Dynamics of rational mappings in dimension 2 and the Renormalization Map for the Iterated Monodromy Group of the Basilica Map

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We will look at the dynamics of a family of birational maps on $\mathbb{R}^2$ and $\mathbb{C}^2$. Then we will discuss a noninvertible rational map acting in dimension 2. This map arises as a renormalization map in the Iterated Monodromy group of the Basilica map.

Rational mappings of $\mathbb{C}^2$

A *rational map* is an *algebraic expression*

$$f = (P_1/Q_1, P_2/Q_2) : \mathbb{C}^2 \dashrightarrow \mathbb{C}^2$$

where the P_j and Q_j are polynomials. We can iterate this by algebraic composition

$$f \circ f = (P_1(f)/Q_1(f), P_2(f)/Q_2(f))$$

to obtain a new rational map.
If $Q_1(z)Q_2(z) \neq 0$, then we may define $f(z) \in \mathbb{C}^2$. If we wish iterate f as a pointwise mapping, then f^2, as well as higher iterates, may be defined only on a subset of $\mathbb{C}^2$.
We will see that there are choices if we want to extend f as a pointwise map of a compactification X of $\mathbb{C}^2$.

Rational mappings of $\mathbb{P}^2 = \mathbb{C}^2 \cup L_\infty$

Theorem (Weierstrass)

If f is a meromorphic map from $\mathbb{P}^2$ to itself, then f is given by homogeneous polynomials f_0, f_1, f_2 of degree d such that $f = [f_0 : f_1 : f_2]$.

A map of the form f gives a rational map $(f_1/f_0, f_2/f_0)$ on $\mathbb{C}^2$. If $\{f_0 = f_1 = f_2 = 0\} = 0$, then f is everywhere holomorphic. We will be particularly interested in the *indeterminacy locus*

$$\mathcal{I} = \{z \in \mathbb{P} : f_0(z) = f_1(z) = f_2(z) = 0\}$$

If $\mathcal{I} = \emptyset$, then f is everywhere holomorphic. We will have $\mathcal{I} = \emptyset$ for "generic" polynomials f_j, but many interesting cases are not generic.

Newton map in dimension 2 – Hubbard and Papadopol

Looking at an interesting non-generic rational map in dimension 2 follows the precedent set by Hubbard and Papadopol. They considered the problem of finding the four points where two plane conics intersect. They showed that you can reduce to the case where the conics are parabolas, thus solving the equations:

$$f_1 = y - x^2 + a = 0, \text{ and } f_2 = x - y^2 + b = 0$$

The Newton iterator for solving $F = (f_1, f_2) = (0, 0)$ is then

$$N_F(x, y) = \mathrm{id} - (DF)^{-1} \cdot F(x, y)$$

They showed that N_F is a rational map of algebraic degree 3 and topological degree 4 and that it has interesting geometric properties. This work presented us with interesting questions and was influential for subsequent work on the dynamics of rational maps.

Invertible maps – Hubbard's work with Hénon maps

A rational map f is said to be *birational* if there is a rational map g such that $f \circ g = \mathrm{id}$, i.e., $f(g(z)) = z$ for z in some open set.
Perhaps the first examples of birational maps of $\mathbb{P}^2$ are the *complex Hénon maps*, which in degree 2 are

$$H(x, y) = (x^2 + c - ay, x)$$

These maps were first studied in the complex domain by Hubbard, who was very influential in initiating and advancing that whole field of research. It is possible to study these maps without looking at the points of indeterminacy because the points of indeterminacy are at infinity, and the line at infinity is invariant under both H and H^{-1}. The indeterminate behavior of H does not influence $\mathbb{C}^2$, and all of the recurrent dynamics of H takes place inside of the compact set $K \subset \mathbb{C}^2$.

Dynamical degree

When we take the algebraic composition of rational maps, we may find common factors, which we remove if we are mapping to $\mathbb{P}^2$

$$f^{\langle n \rangle} = \chi_n f^n$$

Thus we see that $\deg(f^{n+m}) \leq \deg(f^n)\deg(f^m)$ which means that we have a limit, the *dynamical degree*

$$\delta(f) := \lim_{n\to\infty} (\deg(f^n))^{1/n}$$

Thus if $\delta < \deg(f)$, then

$$\deg(\chi_n) \sim \deg(f)^n - \delta^n \sim \deg(f)^n$$

which means that the algebraic composition $f^{\langle n \rangle}$ is “mostly cancellation”. In the case of Hénon maps, we have $\delta(H) = 2 = \deg(H)$.

What is "higher dimension"?

$\mathcal{M}_q = q \times q$ matrices

$J(x_{i,j}) = \left(\frac{1}{x_{i,j}}\right) =$ Cremona involution

$I(A) = A^{-1} =$ matrix inversion

I and J define rational involutions on $\mathbb{P}(\mathcal{M}_q)$. degree$(I) = q - 1$, and degree$(J) = q^2$. $K := I \circ J$ defines a birational map of $\mathbb{P}(\mathcal{M}_q)$. What is the (dynamical) degree of K? Computer is of limited assistance if $q \geq 14$.

- K may be defined to act on many different spaces of matrices – symmetric, cyclic, . . .
- f is defined over the integers, so we can take our matrices to be defined over any field $\mathbb{F}$.
- To check *algebraic stability*, we consider *exceptional* E and use the criterion that $K^n(E) \not\subset \mathcal{I}(K)$ for all $n \geq 0$.
- With Kyounghee Kim, we found it useful to work over the field $\mathbb{F}_7$, the field of 7 elements: K acts on a set with $7(q^2 - 1)$ elements. We showed algebraic stability by finding a point $p_0 \in E$ with $K^n(p_0) \notin \mathcal{I}(K)$ (mod 7), $\forall n$.

"Golden mean" birational map – work with Diller

We will start by discussing a family of maps which are very well behaved and illustrate the difficulties of working with points of indeterminacy.
We consider the family, $a < 0$, $a \neq -1$:

$$f_a(x, y) = \left(y \frac{x+a}{x-1}, x + a - 1 \right),$$

f_a is *reversible*: the involution $(x, y) \mapsto (-y, -x)$ conjugates f_a to its inverse. It will be most convenient to consider f_a as acting on the set $\mathbb{P}^1 \times \mathbb{P}^1$ (which is birationally equivalent to $\mathbb{P}^2$). The lines at infinity $\{\infty\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{\infty\}$ are interchanged, and f^2 acts as translation by $a - 1$ on each of them. (∞, ∞), which is a parabolic fixed point, which governs the global behavior of f_a.
$\deg(f^n)$ is the n-th Fibonacci number, so

$$\delta(f_a) = \frac{1 + \sqrt{5}}{2} \sim 1.618$$

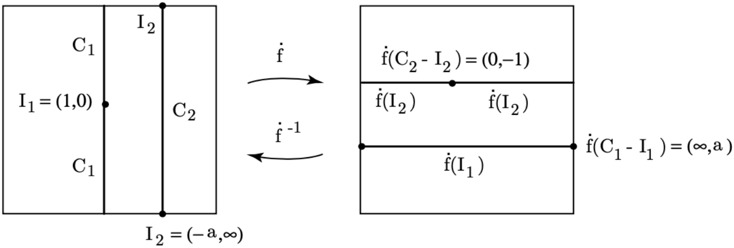
is the "golden ratio".

Blowup/blowdown behavior of f_a on $\mathcal{I} \subset \mathbb{P}^1 \times \mathbb{P}^1$.

The indeterminate points are

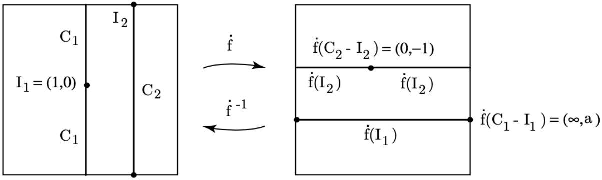
$$\mathcal{I}(f_a) = \{(1,0), (-a,\infty)\} \text{ and } \mathcal{I}(f_a^{-1}) = \{(0,-1), (\infty, a)\}$$

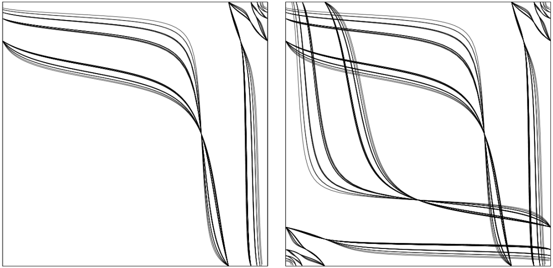
The real locus is $\mathbb{RP}^1 \sim S^1 = [\infty = -\infty, \infty]$



Stable/unstable laminations for $a = -2$

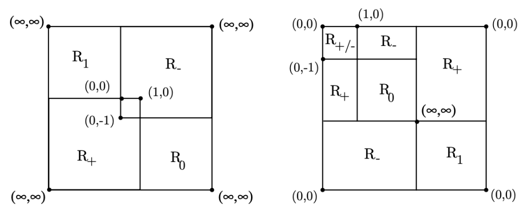
The stable lamination contains crossings, caused by the collapsed vertical lines. The unstable lamination comes from applying the reversing involution $(x, y) \mapsto (-y, -x)$.

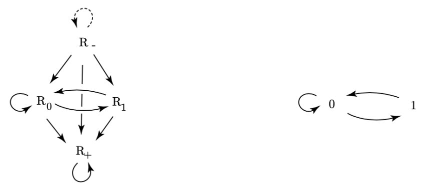




The finite indeterminate points $(1,0)$ and $(0,-1)$ are used to define a partition. The "filtration" properties are shown. The recurrent part gives the "golden mean" subshift.

Theorem (Filtration Properties)





Parabolic basins define the nonwandering set

The forward/backward parabolic basins $\mathcal{B}^{\pm}$ are the points that approach (∞, ∞) locally uniformly in forward/backward time.

$$\mathcal{B}^{+} = \bigcup_{n=0}^{\infty} f^{-n}(R_{+}) \qquad \mathcal{B}^{-} = \bigcup_{n=0}^{\infty} f^{n}(R_{-})$$

We say that a point p is *wandering* if there is a neighborhood U of p such that $U \cap f^n(U - \mathcal{I}(f^n)) = \emptyset$ for all integers $n \neq 0$.
Let Ω denote the nonwandering set, i.e., the complement of the wandering points.

Theorem

$$\Omega \cap \mathbb{R}^2 = \mathbb{R}^2 - \left(\mathcal{B}^{+} \cup \mathcal{B}^{-}\right)$$

Golden Mean Subshift: Coding rectangles

$\Sigma_G =$ bilateral sequences $w = (w_n)_{n\in\mathbb{Z}}$ on symbols 0 and 1 such that the word "11" does not appear anywhere.

If we code an orbit by its itinerary, we see that its code belongs to Σ_G. If it is in an axis at infinity, then it will begin or end with $(10)^\infty$ or $(01)^\infty$.

$\Sigma_G^* =$ elements of Σ_G that do not begin or end in $(01)^\infty$ or $(10)^\infty$

For any finite word $w_{[-m,n]}$ in Σ_G, we define the rectangle

$$R(w_{[-m,n]}) := \text{closure of} \bigcap_{j=-m}^{n} f_a^j(\text{int}(R_{w_j}))$$

Theorem

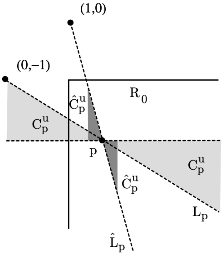
For $w \in \Sigma_G^*$, the rectangle $R(w_{[-m,n]})$ shrinks to a point as $m, n \to \infty$. This gives a map

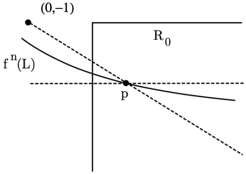
$$R : \Sigma_G^* \to \mathbb{R}^2, \quad w \mapsto R(w)$$

and the image of R is the nonwandering set in $\mathbb{R}^2$.

Points of indeterminacy give Invariant cone field

Cone fields give some (but not uniform) expansion. They allow us to control the boundaries of rectangles so that the rectangles shrink to points.





Theorem

The cone fields are preserved: If p and $f(p)$ both belong to $(R_0 \cup R_1) \cap \mathbb{R}^2$, then

$$Df_a(p)(\hat{\mathcal{C}}_p) \subset \mathcal{C}_{f(p)}$$

Method

1. Choose the right curves (if it is possible)
2. Map them forward
3. Use intersection theory to count intersection points
4. Find enough real points and conclude that there can be no other intersections.

Cohomology: $H^2(\mathbb{P}^1 \times \mathbb{P}^1)$ is generated by V (class of vertical line $\mathbb{P}^1$) and H (horizontal).
The intersection product is defined by

$$V \cdot V = 0, \quad H \cdot H = 0, \quad H \cdot V = 1$$

The push forward is given by

$$f_* = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad f_*^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}$$

with respect to the basis $\langle V, H \rangle$.

Can this method work?

We draw pictures in $\mathbb{R}^2$, but we are actually working with the complexifications.

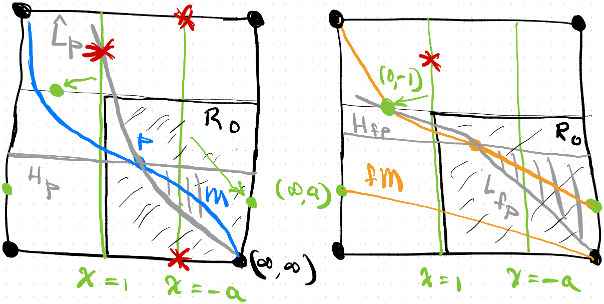
The method involves pulling back cohomology, which normally only involves continuous (topological) maps. This is justified for rational maps in the complex setting: it is essential that singlarities are complex codimension 2, not just real codimension 2.

Intersection theory counts all complex intersections with multiplicity, and multiplicity is always (≥ 1). Multiplicity 1 means transversality.

If γ is a (complex) curve in $\mathbb{P}^1 \times \mathbb{P}^1$, and its class is $\{\gamma\} = mV + nH$, then γ intersects every horizontal line m times and every vertical n times.

If H_1 and H_2 are disjoint horizontal lines, then $\{f^n H_j\} = F_n V + F_{n+1} H$. Thus $f^n H_1 \cdot f^n H_2 = 2F_n F_{n+1}$. We conclude that f is not a homeomorphism, i.e. $A \cdot B \neq f_* A \cdot f_* B$.

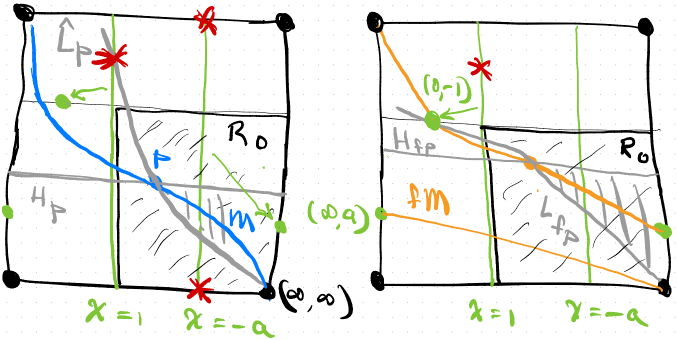
Illustration of method: invariance of cone field



Suppose $p \in R_0$ and $fp \in R_0$.
Choose line M with $p \in M$ and $T_pM \subset \hat{\mathcal{C}}_p$. M is not horizontal or vertical, so $\{M\} = 1V + 1H = [1,1]$. Thus $\{fM\} = f_*\{M\} = [1,2]$
$H_{fp} =$ horizontal, so $\{H_{fp}\} = [0,1]$.

$$H_{fp} \cdot fM = [0,1] \cdot [1,2] = 1$$

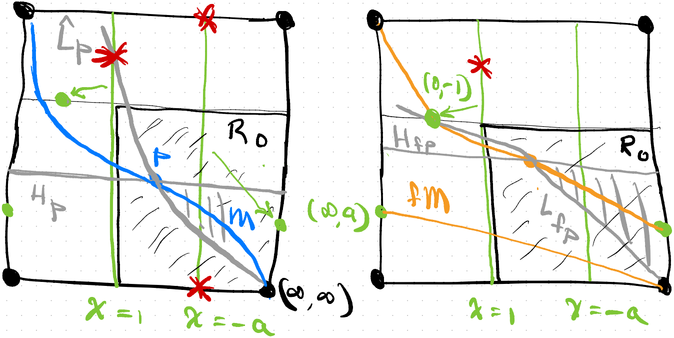
Thus H_{fp} and fM are transverse at fp, and fM is monotone.



Similarly, L_p not vertical or horizontal, so $\{L_{fp}\} = [1,1]$.

$$L_{fp} \cdot fM = [1,1] \cdot [2,1] = 3$$

M must intersect the vertical collapsed line $\{x = -a\}$, so $(0,-1) \in fM$. On the other hand, $(\infty,\infty) \in M$, so $(\infty,\infty) \in fM$. Thus $\{(0,-1), fp, (\infty,\infty)\} \subset L_{fp} \cap fM$, so all intersections are transverse. We conclude that $D_p(f)(T_pM) \notin \partial \mathcal{C}_{fp}$.



We have seen that L_{fp} intersects fM transversally at $\{(0,-1), fp, (\infty,\infty)\} \subset L_{fp} \cap fM$, and these are the only intersections. We know that fM intersects the vertical line at infinity to the right at (∞, a). Thus fM is above L_{fp} on the interval fp to (∞,∞). Since fM is monotone, it lies below H_{fp}.

We conclude that $Df_p(\hat{\mathcal{C}}_p)$ lies in the interior of the cone $\mathcal{C}_{fp}$.

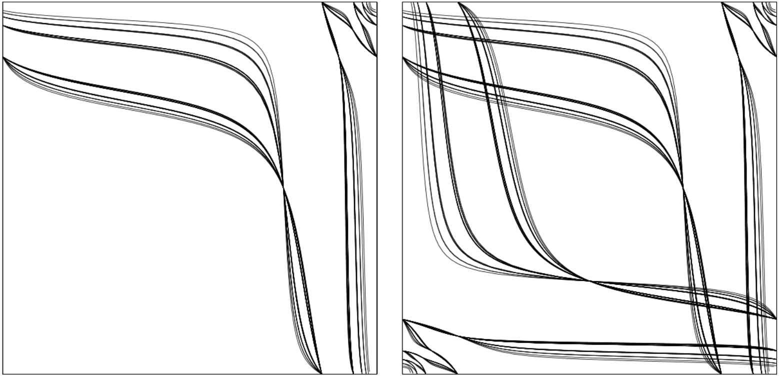
Points of view on currents

- L. Schwartz: k-forms with distribution coefficients $\sum S_\alpha dx^\alpha$
- Geometric measure theory: rectifiable sets and rectifiable currents
- Dynamical systems (Ruelle and Sullivan): *Axiom A maps lead to "geometric currents" which represent the entire stable/unstable laminations.*

  f_a is not hyperbolic, but we may work directly to define laminar currents. These consist of currents of integration over the leaves of the lamination, in coordination with transversal measures measure the lamination itself.
- The family of transversal measures is invariant under holonomy – sliding along the leaves of the lamination. The transversal measures themselves can (probably) be "worked out" as "weighted-balanced" measures in terms of the "generational bands" in the lamination. Like the horseshoe but more interesting, with F_n replacing 2^n.
- It is an interesting question to know any relationship between this current and its complex counterpart.

Measuring the lamination: constructing the product measure

Measures: If we intersect these currents, then we obtain the invariant measure of maximal entropy. The measure is the "product" of the two families of transversal measures. Since the stable/unstable laminations are guided by the cone fields, they are transverse. Transversality makes the product structure geometrically (visually) "obvious" and gives the measure of (maximal) entropy $\log \phi$.



What about the whole family? ($\varepsilon = a$, $\lambda_{\text{real}} =$ complexity)

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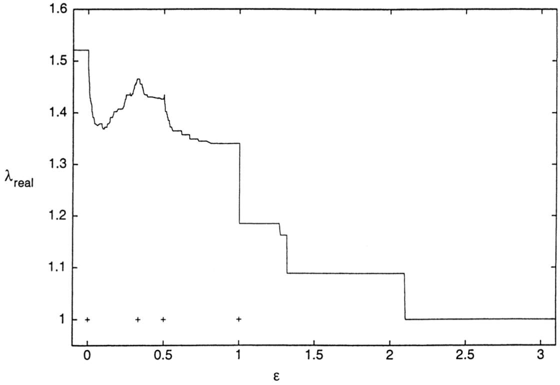

Fig. 1. An estimation of λ_{real} by $\mathcal{A}_{13}^{1/13}$, as a function of the parameter ϵ, in the interval $[-0.1, 3.1]$. The integrable points $\epsilon = -1, 0, \frac{1}{3}, \frac{1}{2}, 1$ are represented by crosses.

The value $a = 3$ seemed to be the minimum and especially interesting. This map was treated by same intersection-theoretic methods in the proceedings of HubbardFest (*Families and Friends*).

Different behaviors at fixed point $p_a = (\frac{1-a}{2}, \frac{a-1}{2})$

- $a < -1$: p_a is saddle point with negative multipliers $\lambda < -1$, $1/\lambda$
- $-1 < a < 0$: p_a is saddle with multipliers $\lambda > 1$, $1/\lambda$
- $a = 0$: The differential is $\sim \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
- $a > 0$: p_a is an elliptic fixed point. The rotation number moves from 0 to 1/2 as $a \to \infty$
- There are "singular" parameters $a = -1, 0, \frac{1}{3}, \frac{1}{2}, 1$ for which the dynamical degree is 1. These maps are integrable, and thus the entropy vanishes for the complexification of f_a, which acts in $\mathbb{CP}^1 \times \mathbb{CP}^1$. This means that the real map must have zero entropy, too. For the other values of a, f_a has dynamical degree ϕ, and the entropy of the complex map f_a is $\log \phi$.
- $a = 3$: For $a = 3$, the rotation number at p_a is $\frac{1}{3}$. For $a \neq 3$ but near 3, f_a is a twist map at p_a; computer pictures show typical KAM behavior at p_a. However, the direction of twist changes from "left" to "right" at $a = 3$.

12 Real Dynamics of a Family of Plane Birational Maps: Trapping Regions and Entropy Zero

Eric Bedford and Jeffrey Diller

1 Introduction

We consider dynamics of the one parameter family of birational maps

$$f = f_a : (x, y) \mapsto \left(y\frac{x + a}{x - 1}, x + a - 1 \right). \tag{1.1}$$

This family was introduced and studied by Abarenkova, Anglès d'Auriac, Boukraa, Hassani, and Maillard, with results published in [A1–7]. We consider here real (as opposed to complex) dynamics, treating f_a as a self-map of $\mathbf{R}^2$, and we restrict our attention to parameters $a > 1$. In order to discuss our main results, we let $\mathcal{B}^+, \mathcal{B}^- \subset \mathbf{R}^2$ be the sets of points with orbits diverging locally uniformly to infinity in forward/backward time, and we take $K \subset \mathbf{R}^2$ to be the set of points $p \in \mathbf{R}^2$ whose full orbits $(f^n(p))_{n\in\mathbf{Z}}$ are bounded.

In [BD05] we studied the dynamics of f_a for the parameter region $a < 0$, $a \neq -1$. In this case, $\mathcal{B}^+$ and $\mathcal{B}^-$ are dense in $\mathbf{R}^2$; and the complement $\mathbf{R}^2 - \mathcal{B}^+ \cup \mathcal{B}^-$ is a non-compact set on which the action of f_a is very nearly hyperbolic and essentially conjugate to the golden mean subshift. In particular, f_a is topologically mixing on $\mathbf{R}^2 - \mathcal{B}^+ \cup \mathcal{B}^-$, and most points therein have unbounded orbits. The situation is quite different when $a > 1$.

Theorem 1.1. *If $a > 1$, then $\mathbf{R}^2 - \mathcal{B}^+ \cup \mathcal{B}^- = K$. Moreover, the set K is compact in $\mathbf{R}^2$, contained in the square $[-a, 1] \times [-1, a]$.*

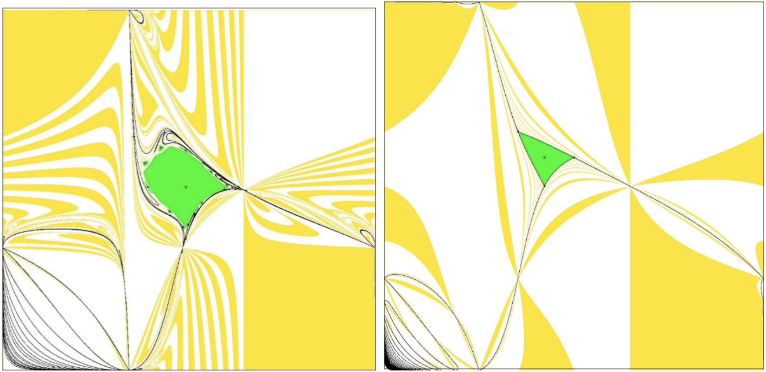

Figure 1. Dynamics of f for parameter values $a = 1.1$ (left) and $a = 2$ (right). Forward orbits of white points escape to infinity traveling up and to the right. Yellow points escape by alternating between the bottom right and upper left corners. The black curves are stable manifolds of a saddle three cycle (the vertices of the 'triangle'). The green regions consist of points whose orbits are bounded in forward and backward time. Reflection about the line $y = -x$ corresponds to replacing f by f^{-1}. It leaves the green region invariant and exchanges stable and unstable manifolds.

Zero entropy for $a = 3$: only two nonwandering points

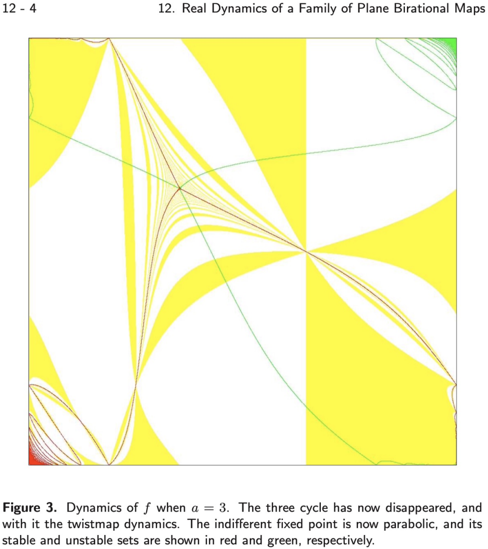

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Figure 3. Dynamics of f when $a = 3$. The three cycle has now disappeared, and with it the twistmap dynamics. The indifferent fixed point is now parabolic, and its stable and unstable sets are shown in red and green, respectively.

Joint work with Grigorchuk, Lyubich and Dang

In connection with the Iterated Monodromy Group for the Basilica map, there is a 1-parameter family of *spectral measures* ω_y. These arise from a sequence of polynomials $P_n(x, y)$. The first is given by

$$P_0 = 2y - x + 2$$

Then we may pass from level n to level $n + 1$:

$$P_{n+1}(x, y) = y^{2^{n+1}} P_n(f(x, y))$$

where f is a "renormalization" map found by Grigorchuk and Žuk. For fixed y the spectral measure is given by $\omega_y = \lim_{n\to\infty} \omega_y^n$ with

$$\omega_y^n := \lim_{n\to\infty} \frac{1}{2^n} \sum_{\{a:P_n(a,y)=0\}} \delta_a$$

Our objectives

- The spectral measures are carried on the real points, so it is of prime interest to consider the real map, i.e., f acting on $\mathbb{R}^2$.

- We will also consider this family in $\mathbb{C}^2$ and study the dynamics of the map f to get a fuller picture of the family of measures $\{\omega_y\}$.

- For this, we construct currents on $\mathbb{R}^2$ and $\mathbb{C}^2$, both.

- This "renormalization map" arises as merely an algebraic substitution, so there is no guarantee that it should be meaningful as pointwise mapping or some other dynamical system.

Complex analysis and complex geometry: positive, closed currents

Positive, closed currents are a useful generalization of complex curves and varieties. They have properties analogous to measures, and they allow us to define a topology which yields a useful compactness property.

Theorem (Lelong)

If T is a positive current on $\mathbb{C}^2$, then T is represented by integration (measures). A set of positive currents with uniformly bounded mass is weak* compact.

Theorem (Lelong)

A variety $V \subset \mathbb{C}^2$ defines a current of integration $[V]$, which is positive and closed. If T is a positive, closed (1,1)-current on $\mathbb{C}^2$, then there is a sequence of varieties V_k such that $\frac{1}{k}[V_k] \to T$ in the weak* topology on currents.

Spectral current

Let us set $\mathbb{L}_f = \{P_0(x, y) = 0\} = \{2y - x + 2 = 0\}$ Given this setup, we are let to consider the *n-th spectral current*

$$S_n := \frac{1}{2^n}[f^{-n}(\mathbb{L}_f)] = \frac{1}{2^n}[P_n = 0]$$

We may recover the measures ω_y^n by slicing the current S_n by the horizontal line at height y.
We will want the existence of a *spectral current*

$$S_{\text{spec}} := \lim_{n\to\infty} S_n$$

If this limit exists, then we will have

$$f^* S_{\text{spec}} = 2S_{\text{spec}}$$

To know the existence of this limit, we look at the mapping itself

$$f(x, y) := \left(-2 + \frac{x(x-2)}{y^2}, \frac{x-2}{y^2}\right)$$

The Sibony current

Theorem (Sibony)

Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ *be a rational map of degree* d *and dynamical degree* $1 < \delta < d$. *Then there is a unique positive, closed current* S *of mass 1 such that* $f^*S = d \cdot S$. *If* T *is any other positive, closed current of mass 1, then* $d^{-n}(f^n)^*T \to S$.

We will see that $\delta(f) < 2$ while $\deg(f) = 2$. Thus we may apply this Theorem to obtain a current S_{sib}, which is the same as the desired spectral current S_{spec}.
Let us recall the factorization $f^{\langle n \rangle} = \chi_n f^n$. In fact the varieties are nested, i.e., $\{\chi_n = 0\} \subset \{\chi_{n+1} = 0\}$, and $\chi_{n+1} = \varphi_n \chi_n$. Sibony showed, in fact, that this current is obtained as an infinite sum:

$$S = \sum_{n=1}^{\infty} d^{-n}[\varphi_n = 0]$$

Thus this current is, in some sense, "purely atomic".

First properties of f

We may write f as a map of $\mathbb{P}^2$ in homogeneous coordinates as

$$F([x:y:z]) = \left[-2y^2 - 2xz + x^2 : (x-2z)z : y^2\right]$$

The algebraic degree of F is 2, and

- $[2:0:1]$, i.e. the point $(2,0) \in \mathbb{C}^2$ is indeterminate
- $\mathbb{L}_c := \{x = 2\} \subset \mathbb{C}^2$ is collapsed, and $f(\mathbb{L}_c) = (-2,0)$
- f has *topological degree* 2, i.e. $\#f^{-1}(\text{generic point}) = 2$

Thus inside $\mathbb{P}^2$ there is one collapsed line and one point of indeterminacy. Also

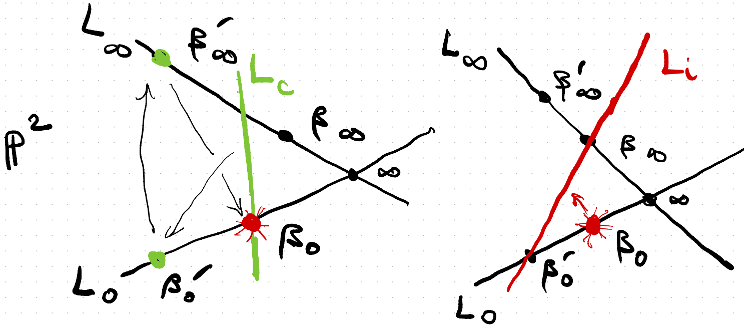
- x-axis $:= L_0 = \{y = 0\} \leftrightarrow L_\infty = \{z = 0\}$ is an invariant sub-system
- x-axis is a line of branch points for F
- The line $\mathbb{L}_f := \{2y - x + 2 = 0\}$ is fixed under $f \circ f$.

A difficulty: S not invariant under the dynamical system

S_{spec} is invariant under f but it is not invariant under f^4 because $\deg(f^4) = 14$, so we cannot have $(f^4)^* S_{\mathrm{spec}} = 16 S_{\mathrm{spec}}$. This loss of degree is "explained" by $f^3(\mathbb{L}_c) \in \mathcal{I}(f)$: if we follow the orbit

$$\mathbb{L}_c = \{x = 2\} \mapsto \beta_0' = (-2, 0) \mapsto \beta_\infty' = [-2 : 1 : 0] \mapsto \beta_0 = (2, 0)$$

$$\beta_0 \dashrightarrow \mathbb{L}_i := \overline{\beta_0' \beta_\infty}$$



Move to a better model: Compactification of $\mathbb{C}^2$ by a blowup of $\mathbb{P}^2$

We follow the orbit

$$\mathbb{L}_c \mapsto \beta_0' \mapsto \beta_\infty' \mapsto \beta_0 \dashrightarrow \mathbb{L}_i := \overline{\beta_0' \beta_\infty}$$

Thus we construct new manifold $\pi : X \to \mathbb{P}^2$ which blows up the points β_0, β_0', β_∞, β_∞'. We lift the map f to a map

$$f_X := \pi^{-1} \circ f \circ \pi : X \dashrightarrow X$$

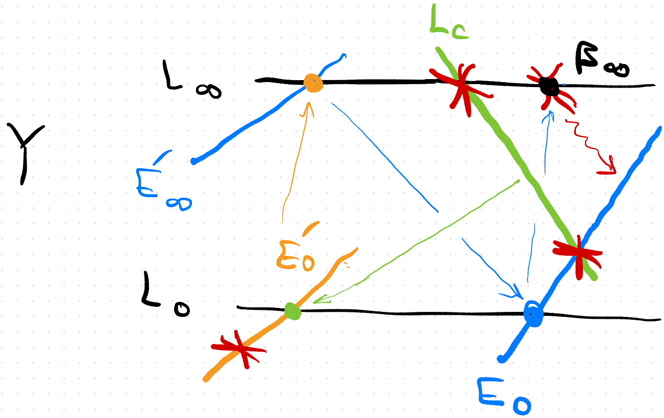
Although f_X is birationally conjugate to the original f, it acts differently in important respects.

Note: In the original formulation, we had $f^2(x, 0) = x^2 - 2$, so there is a subsystem $L_0 \leftrightarrow L_\infty$, and this persists for X, so f_X^2 acts like the Chebyshev map there.

In the new space X, we have added a sub-system $E_0 \leftrightarrow E_\infty$ on which f_X^2 is conjugate to $z \mapsto z^2 - 12$.

Why did we need to blow up β_∞?

The problem before was that f^3(collapsed line) $\in \mathcal{I}(f)$. Try again. We let Y denote $\mathbb{P}^2$ blown up at β_0, β_0' and β_∞', then we let $E_0 = \pi_Y^{-1}(\beta_0)$, $E_0' = \pi_Y^{-1}(\beta_0')$ and $E_\infty' = \pi_Y^{-1}(\beta_\infty')$, and we have and indeterminate point at $L_0 \cap E_0$. So $f_Y^4(\mathbb{L}_c) = \beta_\infty \in \mathcal{I}(f_Y)$.

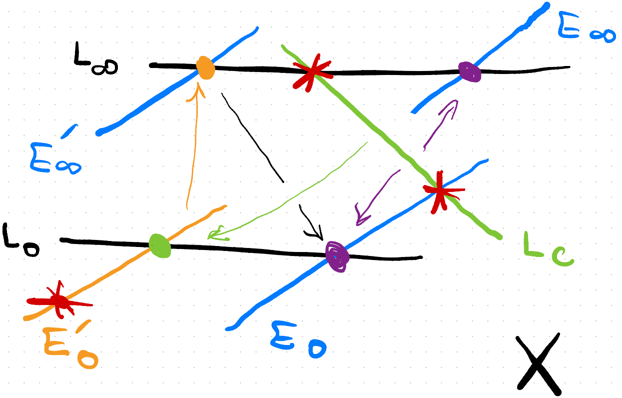


Behavior of f_X acting on X

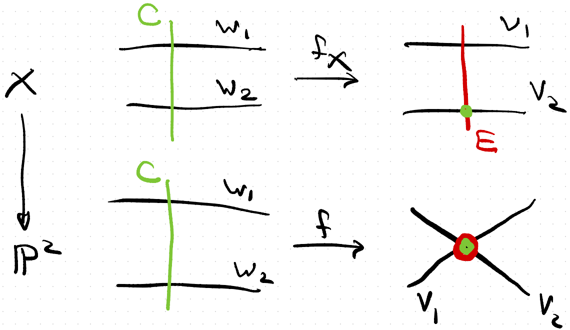
We now have a space which is $\mathbb{P}^2$ blown up at 4 points. The induced map f_X has *2 collapsed lines* ($\mathbb{L}_c$ and E_0') and *3 points of indeterminacy*

$$\mathcal{I}(f_X) = \{\mathbb{L}_c \cap L_\infty, \mathbb{L}_c \cap E_0, E_0' \cap \{x = -2\} = E_0' \cap \text{vertical through } \beta_0'\}$$

We have a new 2-cycle: $E_0 \cap L_0 \leftrightarrow E_\infty \cap L_\infty$ which captures the collapsed orbits, and a new sub-system $E_0 \leftrightarrow E_\infty$.



Difference between f^* and f_X^* as pullbacks of currents



$$f_X^*(V_1) = W_1 + C, \quad f_X^*(V_2) = W_2$$
$$f^*(V_1) = W_1 + C, \quad f^*(V_2) = W_2 + C$$

Proposition

$$f_X^*(S_{sib}) = 2S_{sib} - \frac{5}{6}[\mathbb{L}_c]$$

Blowing up creates new topology

$H^{1,1}(\mathbb{P}^2)$ is generated by the class of a general line in $\mathbb{P}^2$.
$\pi : X \to \mathbb{P}^2$ blows up β_0, β_0', β_∞, β_∞'. We let $E_0 := \pi^{-1}(\beta_0)$ denote the exceptional divisor over β_0, $E_0' := \pi^{-1}(\beta_0')$, etc.
The exceptional divisors are nontrivial elements of $H^{1,1}(X)$, so if $L_X = \pi^*(L)$ denotes the class of a general line in X, then L_X, E_0, E_0', E_∞, E_∞' is a basis of $H^{1,1}(X)$.

Theorem

The induced map $f_X^* : H^{1,1}(X) \to H^{1,1}(X)$ satisfies $(f_X^*)^n = (f_X^n)^*$: the passage to cohomology is functorial if we work on X, i.e., it respects composition.

With respect to this basis, f_X^* is a 5-by-5 matrix. Computing this matrix, we find that the spectral radius of f_X^* is $\delta \sim 1.801$, which is a root of $x^3 - x^2 - 2x + 1$.

Invariant Green current

We let $\theta^+ \in H^{1,1}(X)$ denote the eigenvector corresponding to the eigenvalue δ.

Theorem (Sibony - Guedj)

If η is a Kähler form on X with total mass 1, then there is a limit

$$T_f = \lim_{n\to\infty} \delta^{-n} f_X^{n*}(\eta)$$

which is independent of η. T_f is a positive, closed current on X of total mass 1, and $f_X^*(T_f) = \delta T_f$.

T_f is the Green current and defines an element of the invariant cohomology class θ^+. In particular, T_f is compatible with the dynamics of f:

$$f_X^{*n} T_f = \delta^n T_f$$

Approximation to the spectral current

The line $\mathbb{L}_c = \{x = 2\}$ is collapsed to the point $(-2, 0)$, and $f^2(\beta_0') \in \mathbb{L}_f$. Thus the preimages may be arranged as

$$f^{-n}(\mathbb{L}_f) = f^{-n+3}(\mathbb{L}_c) \cup f_X^{-n}(\mathbb{L}_f)$$

Theorem

$$T_f = \lim_{n\to\infty} T_{f,n} = \lim_{n\to\infty} \delta^{-n} f_X^{*n}[\mathbb{L}_f]$$

Thus we see that the difference between the n-th approximate spectral current and the approximate Sibony current is exponentially small.

$$S_{\mathrm{spec},n} = S_{\mathrm{sib},n-3} + \frac{\delta^n}{2^n} T_{f,n} + \text{smaller error}$$

Invariant cone field

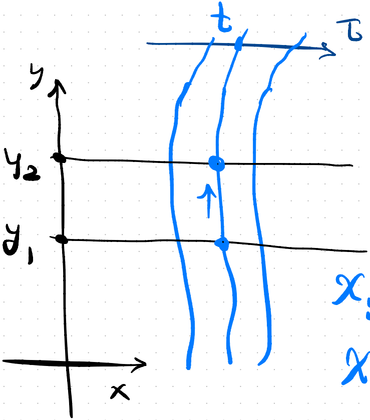
The spectral measures are real, so we will now try to describe S_{spec} as a real current supported on the real points $X_{\mathbb{R}}$ of X. The manifold $X_{\mathbb{R}}$ is the same as starting with the real projective space and blowing up points there. In analogy with the cone field for f_a, we can define a cone field on X_R, using lines that pass through the points of blowup β_0, β_0', β_∞, and β_∞'.

Theorem

The horizontal cone field is forward invariant under f and the vertical cone field is backward invariant.

The real preimages $f^{-n}(\mathbb{L}_f) \cap \mathbb{R}^2$ (for the spectral current), as well as $f^{-n}(\mathbb{L}_c) \cap \mathbb{R}^2$ (for the Sibony current) are "guided" by the cone field, and so each connected component is an arc which may be written "vertically", i.e., as a graph $\{x = \psi(y)\}$ over the y-axis. These graphs are pairwise disjoint, so we may write the real slice of the spectral current as a laminar current.

The n-th approximate real Sibony current



τ = transversal

$$S_n = \int_t [\gamma_t]\, \omega_y^n(t)$$

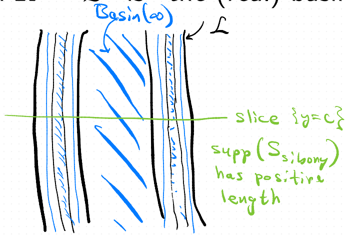
$\chi_{y_1 y_2}$ = holonomy map

$$\chi_*(\omega_{y_1}) = \omega_{y_2}$$

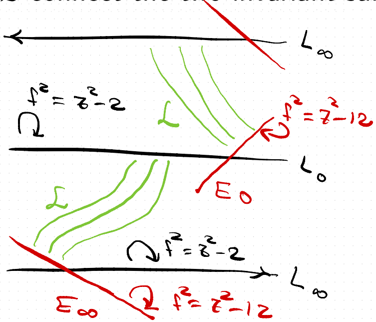
The approximate spectral measures ω_y^n are the slice measures of the approximate Sibony current. They are connected to each other by the holonomy map.

We let $\mathcal{L}$ denote the closure of all of these graphs. Our goals:

- $\mathcal{L}$ is a lamination, and $\mathbb{R}^2 - \mathcal{L}$ "is" the (real) basin of attraction of ∞.



- To what extent does $\mathcal{L}$ connect the two invariant sub-systems?



See these questions better in $\mathbb{P}^1 \times \mathbb{P}^1$?

Consider the birational map $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ given by

$$\varphi(x, y) = (u, v) = \left(\frac{x - 2y - 2}{y}, x - 2y + 2 \right)$$

and conjugate to a new form of f:

$$h(u, v) = \varphi \circ f \circ \varphi = \left(v \frac{u + 4}{u + 2}, u(u + 4) \right)$$

We note similarities with

$$f_a(x, y) = \left(y \frac{x + a}{x - 1}, x + a - 1 \right)$$

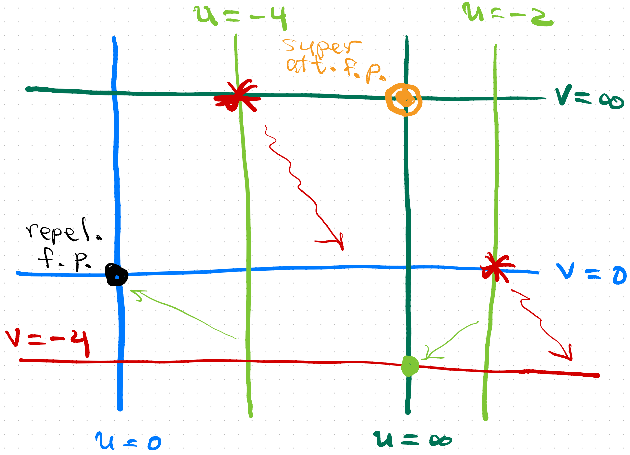
and f_a maps vertical lines to horizontal.

Blowup/blowdown behavior in new coordinates

Blue subsystem (u, v-axes): h^2 conjugate to $z \mapsto z^2 - 12$

Dark green subsystem ($u = \infty$, $v = \infty$): h^2 conjugate to $z \mapsto z^2 - 2$

Light green is collapsed. And we do not have $(h^3)^* \neq (h^*)^3$ because $\{u = -2\} \mapsto \{u = \infty, v = -4\} \mapsto \{u = -4, v = \infty\} \in \mathcal{I}(h)$



Happy birthday !!