Towards Transcendental Thurston Theory

Prochorov Nikolai
Université d'Aix-Marseille

June 10, 2025

Rational dynamics

- Branched self-covers of S^2 are topological analogues of rational maps.
- Rational map $g\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is called postcritically finite (pcf) if each critical value of g has a finite orbit.
- Thurston map is a postcritically finite branched self-cover of S^2.

Question: When is a Thurston map dynamically equivalent to a pcf rational map?

Answer: Thurston's topological characterization of pcf rational maps (W.Thurston'80s, Douady-Hubbard'93).

Rational dynamics

- Branched self-covers of S^2 are topological analogues of rational maps.
- Rational map $g\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is called postcritically finite (pcf) if each critical value of g has a finite orbit.
- Thurston map is a postcritically finite branched self-cover of S^2.

Question: When is a Thurston map dynamically equivalent to a pcf rational map?

Answer: Thurston's topological characterization of pcf rational maps (W.Thurston'80s, Douady-Hubbard'93).

Rational dynamics

- Branched self-covers of S^2 are topological analogues of rational maps.
- Rational map $g\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is called postcritically finite (pcf) if each critical value of g has a finite orbit.
- Thurston map is a postcritically finite branched self-cover of S^2.

Question: When is a Thurston map dynamically equivalent to a pcf rational map?

Answer: Thurston's topological characterization of pcf rational maps (W.Thurston'80s, Douady-Hubbard'93).

Rational dynamics

- Branched self-covers of S^2 are topological analogues of rational maps.
- Rational map $g\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is called postcritically finite (pcf) if each critical value of g has a finite orbit.
- Thurston map is a postcritically finite branched self-cover of S^2.

Question: When is a Thurston map dynamically equivalent to a pcf rational map?

Answer: Thurston's topological characterization of pcf rational maps (W.Thurston'80s, Douady-Hubbard'93).

Rational dynamics

- Branched self-covers of S^2 are topological analogues of rational maps.
- Rational map $g\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is called postcritically finite (pcf) if each critical value of g has a finite orbit.
- Thurston map is a postcritically finite branched self-cover of S^2.

Question: When is a Thurston map dynamically equivalent to a pcf rational map?

Answer: Thurston's topological characterization of pcf rational maps (W.Thurston'80s, Douady-Hubbard'93).

Transcendental dynamics

Studies of dynamics of transcendental entire maps and meromorphic maps.

Important achievements:

1. Very satisfactory results for some explicit families, for example exponential family $z \mapsto \lambda \exp(z)$, $\lambda \in \mathbb{C}^*$.
2. Significant progress in understanding dynamics of finite type transcendental maps, i.e., those with a finite set of singular values.

Transcendental dynamics

Studies of dynamics of transcendental entire maps and meromorphic maps.

Important achievements:

1. Very satisfactory results for some explicit families, for example exponential family $z \mapsto \lambda \exp(z)$, $\lambda \in \mathbb{C}^*$.
2. Significant progress in understanding dynamics of finite type transcendental maps, i.e., those with a finite set of singular values.

Transcendental dynamics

Studies of dynamics of transcendental entire maps and meromorphic maps.

Important achievements:

1. Very satisfactory results for some explicit families, for example exponential family $z \mapsto \lambda \exp(z)$, $\lambda \in \mathbb{C}^*$.
2. Significant progress in understanding dynamics of finite type transcendental maps, i.e., those with a finite set of singular values.

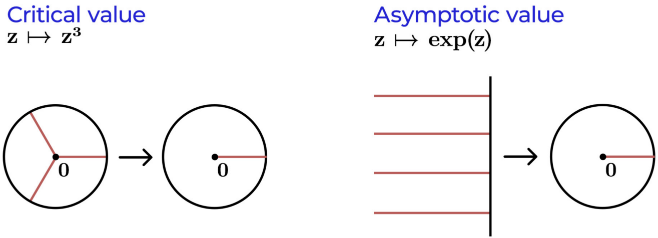
Transcendental dynamics

Studies of dynamics of transcendental entire maps and meromorphic maps.

Important achievements:

1. Very satisfactory results for some explicit families, for example exponential family $z \mapsto \lambda \exp(z)$, $\lambda \in \mathbb{C}^*$.
2. Significant progress in understanding dynamics of finite type transcendental maps, i.e., those with a finite set of singular values.

Example (Singular values).



Transcendental Thurston Theory

Meromorphic map $g\colon \mathbb{C} \to \widehat{\mathbb{C}}$ is called postsingularly finite (psf) is it has finitely many singular values, and each of them has a finite forward orbit.

Goal: Topological characterization of psf meromorphic maps.

Known results:

- Topological characterization of psf exponential maps (Hubbard-Schleicher-Shishikura'09).
- Further evidence towards an analogous characterization (Shemyakov'22, Mukundan-NP-Reinke'24, and D'Souza'25).

Transcendental Thurston Theory

Meromorphic map $g\colon \mathbb{C} \to \widehat{\mathbb{C}}$ is called postsingularly finite (psf) is it has finitely many singular values, and each of them has a finite forward orbit.

Goal: Topological characterization of psf meromorphic maps.

Known results:

- Topological characterization of psf exponential maps (Hubbard-Schleicher-Shishikura'09).
- Further evidence towards an analogous characterization (Shemyakov'22, Mukundan-NP-Reinke'24, and D'Souza'25).

Transcendental Thurston Theory

Meromorphic map $g\colon \mathbb{C} \to \widehat{\mathbb{C}}$ is called postsingularly finite (psf) is it has finitely many singular values, and each of them has a finite forward orbit.

Goal: Topological characterization of psf meromorphic maps.

Known results:

- Topological characterization of psf exponential maps (Hubbard-Schleicher-Shishikura'09).
- Further evidence towards an analogous characterization (Shemyakov'22, Mukundan-NP-Reinke'24, and D'Souza'25).

Transcendental Thurston Theory

Meromorphic map $g\colon \mathbb{C} \to \widehat{\mathbb{C}}$ is called postsingularly finite (psf) is it has finitely many singular values, and each of them has a finite forward orbit.

Goal: Topological characterization of psf meromorphic maps.

Known results:

- Topological characterization of psf exponential maps (Hubbard-Schleicher-Shishikura'09).
- Further evidence towards an analogous characterization (Shemyakov'22, Mukundan-NP-Reinke'24, and D'Souza'25).

Setup

Topologically holomorphic maps

Definition.

Let X and Y be topological surfaces. A map $f\colon X \to Y$ is called topologically holomorphic if f is continuous, open, and discrete.

We consider topologically holomorphic maps $f\colon X \to S^2$, where $X = S^2$ or $X = S^2 \setminus \{e\}$, and f is of parabolic type.

Such maps are denoted $f\colon S^2 \dashrightarrow S^2$.

Topologically holomorphic maps

Definition.

Let X and Y be topological surfaces. A map $f: X \to Y$ is called topologically holomorphic if f is continuous, open, and discrete.

We consider topologically holomorphic maps $f: X \to S^2$, where $X = S^2$ or $X = S^2 \setminus \{e\}$, and f is of parabolic type.

Such maps are denoted $f: S^2 \dashrightarrow S^2$.

Thurston maps

Definition.

Topologically holomorphic map $f: S^2 \dashrightarrow S^2$ is called a Thurston map if f is postsingularly finite.

Example.

- psf meromorphic maps;
- $\varphi \circ g$, where g is a psf meromorphic map, $\varphi \in \mathrm{Homeo}(\widehat{\mathbb{C}})$, and $\varphi(P_g) = P_g$, where P_g is the postsingular set of g—the union of all forward orbits of singular values of f.

Thurston maps

Definition.

Topologically holomorphic map $f\colon S^2 \dashrightarrow S^2$ is called a Thurston map if f is postsingularly finite.

Example.

- psf meromorphic maps;
- $\varphi \circ g$, where g is a psf meromorphic map, $\varphi \in \mathrm{Homeo}(\widehat{\mathbb{C}})$, and $\varphi(P_g) = P_g$, where P_g is the postsingular set of g—the union of all forward orbits of singular values of f.

Thurston maps

Definition.

Topologically holomorphic map $f: S^2 \dashrightarrow S^2$ is called a Thurston map if f is postsingularly finite.

Example.

- psf meromorphic maps;
- $\varphi \circ g$, where g is a psf meromorphic map, $\varphi \in \mathrm{Homeo}(\widehat{\mathbb{C}})$, and $\varphi(P_g) = P_g$, where P_g is the postsingular set of g—the union of all forward orbits of singular values of f.

Combinatorial equivalence

Definition.

Two Thurston maps f and g are combinatorially equivalent if there exist homeomorphisms $h_0, h_1 \colon S^2 \to S^2$ that are isotopic rel. P_f and satisfy $h_0 \circ f = g \circ h_1$.

$$\begin{array}{ccc} (S^2, P_f) & \xrightarrow{h_1} & (S^2, P_g) \\ \downarrow f & & \downarrow g \\ (S^2, P_f) & \xrightarrow{h_0} & (S^2, P_g) \end{array}$$

Characterization problem

Definition.

A Thurston map f is said to be realized if f is combinatorially equivalent to a psf meromorphic map. Otherwise, it is called obstructed.

Characterization problem

Definition.

A Thurston map f is said to be realized if f is combinatorially equivalent to a psf meromorphic map. Otherwise, it is called obstructed.

When given Thurston map is realized?

Obstructions

Let f be a Thurston map with a postsingular set P_f.

Definition.

A simple closed essential curve $\gamma \subset S^2 \backslash P_f$ is called a Levy cycle for f if, for some $n \geqslant 1$, there exists a simple closed curve $\gamma' \subset f^{-n}(\gamma)$ such that

- γ and γ' are homotopic in $S^2 \backslash P_f$,
- $\deg(f^{\circ n}|\gamma' \colon \gamma' \to \gamma) = 1$.

If $n = 1$, γ is called a Levy fixed curve.

Obstructions

Let f be a Thurston map with a postsingular set P_f.

Definition.

A simple closed essential curve $\gamma \subset S^2 \backslash P_f$ is called a Levy cycle for f if, for some $n \geqslant 1$, there exists a simple closed curve $\gamma' \subset f^{-n}(\gamma)$ such that

- γ and γ' are homotopic in $S^2 \backslash P_f$,
- $\deg(f^{\circ n}|\gamma' \colon \gamma' \to \gamma) = 1$.

If $n = 1$, γ is called a Levy fixed curve.

Obstructions

Let f be a Thurston map with a postsingular set P_f.

Definition.

A simple closed essential curve $\gamma \subset S^2 \backslash P_f$ is called a Levy cycle for f if, for some $n \geqslant 1$, there exists a simple closed curve $\gamma' \subset f^{-n}(\gamma)$ such that

- γ and γ' are homotopic in $S^2 \backslash P_f$,
- $\deg(f^{\circ n}|\gamma' \colon \gamma' \to \gamma) = 1$.

If $n = 1$, γ is called a Levy fixed curve.

Obstructions

Proposition.

Any Thurston map having a Levy cycle is obstructed.

Obstructions

Proposition.

Any Thurston map having a Levy cycle is obstructed.

Theorem (Hubbard-Schleicher-Shishikura'09).

An exponential Thurston map is realized if and only if it has no Levy cycle.

Main result

Main result

Theorem (NP'24).

Let f be a Thurston map satisfying the following conditions:

(A) $|P_f| = 4$, and

(B) there exists $B \subset P_f$ such that $|B| = 3$, $S_f \subset B$, and $|\overline{f^{-1}(B)} \cap P_f| = 3$.

Then f is realized if and only if it has no Levy fixed curve.

Main result

Theorem (NP'24).

Let f be a Thurston map satisfying the following conditions:

(A) $|P_f| = 4$, and

(B) there exists $B \subset P_f$ such that $|B| = 3$, $S_f \subset B$, and $|\overline{f^{-1}(B)} \cap P_f| = 3$.

Then f is realized if and only if it has no Levy fixed curve.

Main result

Theorem (NP'24).

Let f be a Thurston map satisfying the following conditions:

(A) $|P_f| = 4$, and

(B) there exists $B \subset P_f$ such that $|B| = 3$, $S_f \subset B$, and $|\overline{f^{-1}(B)} \cap P_f| = 3$.

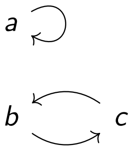
Then f is realized if and only if it has no Levy fixed curve.

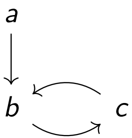
Examples

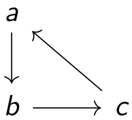
Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be a Thurston map with $S_f = \{a, b, \infty\}$ and $P_f = \{a, b, c, \infty\}$.

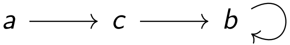
Examples

Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a Thurston map with $S_f = \{a, b, \infty\}$ and $P_f = \{a, b, c, \infty\}$.



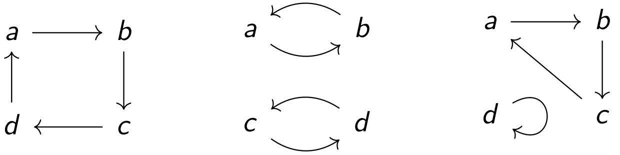






Examples

Let $f\colon S^2 \dashrightarrow S^2$ be a Thurston map with $|S_f| \leqslant 3$ and $P_f = \{a, b, c, d\}$.



Remarks

1. The first Thurston-type criterion applicable to uncountably many pairwise combinatorially inequivalent Thurston maps, both realized and obstructed.

2. This criterion also applies to non-entire transcendental Thurston maps and Thurston maps with several essential singularities;

3. The developed methods have led to further results within the framework of transcendental Thurston theory.

Remarks

1. The first Thurston-type criterion applicable to uncountably many pairwise combinatorially inequivalent Thurston maps, both realized and obstructed.

2. This criterion also applies to non-entire transcendental Thurston maps and Thurston maps with several essential singularities;

3. The developed methods have led to further results within the framework of transcendental Thurston theory.

Remarks

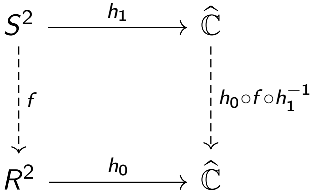
1. The first Thurston-type criterion applicable to uncountably many pairwise combinatorially inequivalent Thurston maps, both realized and obstructed.
2. This criterion also applies to non-entire transcendental Thurston maps and Thurston maps with several essential singularities;
3. The developed methods have led to further results within the framework of transcendental Thurston theory.

Idea of proof

Recalling combinatorial equivalence

Let f be a Thurston map.

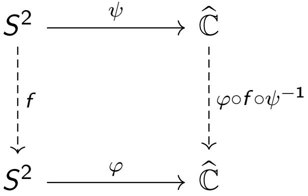
Goal: Find two homeomorphisms $h_0, h_1 \colon S^2 \to \widehat{\mathbb{C}}$ such that h_0 and h_1 are isotopic rel. P_f and $g = h_0 \circ f \circ h_1^{-1} \colon \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$ is holomorphic.



Pullback

Let f be a Thurston map and $\varphi\colon S^2 \to \widehat{\mathbb{C}}$ be a homeomorphism. Then

- there exists a homeomorphism $\psi\colon S^2 \to \widehat{\mathbb{C}}$ such that $\varphi \circ f \circ \psi^{-1}\colon \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$ is holomorphic;

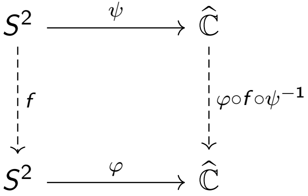


- ψ is unique up to post-composition with an affine transformation;

Pullback

Let f be a Thurston map and $\varphi\colon S^2 \to \widehat{\mathbb{C}}$ be a homeomorphism. Then

- there exists a homeomorphism $\psi\colon S^2 \to \widehat{\mathbb{C}}$ such that $\varphi \circ f \circ \psi^{-1}\colon \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$ is holomorphic;



- ψ is unique up to post-composition with an affine transformation;

Iterating pullback

$$
\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
S^2 & \xrightarrow{\varphi_3} & \widehat{\mathbb{C}} \\
\downarrow f & & \downarrow \varphi_2 \circ f \circ \varphi_3^{-1} \\
S^2 & \xrightarrow{\varphi_2} & \widehat{\mathbb{C}} \\
\downarrow f & & \downarrow \varphi_1 \circ f \circ \varphi_2^{-1} \\
S^2 & \xrightarrow{\varphi_1} & \widehat{\mathbb{C}} \\
\downarrow f & & \downarrow \varphi_0 \circ f \circ \varphi_1^{-1} \\
S^2 & \xrightarrow{\varphi_0} & \widehat{\mathbb{C}}
\end{array}
$$

Teichmüller space

Let $A \subset S^2$ be a finite set.

Two homeomorphisms $\varphi \colon S^2 \to \widehat{\mathbb{C}}$ and $\psi \colon S^2 \to \widehat{\mathbb{C}}$ are said to be equivalent if there exists a Möbius transformation M such that $M \circ \varphi$ and ψ are isotopic rel. A.

$$\begin{array}{ccc} & (S^2, A) & \\ {}^{\varphi}\swarrow & & \searrow^{\psi} \\ (\widehat{\mathbb{C}}, \varphi(A)) & \xrightarrow{\quad M \quad} & (\widehat{\mathbb{C}}, \psi(A) = (M \circ \varphi)(A)) \end{array}$$

Definition.

The Teichmüller space $\mathcal{T}_A$ of the sphere S^2 with the marked set A is

$$\mathcal{T}_A := \{\varphi | \varphi \colon S^2 \to \widehat{\mathbb{C}} \text{ homeomorphism}\} / \sim .$$

Teichmüller space

Let $A \subset S^2$ be a finite set.

Two homeomorphisms $\varphi \colon S^2 \to \widehat{\mathbb{C}}$ and $\psi \colon S^2 \to \widehat{\mathbb{C}}$ are said to be equivalent if there exists a Möbius transformation M such that $M \circ \varphi$ and ψ are isotopic rel. A.

$$\begin{array}{ccc} & (S^2, A) & \\ \swarrow^{\varphi} & & \searrow^{\psi} \\ (\widehat{\mathbb{C}}, \varphi(A)) & \xrightarrow{\quad M \quad} & (\widehat{\mathbb{C}}, \psi(A) = (M \circ \varphi)(A)) \end{array}$$

Definition.

The Teichmüller space $\mathcal{T}_A$ of the sphere S^2 with the marked set A is

$$\mathcal{T}_A := \{\varphi | \varphi \colon S^2 \to \widehat{\mathbb{C}} \text{ homeomorphism}\}/ \sim .$$

Teichmüller space

Let $A \subset S^2$ be a finite set.

Two homeomorphisms $\varphi\colon S^2 \to \widehat{\mathbb{C}}$ and $\psi\colon S^2 \to \widehat{\mathbb{C}}$ are said to be equivalent if there exists a Möbius transformation M such that $M \circ \varphi$ and ψ are isotopic rel. A.

$$\begin{array}{ccc} & (S^2, A) & \\ \swarrow^{\varphi} & & \searrow^{\psi} \\ (\widehat{\mathbb{C}}, \varphi(A)) & \xrightarrow{\quad M \quad} & (\widehat{\mathbb{C}}, \psi(A) = (M \circ \varphi)(A)) \end{array}$$

Definition.

The Teichmüller space $\mathcal{T}_A$ of the sphere S^2 with the marked set A is

$$\mathcal{T}_A := \{\varphi | \varphi\colon S^2 \to \widehat{\mathbb{C}} \text{ homeomorphism}\}/ \sim .$$

Pullback map

Definition.

Let f be a Thurston map and $\tau = [\varphi] \in \mathcal{T}_{P_f}$. Then the pullback map $\sigma_f : \mathcal{T}_{P_f} \to \mathcal{T}_{P_f}$ is defined by $\sigma_f([\varphi]) = [\psi]$, where

$$\begin{array}{ccc} (S^2, P_f) & \xrightarrow{\psi} & (\widehat{\mathbb{C}}, \psi(P_f)) \\ \Big\downarrow f & & \Big\downarrow \varphi\circ f\circ\psi^{-1} \\ (S^2, P_f) & \xrightarrow{\varphi} & (\widehat{\mathbb{C}}, \varphi(P_f)) \end{array}$$

Proposition.

Thurston map f is realized if and only if the pullback map σ_f has a fixed point in the Teichmüller space $\mathcal{T}_{P_f}$.

Pullback map

Definition.

Let f be a Thurston map and $\tau = [\varphi] \in \mathcal{T}_{P_f}$. Then the pullback map $\sigma_f : \mathcal{T}_{P_f} \to \mathcal{T}_{P_f}$ is defined by $\sigma_f([\varphi]) = [\psi]$, where

$$\begin{array}{ccc} (S^2, P_f) & \xrightarrow{\psi} & (\widehat{\mathbb{C}}, \psi(P_f)) \\ \downarrow f & & \downarrow \varphi \circ f \circ \psi^{-1} \\ (S^2, P_f) & \xrightarrow{\varphi} & (\widehat{\mathbb{C}}, \varphi(P_f)) \end{array}$$

Proposition.

Thurston map f is realized if and only if the pullback map σ_f has a fixed point in the Teichmüller space $\mathcal{T}_{P_f}$.

General properties

Proposition.

Teichmüller space $\mathcal{T}_{P_f}$ is a complex manifold of dimension $|P_f| - 3$ and σ_f is a holomorphic map.

Proposition.

Teichmüller space $\mathcal{T}_{P_f}$ has a natural complete metric and σ_f is 1-Lipschitz.

Example.

If $|P_f| = 4$, then $\mathcal{T}_{P_f} \sim \mathbb{D}$.

General properties

Proposition.

Teichmüller space $\mathcal{T}_{P_f}$ is a complex manifold of dimension $|P_f| - 3$ and σ_f is a holomorphic map.

Proposition.

Teichmüller space $\mathcal{T}_{P_f}$ has a natural complete metric and σ_f is 1-Lipschitz.

Example.

If $|P_f| = 4$, then $\mathcal{T}_{P_f} \sim \mathbb{D}$.

General properties

Proposition.

Teichmüller space $\mathcal{T}_{P_f}$ is a complex manifold of dimension $|P_f| - 3$ and σ_f is a holomorphic map.

Proposition.

Teichmüller space $\mathcal{T}_{P_f}$ has a natural complete metric and σ_f is 1-Lipschitz.

Example.

If $|P_f| = 4$, then $\mathcal{T}_{P_f} \sim \mathbb{D}$.

Moduli space

Definition.

The moduli space $\mathcal{M}_A$ of the sphere S^2 with the finite marked set A is defined as

$$\mathcal{M}_A := \{\eta | \eta \colon A \to \widehat{\mathbb{C}} \text{ injective}\} / \approx,$$

where $\eta_1 \approx \eta_2$ if there exists a Möbius transformation M such that $M \circ \eta_1 = \eta_2$.

$\pi \colon \mathcal{T}_A \to \mathcal{M}_A$ universal holomorphic covering

$$\pi([\varphi]) = [\varphi | A].$$

Example.

If $|A| = 4$, then $\mathcal{M}_A \sim \widehat{\mathbb{C}} - \{0, 1, \infty\}$.

Moduli space

Definition.

The moduli space $\mathcal{M}_A$ of the sphere S^2 with the finite marked set A is defined as

$$\mathcal{M}_A := \{\eta | \eta\colon A \to \widehat{\mathbb{C}} \text{ injective}\}/\approx,$$

where $\eta_1 \approx \eta_2$ if there exists a Möbius transformation M such that $M \circ \eta_1 = \eta_2$.

$$\pi\colon \mathcal{T}_A \to \mathcal{M}_A \text{ universal holomorphic covering}$$

$$\pi([\varphi]) = [\varphi|A].$$

Example.

If $|A| = 4$, then $\mathcal{M}_A \sim \widehat{\mathbb{C}} - \{0, 1, \infty\}$.

Proposition.

Let f be a Thurston map satisfying conditions (B). Then there exists

- an open and connected set $W \subset \mathcal{M}_{P_f}$, and
- a holomorphic covering $G\colon W \to \mathcal{M}_{P_f}$

such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{T}_{P_f} & \xrightarrow{\sigma_f} & \sigma_f(\mathcal{T}_{P_f}) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathcal{M}_{P_f} & \xleftarrow[G]{} & W \end{array}$$

Iteration on the unit disk

Theorem (NP'24).

Let $h\colon \mathbb{D} \to \mathbb{D}$ be a holomorphic map, and $\pi\colon \mathbb{D} \to \Sigma = \widehat{\mathbb{C}} - \{0, 1, \infty\}$ and $g\colon U \to \Sigma$ are non-injective holomorphic covering maps, where $U \subset \Sigma$ is a domain of $\widehat{\mathbb{C}}$. Suppose the following diagram commutes:

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{\;h\;} & h(\mathbb{D}) \\ \big\downarrow{\scriptstyle\pi} & & \big\downarrow{\scriptstyle\pi} \\ \Sigma & \xleftarrow[\;g\;]{} & U \end{array}$$

Then exactly one of the following two possibilities is satisfied:

1. $(h^{\circ n}(z))$ converges to the unique fixed point of h for every $z \in \mathbb{D}$,
2. $(\pi(h^{\circ n}(z)))$ converges to the same repelling fixed point $x \in \{0, 1, \infty\}$ of the map g, regardless of the choice of $z \in \mathbb{D}$.

Iteration on the unit disk

Theorem (NP'24).

Let $h\colon \mathbb{D} \to \mathbb{D}$ be a holomorphic map, and $\pi\colon \mathbb{D} \to \Sigma = \widehat{\mathbb{C}} - \{0, 1, \infty\}$ and $g\colon U \to \Sigma$ are non-injective holomorphic covering maps, where $U \subset \Sigma$ is a domain of $\widehat{\mathbb{C}}$. Suppose the following diagram commutes:

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{\ h\ } & h(\mathbb{D}) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \Sigma & \xleftarrow[\ g\ ]{} & U \end{array}$$

Then exactly one of the following two possibilities is satisfied:

1. $(h^{\circ n}(z))$ converges to the unique fixed point of h for every $z \in \mathbb{D}$,
2. $(\pi(h^{\circ n}(z)))$ converges to the same repelling fixed point $x \in \{0, 1, \infty\}$ of the map g, regardless of the choice of $z \in \mathbb{D}$.

Iteration on the unit disk

Theorem (NP'24).

Let $h\colon \mathbb{D} \to \mathbb{D}$ be a holomorphic map, and $\pi\colon \mathbb{D} \to \Sigma = \widehat{\mathbb{C}} - \{0, 1, \infty\}$ and $g\colon U \to \Sigma$ are non-injective holomorphic covering maps, where $U \subset \Sigma$ is a domain of $\widehat{\mathbb{C}}$. Suppose the following diagram commutes:

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{h} & h(\mathbb{D}) \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi} \\ \Sigma & \xleftarrow[g]{} & U \end{array}$$

Then exactly one of the following two possibilities is satisfied:

1. $(h^{\circ n}(z))$ converges to the unique fixed point of h for every $z \in \mathbb{D}$,
2. $(\pi(h^{\circ n}(z)))$ converges to the same repelling fixed point $x \in \{0, 1, \infty\}$ of the map g, regardless of the choice of $z \in \mathbb{D}$.

Degenerations

Observation

Let f be a Thurston map satisfying conditions (A) and (B).

Then the orbit $(\sigma_f^{\circ n}(\tau))$ leaves every compact set of the Teichmüller space $\mathcal{T}_{P_f}$ if and only if its projection $(\pi(\sigma_f^{\circ n}(\tau)))$ does so in the moduli space $\mathcal{M}_{P_f}$.

Relative version of Thurston's result

Marked Thurston maps

Let $f\colon S^2 \dashrightarrow S^2$ be a Thurston map.

- Let $A \subset S^2$ be a finite set such that $P_f \subset A$ and for every $a \in A$ either $f(a) \in A$ or a is an essential singularity of f;
- Pair (f, A) is called a marked Thurston map.

Definition.

Two marked Thurston maps $f\colon (S^2, A) \circlearrowleft$ and $g\colon (S^2, B) \circlearrowleft$ are combinatorially equivalent if there exist two homeomorphisms $h_0, h_1\colon S^2 \to S^2$ such that h_0 is isotopic rel. A to h_1, and $h_0 \circ f = g \circ h_1$.

Marked Thurston maps

Let $f\colon S^2 \dashrightarrow S^2$ be a Thurston map.

- Let $A \subset S^2$ be a finite set such that $P_f \subset A$ and for every $a \in A$ either $f(a) \in A$ or a is an essential singularity of f;
- Pair (f, A) is called a marked Thurston map.

Definition.

Two marked Thurston maps $f\colon (S^2, A) \circlearrowleft$ and $g\colon (S^2, B) \circlearrowleft$ are combinatorially equivalent if there exist two homeomorphisms $h_0, h_1\colon S^2 \to S^2$ such that h_0 is isotopic rel. A to h_1, and $h_0 \circ f = g \circ h_1$.

Relative version of Thurston's theorem

How are the properties of marked Thurston maps $f\colon (S^2, A) \circlearrowleft$ and $f\colon (S^2, B) \circlearrowleft$ related when $B \subset A$?

Relative version of Thurston's theorem

Suppose the Thurston map $f\colon (S^2, B) \circlearrowleft$ is realized. What can be said about the realizability of the Thurston map $f\colon (S^2, A) \circlearrowleft$, where $B \subset A$?

Theorem (NP'24; Bartholdi-Dudko'21; Selinger-Yampolski'15).

Let $f\colon (S^2, A) \circlearrowleft$ and $f\colon (S^2, B) \circlearrowleft$ are marked Thurston maps with $B \subset A$. Then $f\colon (S^2, A) \circlearrowleft$ is realized if and only if $f\colon (S^2, B) \circlearrowleft$ is realized and $f\colon (S^2, A) \circlearrowleft$ has no Levy cycle.

The theorem addresses the following three cases in a unified way:

1. f is a finite degree map that is not a $(2, 2, 2, 2)$-map;
2. f is a $(2, 2, 2, 2)$-map;
3. f is a transcendental map.

Relative version of Thurston's theorem

Suppose the Thurston map $f\colon (S^2, B) \circlearrowleft$ is realized. What can be said about the realizability of the Thurston map $f\colon (S^2, A) \circlearrowleft$, where $B \subset A$?

Theorem (NP'24; Bartholdi-Dudko'21; Selinger-Yampolski'15).

Let $f\colon (S^2, A) \circlearrowleft$ and $f\colon (S^2, B) \circlearrowleft$ are marked Thurston maps with $B \subset A$. Then $f\colon (S^2, A) \circlearrowleft$ is realized if and only if $f\colon (S^2, B) \circlearrowleft$ is realized and $f\colon (S^2, A) \circlearrowleft$ has no Levy cycle.

The theorem addresses the following three cases in a unified way:

1. f is a finite degree map that is not a $(2,2,2,2)$-map;
2. f is a $(2,2,2,2)$-map;
3. f is a transcendental map.

Relative version of Thurston's theorem

Suppose the Thurston map $f\colon (S^2, B) \circlearrowleft$ is realized. What can be said about the realizability of the Thurston map $f\colon (S^2, A) \circlearrowleft$, where $B \subset A$?

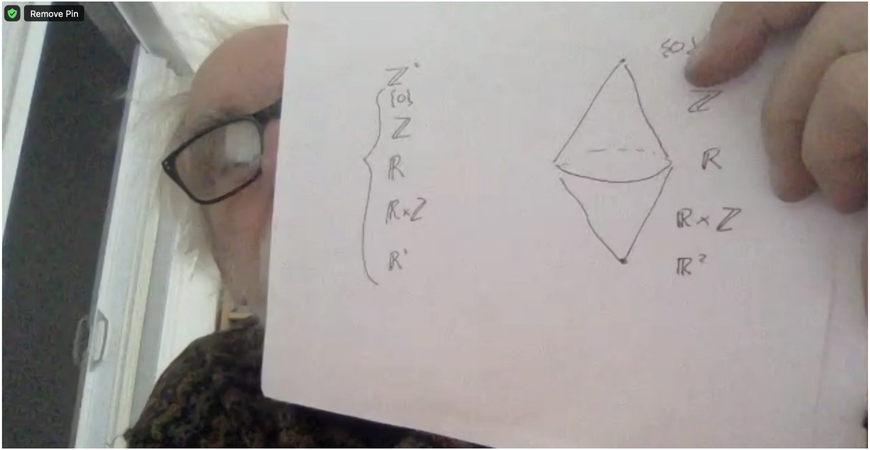
Theorem (NP'24; Bartholdi-Dudko'21; Selinger-Yampolski'15).

Let $f\colon (S^2, A) \circlearrowleft$ and $f\colon (S^2, B) \circlearrowleft$ are marked Thurston maps with $B \subset A$. Then $f\colon (S^2, A) \circlearrowleft$ is realized if and only if $f\colon (S^2, B) \circlearrowleft$ is realized and $f\colon (S^2, A) \circlearrowleft$ has no Levy cycle.

The theorem addresses the following three cases in a unified way:

1. f is a finite degree map that is not a $(2, 2, 2, 2)$-map;
2. f is a $(2, 2, 2, 2)$-map;
3. f is a transcendental map.

Happy birthday, Hamal!



Getting a Levy fixed curve

Proposition.

Let f be a Thurston map with $|P_f| = 4$, and $\tau = [\varphi] \in \mathcal{T}_{P_f}$, $\sigma_f(\tau) = [\psi] \in \mathcal{T}_{P_f}$, so that the map $g := \varphi \circ f \circ \psi^{-1} : \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$ is holomorphic.

Suppose that there exists an annulus $U \subset \widehat{\mathbb{C}}$ such that:

- each connected component of $\widehat{\mathbb{C}} - U$ contains two points of $\psi(A)$;
- $\mathrm{mod}\,(U) \geqslant M(d_0)$, where $d_0 = d_{\mathcal{T}}(\tau, \sigma_f(\tau))$;
- g is injective on U.

Then f has a Levy fixed curve.

Getting a Levy fixed curve

Proposition.

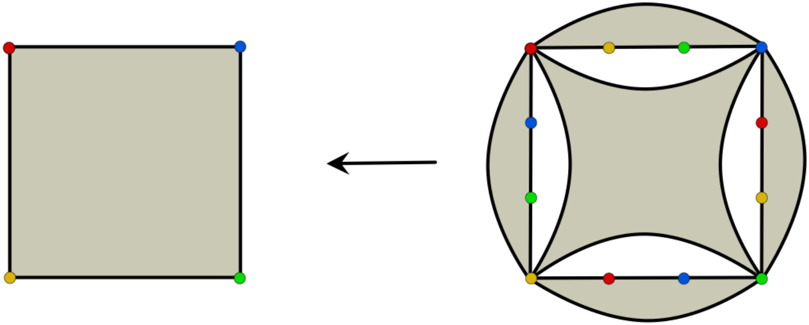
Let f be a Thurston map with $|P_f| = 4$, and $\tau = [\varphi] \in \mathcal{T}_{P_f}$, $\sigma_f(\tau) = [\psi] \in \mathcal{T}_{P_f}$, so that the map $g := \varphi \circ f \circ \psi^{-1} : \widehat{\mathbb{C}} \dashrightarrow \widehat{\mathbb{C}}$ is holomorphic.

Suppose that there exists an annulus $U \subset \widehat{\mathbb{C}}$ such that:

- each connected component of $\widehat{\mathbb{C}} - U$ contains two points of $\psi(A)$;
- $\operatorname{mod}(U) \geqslant M(d_0)$, where $d_0 = d_T(\tau, \sigma_f(\tau))$;
- g is injective on U.

Then f has a Levy fixed curve.

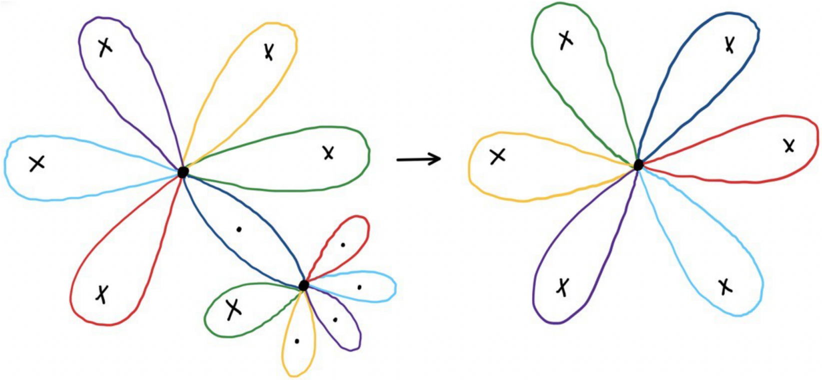
Combinatorial model



$$f(z) = \frac{3z^5 - 20z}{5z^4 - 12}$$

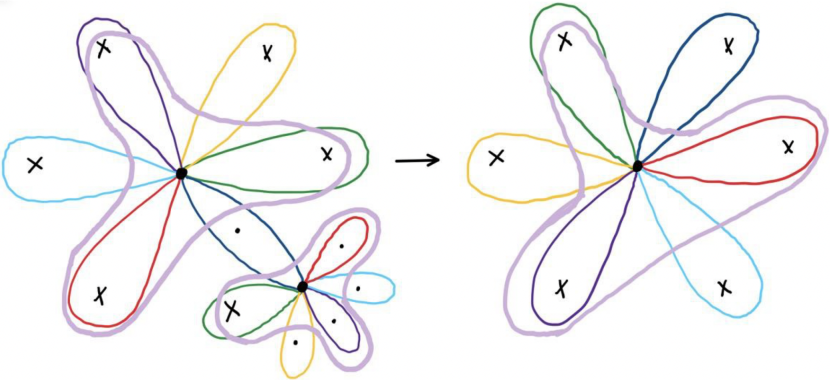
$$C_f = P_f = \{\pm 1 \pm i\}$$

Obstructed example

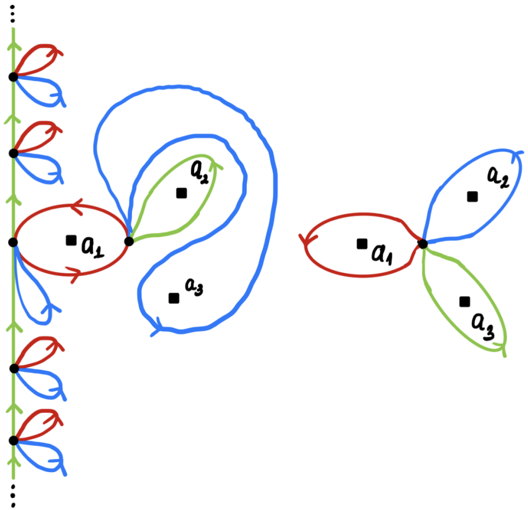


$$x_1 \longrightarrow x_2 \longrightarrow x_3 \longrightarrow x_4 \longrightarrow x_5 \longrightarrow x_6$$

Example of a Levy cycle



Obstructed example



Example of a Levy cycle