Transcendental Thurston Theory in Toulouse
A topological characterization of postsingularly finite meromorphic functions

Dierk Schleicher

Université Aix-Marseille, France

Toulouse, 13 June 2025

3 The theorem (joint with Sergey Shemyakov)

Theorem 1 (Meromorphic Thurston theorem)

Let $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a postsingularly finite topological branched cover with hyperbolic orbifold modeled after a finite composition of

- *structurally finite transcendental maps (any degree); and*
- *rational maps.*

Then g is equivalent to a meromorphic function if and only if g does not admit a multicurve obstruction.

- this yields very large class of maps to which the theorem applies
- includes a new proof of Thurston's rational maps theorem
- also applies to further transcendental families if their only "efficient limits" are rational

4 Thurston theory

Thurston theory ("old part", well known)

- topology has geometry for 3-manifolds
- surface homeomorphisms have geometry
- topology and geometry of *rational* maps on the Riemann sphere

In all cases: exact conditions when topological objects have geometric realization

big panorama of research, four volumes of Hubbard book

Transcendental dynamics
joint with Sergey Shemyakov, in collaboration with Marseille dynamics team, especially Kostiantyn Drach, Bernhard Reinke, and Nikolai Prochorov

- **Project:** extension of Thurston's rational map theorem to large classes of transcendental maps

5 The question addressed by Thurston

Describe postcritically finite rational maps among all topological branched self-covers of the sphere that "want to be" holomorphic.

Purpose: topological covers have much greater flexibility, can be constructed and classified more easily: typically the classification of any family of rational maps starts with a classification of all topological maps and then restricts to the holomorphic ones.

Most prominent examples: classification of polynomials (Poirier, by Hubbard trees) and of Newton rational maps (Lodge–Mikulich–S).

Postcritically finite (pcf): all branch points have finite orbits (periodic or preperiodic); the postcritical set P is the union of all these forward orbits — finite!

Importance of pcf maps: easier to work on. "Search for your lost keys under the lamp post where the light is — but there is a lamp post at every important street intersection!"

6 Thurston's Fundamental Theorem

Theorem (Characterization of Rational Maps: Thurston–Douady–Hubbard, 1980's–1993)

A postcritically finite branched self-cover of the sphere with hyperbolic orbifold is realized by a rational map if and only if it does not admit a multicurve obstruction

Definition. A topological pcf branched cover f with postcritical set $P = P_f$ is *realized* by a rational map g if g satisfies the commutative diagram so that $\phi_1 = \phi_0$ on P_f, and they are homotopic rel P_f.

$$\begin{array}{ccc} (\mathbb{S}^2, P_f) & \xrightarrow{\phi_1} & (\overline{\mathbb{C}}, P_{g,1}) \\ \downarrow f & & \downarrow g \\ (\mathbb{S}^2, P_f) & \xrightarrow{\phi_0} & (\overline{\mathbb{C}}, P_{g,0}) \end{array}$$

Multicurve obstruction: finitely many "essential" curves in $\mathbb{S}^2 \setminus P$ with controlled mapping properties (can be explicitly verified).
This is a "negative result", complemented recently by positive result (Dylan Thurston)

6 a Thurston's Fundamental Theorem

Theorem (Characterization of Rational Maps: Thurston–Douady–Hubbard, 1980's–1993)

A postcritically finite branched self-cover of the sphere with hyperbolic orbifold is realized by a rational map if and only if it does not admit a multicurve obstruction

Definition. A topological pcf branched cover f with postcritical set $P = P_f$ is *realized* by a rational map g if g is topologically conjugate to a map $\tilde{f}$ that is homotopic to f rel P.

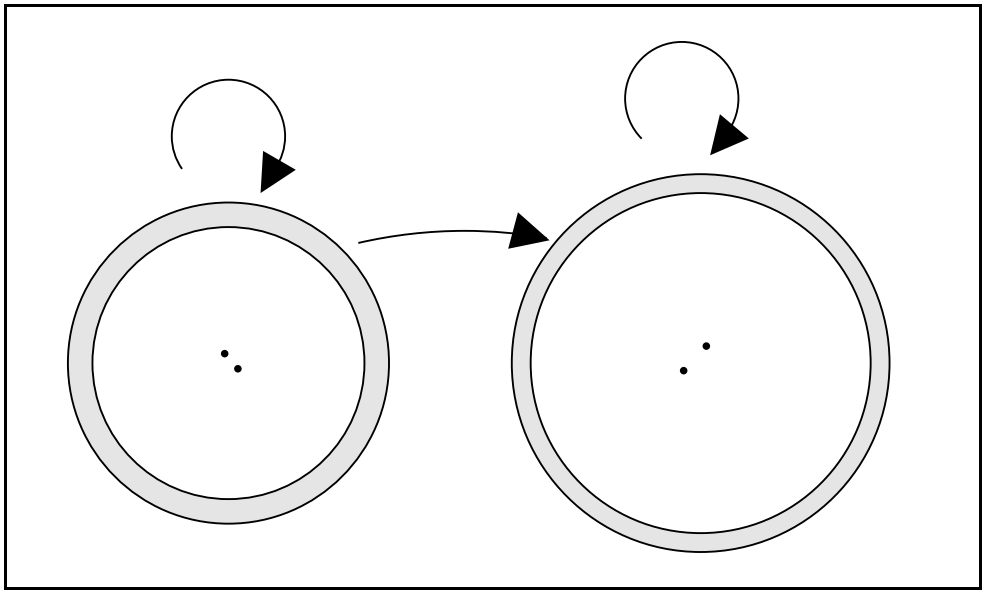
$$\begin{array}{ccc} (\mathbb{S}^2, P_f) & \xrightarrow{\phi_1} & (\overline{\mathbb{C}}, P_{g,1}) \\ \downarrow{\scriptstyle \psi \circ f} & & \downarrow{\scriptstyle g} \\ (\mathbb{S}^2, P_f) & \xrightarrow{\phi_1} & (\overline{\mathbb{C}}, P_{g,0}) \end{array}$$

In other words, $\psi = id$ on P, and ψ homotopic to id rel P.

Multicurve obstruction: finitely many "essential" curves in $\mathbb{S}^2 \setminus P$ with controlled mapping properties (can be explicitly verified).

This is a "negative result", complemented recently by positive result (Dylan Thurston)

7 Multicurve obstructions



$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Example of multicurve obstruction consisting of two “essential” curves in complement of postsingular set (marked by dots). When these curves are represented by complex annuli in $\mathbb{C} \setminus P$, ever more preimages of the left annulus will “pile up” in the homotopy class on the left, leading to infinite modulus by Grötzsch inequality

8 Postsingularly finite exponential functions

The first transcendental version of Thurston theory was developed by Hubbard–S–Shishikura 1995–2009:

Theorem (Thurston theory for exponential maps, [HSS] 2009)

A postsingularly finite topological exponential function is "realized" by a holomorphic exponential function if and only if it does not admit a Levy cycle.

Corollary: classification of postsingularly finite exponential maps (Laubner–S–Vicol, 2009 in terms of spiders; Pfrang–Rothgang–S. 2016 in terms of Hubbard trees).

Further theorems:

Nikolai Prochorov: two uncountable families of transcendental maps

Kostya Bogdanov: large families with infinite singular orbits

9 Transcendental mappings, singular values

Common wisdom: holomorphic dynamics is controlled by dynamics of singular values.

Definition: A *regular value* of a holomorphic map $f : \mathbb{C} \to \mathbb{C}$ is a point $w \in \mathbb{C}$ that has a neighborhood V so that all branches of f^{-1} restricted to V are univalent. Every other $w \in \mathbb{C}$ is called a *singular value*.

Two basic types of singular values:

Critical value: if $f'(z) = 0$, then $w = f(z)$ is a *critical value*.

Asymptotic value: if there is a curve $\gamma : [0, \infty) \to \mathbb{C}$ with $\gamma(t) \to \infty$ as $t \to \infty$ and $f(\gamma(t)) \to w$, then w is an *asymptotic value*.

A transcendental map is called of *finite type* (or: in the Speiser class) if it only has finitely many singular values.

10 Structurally finite transcendental mappings

"Maps of finite type" means restriction on target plane (finitely many singular values). We also need finiteness condition on the source plane (automatic in Thurston's context of rational maps).

A point z with $f'(z) = 0$ is known as *critical point*.
Equivalence class of homotopic curves γ is *asymptotic track*.

Definition: A holomorphic map is called *structurally finite* if it has

- only finitely many critical points
- only finitely many asymptotic tracts
- and only finitely many poles (if it is not entire anyway).

Theorem (Nevanlinna/Elfving): A meromorphic transcendental map is structurally finite if and only if it can be written as

$$g = \int s(\zeta) \exp(p(\zeta))\, d\zeta$$

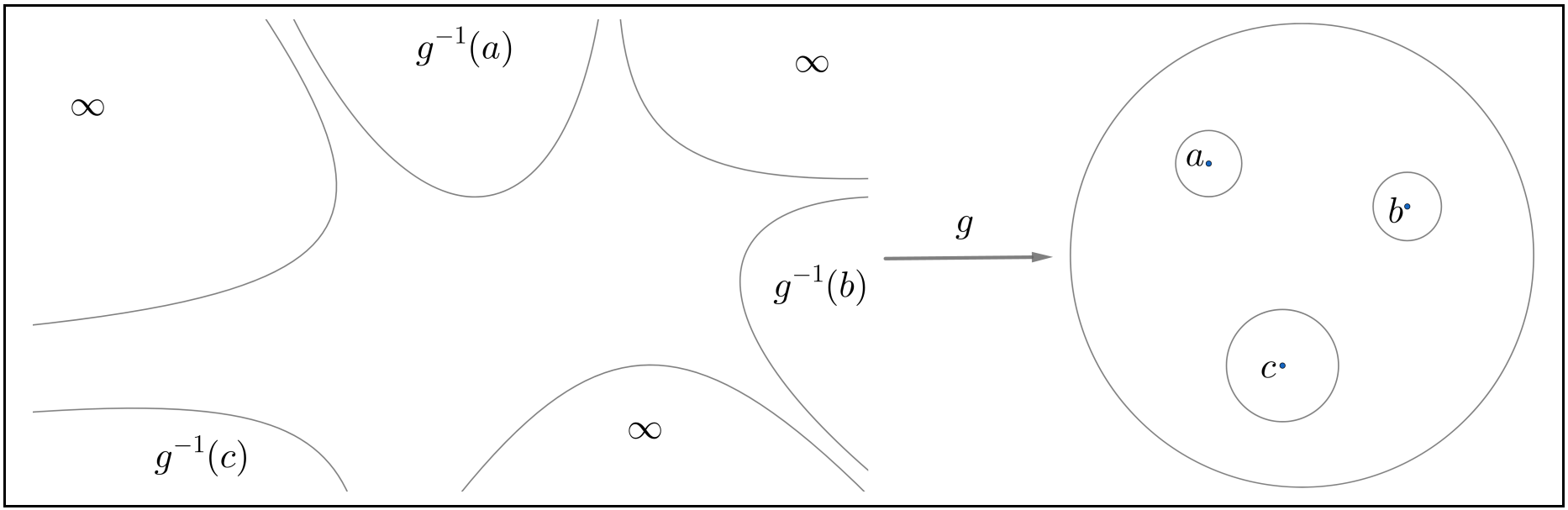
where p is a polynomial and s is a rational function
(note: condition on residues for integral to exist!)

11 Structurally finite maps

Structurally finite maps and their asymptotic tracts. Given

$$g = \int s(\zeta) \exp(p(\zeta))\, d\zeta$$

so that $\deg p = d$, then g has $2d$ asymptotic tracts: half of them over ∞, the other half over finite asymptotic values.



12 The σ-map in Teichmüller space

Fundamental idea of Thurston: "topological candidate function" f induces a map σ on a finite-dimensional Teichmüller space.

Let $P_f \ni \infty$ be the finite set of orbits of branch points of f and $S := (\mathbb{S}^2, P_f)$: sphere with finitely many marked points. Then f induces self-map on $\mathrm{Teich}(S)$: given $\phi_0 : S \to (\overline{\mathbb{C}}, \phi_n(P))$, then

$$\begin{array}{ccc} (\mathbb{S}^2, P_f) & \xrightarrow{\phi_1} & (\overline{\mathbb{C}}, P_{g,1}) \\ \Big\downarrow f & & \Big\downarrow g \\ (\mathbb{S}^2, P_f) & \xrightarrow{\phi_0} & (\overline{\mathbb{C}}, P_{g,0}) \end{array}$$

by pull-back of complex structures and the uniformization theorem. Here $P_{g,i} = \phi_i(P_f)$. The map g is holomorphic and in the space of functions where we search solutions.

Fundamental observation: $\phi_0 \sim \phi_1$ (homotopy rel P_f) $\iff \exists$ fixed point in Teich $\iff$ f is equivalent to rational map g.

Known properties: The map σ is locally uniformly contracting, so either have fixed point in Teich space, or iteration of σ started at any base point must leave compacts of Teich space

13 Cotangent space: quadratic differentials

$$\begin{array}{ccc} (\mathbb{S}^2, P_f) & \xrightarrow{\phi_1} & (\overline{\mathbb{C}}, P_{g,1}) \\ \Big\downarrow f & & g \Big\downarrow \\ (\mathbb{S}^2, P_f) & \xrightarrow{\phi_0} & (\overline{\mathbb{C}}, P_{g,0}) \end{array}$$

σ-map on Teich space:
$[\phi_1] := \sigma([\phi_0])$

Contraction of σ measured in terms of cotangent space:

$$\|d\sigma[\phi_0]\| = \sup_q \frac{\|g_* q\|}{\|q\|} \qquad \in [0, 1]$$

where g is defined by the diagram above: for us, a structurally finite transcendental map.

Here $q = q(z)\, dz^2$ is a holomorphic quadratic differential on $(\overline{\mathbb{C}}, P_{g,1})$ so that $\|q\| := \int_{\mathbb{C}} |q(z)|\, dx\, dy = 1$; this implies that all isolated singularities are at most finite poles, so $q(z)$ is a rational function with only simple poles and only at $P_{g,1}$.

14 Approach to proof: efficient sequences

Known: σ is strictly contracting: $\|d\sigma\| < 1$ everywhere.

Overall idea: if σ is uniformly contracting (i.e. $\|d\sigma\| < 1 - \eta$), then σ has a fixed point. If not, how can it fail to be uniformly contracting?

Definition. A sequence (g_n, q_n) is called *efficient* if $\dfrac{\|(g_n)_* q_n\|}{\|q_n\|} \nearrow 1$.

Here g_n are structurally finite transcendental mappings and q_n are holomorphic quadratic differentials with bounded number of poles, all of which simple.

Conceptual claim. An efficient sequence is possible only if all the "mass" $\|q_n\|$ is distributed along *fat* annuli that map forward to image annuli under g_n in a way that is predicted by the *multicurve matrix*.

$\Longrightarrow$ we either have definite cancellation (hence contraction) or good control of the mass of q_n and Thurston obstruction.

15 Limits / divergence of structurally finite maps

Consider a sequence of structurally finite maps of given degrees:

$$g_n = \int s_n(\zeta) \exp(P_n(\zeta))\, d\zeta$$

where the s_n (rational maps) and the P_n (polynomials) have fixed degrees. Write $s_n = A_n/B_n$ (polynomials).

We call a polynomial *coefficient-normalized* if all coefficients are in $\overline{\mathbb{D}}$, and at least one coefficient has absolute value 1.

Write $P_n = \lambda_n p_n$ with p_n coefficient-normalized and $\lambda_n > 0$.

Multiply s_n (and hence g_n) by real coefficient so that A_n and B_n are coefficient-normalized. This has no impact on cancellation of q_n.

Observation. Every sequence g_n has a subsequence in which p_n, A_n and B_n (and hence s_n) converge (possibly to maps of lower degrees). The only possible divergence is when $\lambda_n \to \infty$ or $\lambda_n \to 0$.

16 Limits / divergence of structurally finite maps (II)

After extracting a subsequence, we have a sequence

$$g_n = \int s_n(\zeta) \exp(\lambda_n p_n(\zeta))\, d\zeta$$

with $s_n \to s_\infty$ and $p_n \to p_\infty$ (drop of degrees possible), and λ_n converges to a limit in $[0, \infty]$.

Three possible cases:

the large case when $\lambda_n \to \infty$: the case when we "converge to the essential singularity";

the balanced case when $\lambda_n \to \lambda_\infty \in (0, \infty)$: here g_n converges to a limiting structurally finite function;

the small case when $\lambda_n \to 0$: here the g_n converge to a rational limit, uniformly in $\mathbb{C}$.

$\Longrightarrow$ in an efficient sequence $\langle g_n, q_n \rangle$, can find limits / controlled divergence of g_n; need same for q_n

17 Divergence/decomposition of quadratic differentials

The quadratic differentials q_n have the form $q_n = q_n(z)\,dz^2$ with rational maps $q_n(z)$ that have only simple poles only at $\phi_n(P_f)$. We only need them normalized so that $\int_{\mathbb{C}} |q_n(z)|\,dx\,dy = 1$.

They can diverge only when poles collide, and only in a controlled way.

Decomposition Theorem [HSS]. Consider a sequence of normalized quadratic differentials q_n with fixed number of simple poles. Then, up to extracting a subsequence, we can write $q_n = \sum_i q_n^i + \sum_j q_n^j$ so that all ingredients are integrable quadratic differentials on $\overline{\mathbb{C}}$ and so that

- thick case: there are affine Möbius transformations M_n^i so that $(M_n^i)_* q_n^i$ converge to integrable limit differentials q_∞^i;
- thin case: there are M_n^j so that $(M_n^j)_* q_n^j \approx \alpha_j dz^2/z^2$ on fat annuli in $\mathbb{C}^*$ so that the moduli diverge;
- all q_n^i and q_n^j are "mutually singular" in the sense that they have essentially disjoint support.

18 The different cases to consider

Since $(g_n)_* q_n$ is linear in q_n, and different components of q_n have essentially disjoint support, consider push-forward on components of q_n separately. This yields six possible cases:

g_n	quadratic differential **thick component**	quadratic differential **thin component**
large case	cancellation near ∞	cancellation near ∞
balanced case	limit and cancellation	N/A
small case	rational limit, cancellation	mass on annuli

19 The balanced case

We have the maps

$$g_n = \int s_n(\zeta) \exp(\lambda_n p_n(\zeta))\, d\zeta$$

with $\lambda_n \to \lambda_\infty \in (0, \infty)$, while as always $s_n \to s_\infty$ and $p_n \to p_\infty$; thus

$$g_n \longrightarrow g_\infty = \int s_\infty(\zeta) \exp \lambda_\infty p_n(\zeta)\, d\zeta$$

In the thick case, the quadratic differentials converge to a limiting differential: $q_n \to q_\infty$, and thus

$$\langle g_n, q_n \rangle \longrightarrow \langle g_\infty, q_\infty \rangle \ .$$

Since g_∞ is transcendental and q_∞ is rational, there is definite cancellation in the limit $\implies$ no efficient sequence!

There is no thin balanced case: most of a fat annulus drifts to 0 ($\to$ small case) or to ∞ ($\to$ large case), so almost all mass is in small or large case

20 The large case (near essential singularity)

Have $g_m = \int s_m(\zeta) e^{\lambda_m p_m(\zeta)}\, d\zeta$ with $s_m \to s_\infty$, $p_m \to p_\infty$, $\lambda_m \to \infty$.

Idea (thick case): fix a domain U where $\operatorname{Re} p_\infty(z) < -2$, hence $\operatorname{Re} p_m(z) < -1$, $\operatorname{Re} \lambda_m p_m \to -\infty$: thus U is in asymptotic tract over finite asymptotic value. Injectivity radius of g_m small when λ_m large.

If $\langle g_m, q_m \rangle$ is efficient, it must be efficient on U. Set $\lambda_m = 2^{e_m}$.

Integration by parts:

$$\int^z s_m(\zeta) e^{\lambda_m p_m(\zeta)}\, d\zeta = e^{\lambda_m p_m(z)} \cdot \frac{1}{\lambda_m} \cdot \frac{s_m(z)}{p_m'(z)} \cdot (1 + o(1))$$

$$= e^{\lambda_m p_m(z) + h_m(z)} = e^{\lambda_m \left(p_m(z) + h_m(z)/\lambda_m \right)} = \left(e^{p_m(z) + h_m(z)/\lambda_m} \right)^{2^{e^m}}$$

map factors through exp on U: definite cancellation in limit

thin case: cut fat annulus into diverging number of standard annuli with bounded modulus; apply argument to each: definite loss of (relative) mass on each sub-annulus

21 The large case (near essential singularity)

Have $g_m = \int s_m(\zeta) e^{\lambda_m p_m(\zeta)}\, d\zeta$ with $s_m \to s_\infty$, $p_m \to p_\infty$, $\lambda_m \to \infty$.

Idea (thick case): fix a domain U where $\operatorname{Re} p_\infty(z) < -2$, hence $\operatorname{Re} p_m(z) < -1$, $\operatorname{Re} \lambda_m p_m \to -\infty$: thus U is in asymptotic tract over finite asymptotic value. Injectivity radius of g_m small when λ_m large.

If $\langle g_m, q_m \rangle$ is efficient, it must be efficient on U. Set $\lambda_m = 2^{e_m}$.

Integration by parts:

$$\int^z s_m(\zeta) e^{\lambda_m p_m(\zeta)}\, d\zeta = e^{\lambda_m p_m(z)} \cdot \frac{1}{\lambda_m} \cdot \frac{s_m(z)}{p_m'(z)} \cdot (1 + o(1))$$

$$= e^{\lambda_m p_m(z) + h_m(z)} = e^{\lambda_m \left(p_m(z) + h_m(z)/\lambda_m \right)} = \left(e^{p_m(z) + h_m(z)/\lambda_m} \right)^{2^{e_m}}$$

map factors through exp on U: definite cancellation in limit

thin case: cut fat annulus into diverging number of standard annuli with bounded modulus; apply argument to each:
definite loss of (relative) mass on each sub-annulus

22 Conclusion so far: the small case

If g_n is a sequence of structurally finite transcendental (entire or meromorphic) functions and q_n is a sequence of quadratic differentials, such that $\langle g_n, q_n \rangle$ is efficient,
then we cannot be in the large or balanced case, so we must be in small case:

$$g_n = \int s_n(\zeta) \exp(\lambda_n p_n(\zeta))\, d\zeta$$

with $s_n \to s_\infty$ and $p_n \to p_\infty$, while $\lambda_n \to 0$.

So uniformly on compacts, $g_n \to \int s_\infty(\zeta)\, d\zeta =: g$, a rational function.

$\Longrightarrow$ "in spirit", we are in the Thurston–Douady–Hubbard-situation.

The original proof does not apply here $\longrightarrow$ need a new proof for the rational maps theorem that is stable under perturbations w.r.t mass

Our approach: consider quadratic differential, either thick or thin case. These cases remain invariant under finite push-forwards (don't mix)!

23 The rational case thick case

In this case, have a limiting rational map g and a limiting quadratic differential q, perfect efficiency: $\|g_* q\| = \|q\|$.

Lemma: This is possible only if

- $f_* q$ has strictly fewer poles than q; or
- g has degree 1; or
- q has exactly 4 poles, all critical points of g are simple and disjoint from poles, $g_* q$ has 4 poles, and the full preimage of the 4 image poles are the critical points of g and the poles
  ("Lattès without dynamics")

Proof: Count preimages of image poles. □

24 The rational thick case, II

We are investigating a particular thick component of quadratic differentials, not necessarily global original map. The original map does not have degree 1 and is not Lattès, but a component map may.

Possibilities in the perfectly efficient small thick case:

- fewer poles in image: cannot happen often
- limiting map has degree 1: the separating annuli are essential
- limiting map Lattès: the separating annuli are essential. Lattès maps have no superattracting cycles, so they are on repelling cycles, hence degree one.

So we have essential fat annuli that map forward injectively $\Longrightarrow$ under push-forward dynamics obtain Levy cycles

This argument is stable under small perturbations of map or quadratic differential: also applies in the limiting small case of transcendental maps

25 The rational thin case

Last case to consider: rational map g, most mass of quadratic differential is on fat annulus, say $\{z \in \mathbb{C}\colon 1/r < |z| < r\}$

quadratic differential close to $c\,dz^2/z^2$

Bounded number of critical points cut annulus into bounded number of fat annuli.

Remove padding on each: $\Longrightarrow$ map sends annulus as $k:1$ cover to image annulus

Conclusion: annuli with mass map forward (except for bounded padding) onto image annuli as expected. Every image annulus may have many preimage annuli, but only finitely many with mass.

$\Longrightarrow$ annuli with mass map forward as expected by combinatorics. Moduli of annuli described by Thurston matrix.

26 back in the days: 2004 — five generations

27 Currently: four generations

28 ... and into the future

Best wishes for many years to come, Hamal!

29 ... and into the future

Best wishes for many years to come, Hamal!