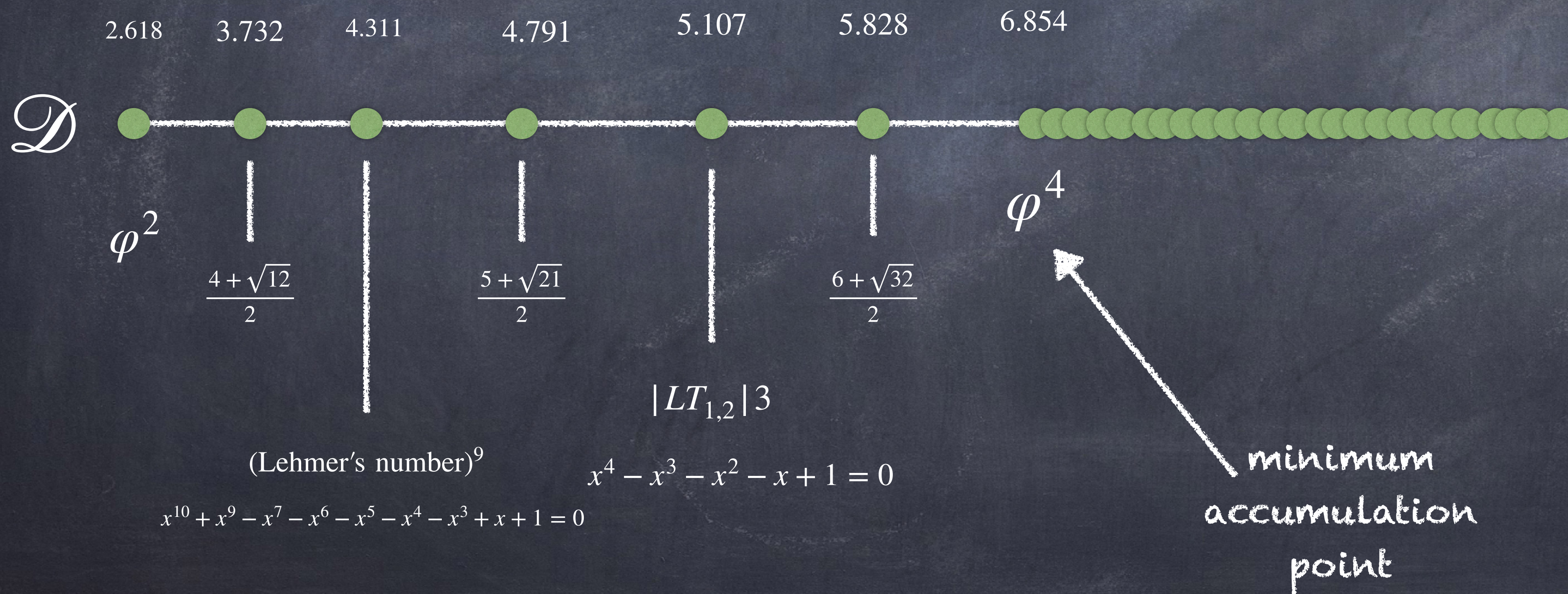
The set  $\mathcal{D} = \{\lambda(\phi)^{2g-2+n(\phi)}; \phi \in \text{Mod}(g)\}$  of normalized dilatations of « fully-punctured » pseudo-Anosov map is:



# Minimum dilatation problem

Theorem (Hironaka-Tsang; Tsang 2024)

The set  $\mathcal{D} = \{\lambda(\phi)^{2g-2+n(\phi)}; \phi \in \text{Mod}(g)\}$  of normalized dilatations of « fully-punctured » pseudo-Anosov map is:



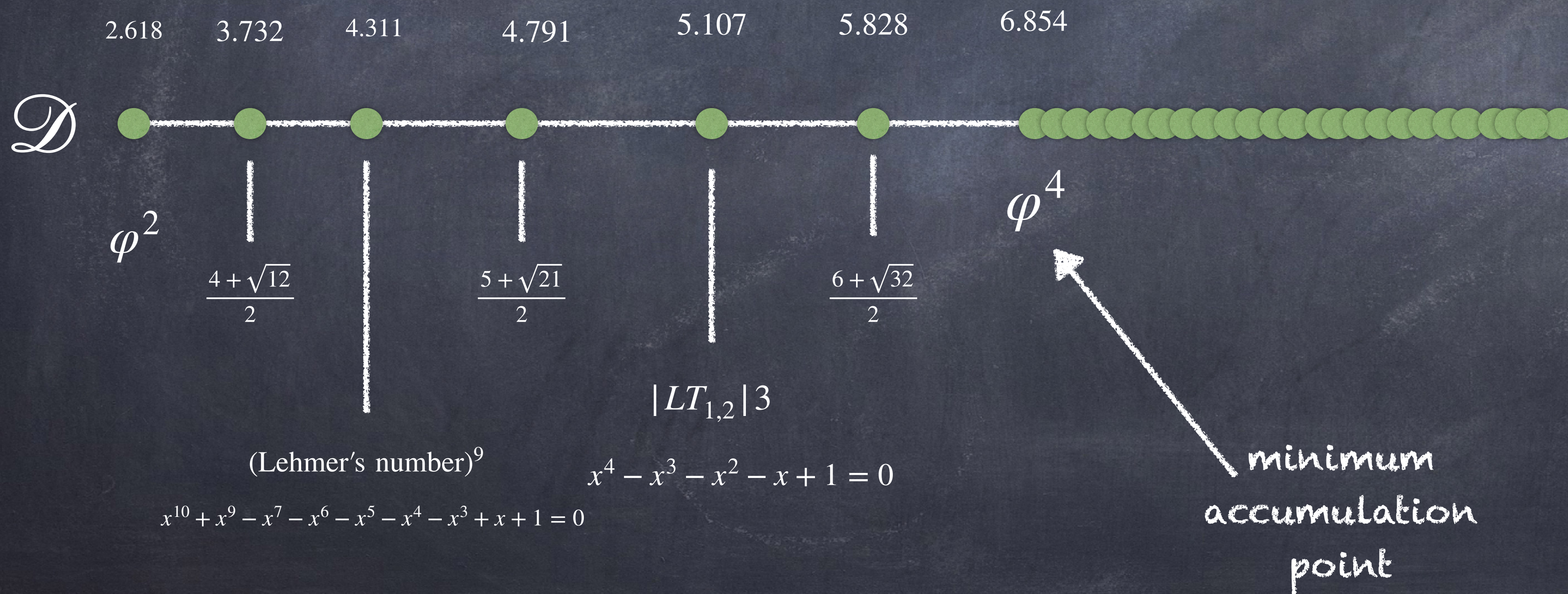
Conjecture

$$\lambda_{\min}^{2g-2} \xrightarrow{g \rightarrow \infty} \varphi^4$$

# Minimum dilatation problem

Theorem (Hironaka-Tsang; Tsang 2024)

The set  $\mathcal{D} = \{\lambda(\phi)^{2g-2+n(\phi)}; \phi \in \text{Mod}(g)\}$  of normalized dilatations of « fully-punctured » pseudo-Anosov map is:



Conjecture

$$\lambda_{\min}^{2g-2} \xrightarrow{g \rightarrow \infty} \varphi^4$$

And probably,  
 $n(\phi_{\min}) = 4g - 4$

thus

$$\lambda^{2g-2} \geq \varphi^{4/3}$$

# Minimum dilatation problem

Theorem (L-Liechti-Tsang 2025)

If  $\phi$  is an orientation-reversing « fully-punctured » pA with at least two puncture orbit then the normalized dilatation satisfies

$$\lambda(\phi)^{|X(S)|} \geq \sigma^2$$

$\sigma = 1 + \sqrt{2}$  is the silver ratio. The inequality is sharp.

# Minimum dilatation problem

Conjecture (Silver conjecture) (L-Liechti-Tsang)

The minimal accumulation point of the set of normalized stretch factors of orientation-reversing fully-punctured pseudo-Anosov maps is  $\sigma^2$

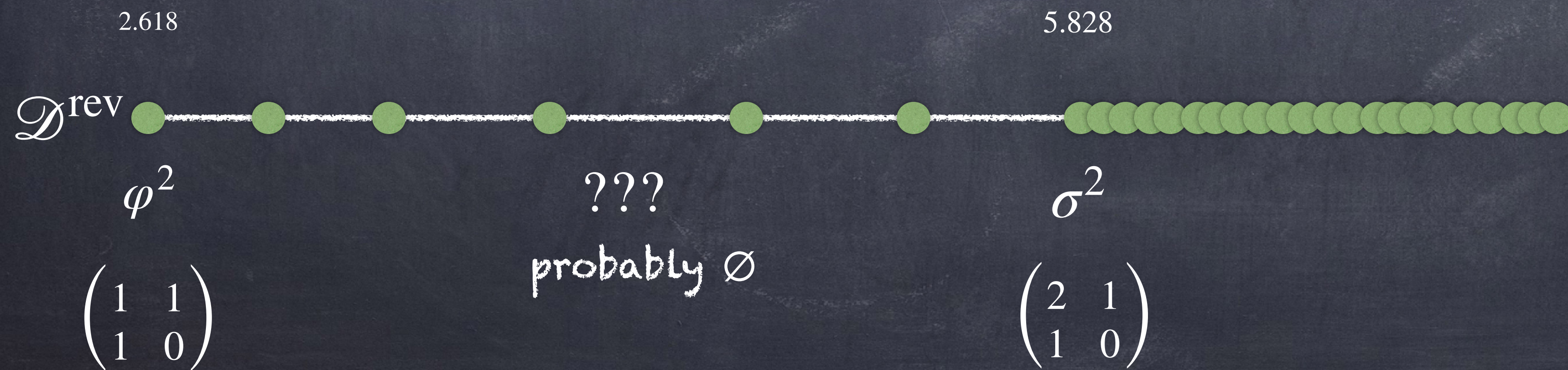
In other words, the spectrum  $\mathcal{D}^{\text{rev}} \cap (\varphi^2, \sigma^2)$  consists only on finitely many isolated values in the range  $(\varphi^2, \sigma^2)$

# Minimum dilatation problem

Conjecture (Silver conjecture) (L-Liechti-Tsang)

The minimal accumulation point of the set of normalized stretch factors of orientation-reversing fully-punctured pseudo-Anosov maps is  $\sigma^2$

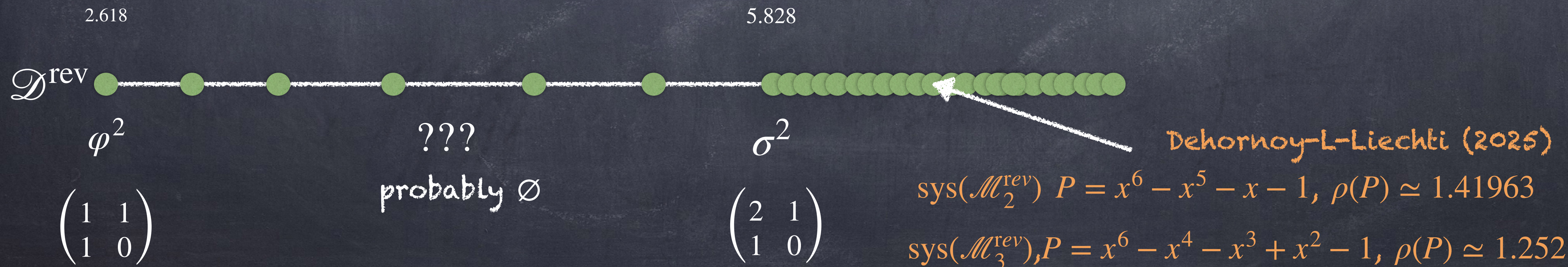
In other words, the spectrum  $\mathcal{D}^{\text{rev}} \cap (\varphi^2, \sigma^2)$  consists only on finitely many isolated values in the range  $(\varphi^2, \sigma^2)$



# Minimum dilatation problem

Conjecture (Silver conjecture) (L-Liechti-Tsang)

The minimal accumulation point of the set of normalized stretch factors of orientation-reversing fully-punctured pseudo-Anosov maps is  $\sigma^2$



Happy Birthday!

