Twisted holomorphic 1-forms, dynamics, and the framed mapping class group

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Part 1: Dilation surfaces and dynamics

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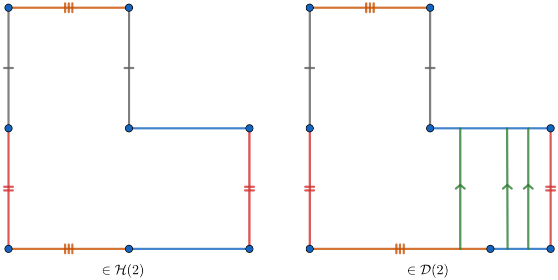

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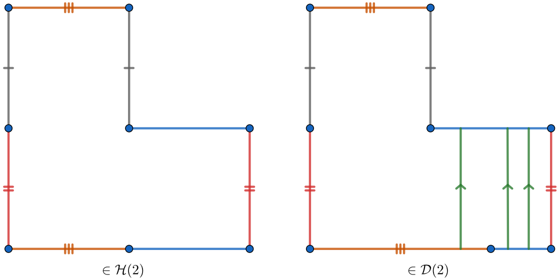

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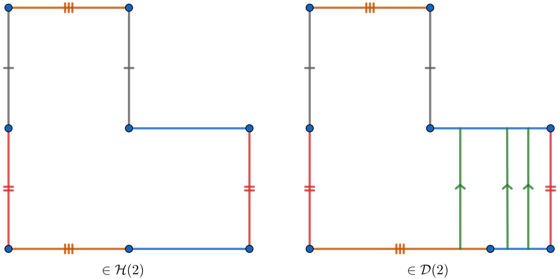

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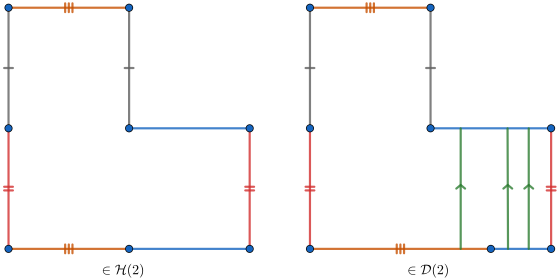

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Theorem (Calderon-Salter)

The inclusion is a π_1-surjection.

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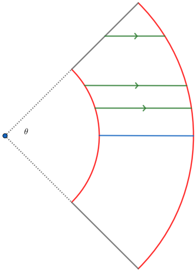

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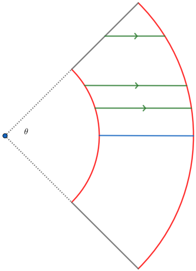

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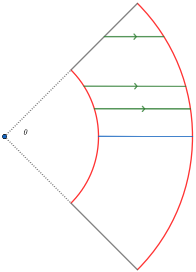

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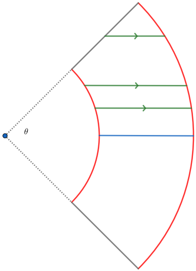

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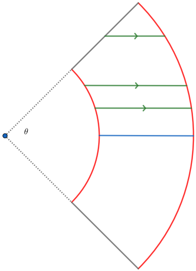

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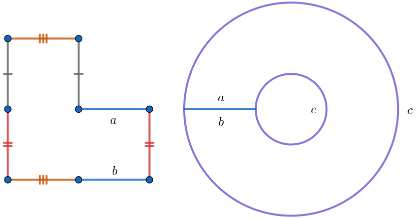

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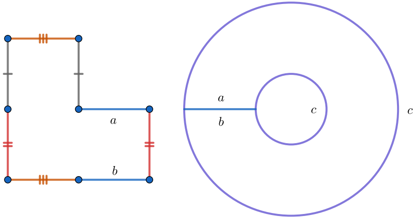

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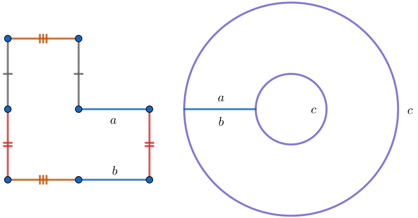

These triangulations can be used to topologize $\mathcal{D}(2g-2)$. Unlike $\mathcal{H}(2g-2)$, it is not obviously a linear manifold.

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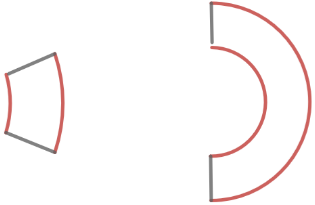
Proof Idea: Assume no Reeb tori. Applying g_t makes a horizontal dilation cylinder become almost Reeb.



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Such a measure exists for $\mathcal{H}(2g-2)$ by work of Masur and Veech.

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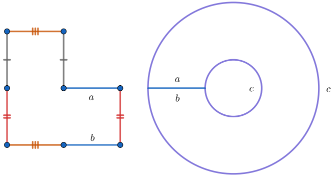

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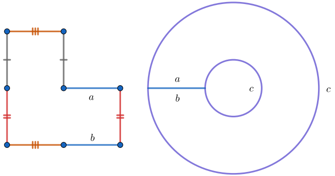

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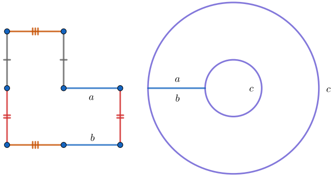

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*most means a cocountable subset of an open dense set.

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Theorem (A-, Salter)

$\mathcal{D}(2g-2) - \mathcal{H}(2g-2)$ admits an $\mathrm{SL}(2,\mathbb{R})$-invariant Lebesgue class measure.

Part 2: Different Perspectives on Dilation Surfaces

The topology of $\mathcal{D}(2g-2)$

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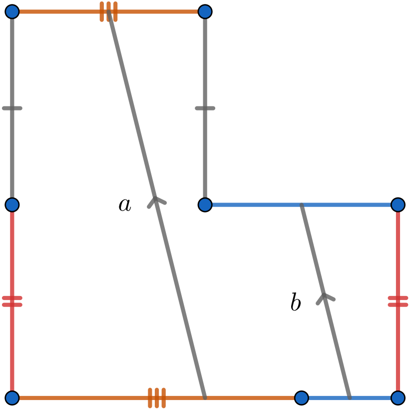

$\chi(a) = 2$ $\chi(b) = \frac{1}{2}$

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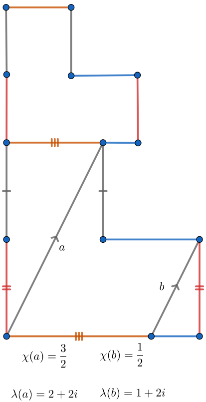

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Theorem

There is a bijective correspondence between χ-twisted holomorphic 1-forms on genus g Riemann surfaces and dilation surfaces with holonomy homomorphism χ.

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3. A Riemann surface with a vector field that vanishes at finitely many points up to homotopy through vector fields that continue to vanish at the same collection of points.

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The moduli space of such surfaces is a $\mathbb{C}^\times$-bundle over a $K(\pi, 1)$ where π is the framed mapping class group (equivalently over the framed mapping class group cover of $\mathcal{M}_{g,1}$).

$\mathrm{SL}(2,\mathbb{R})$-invariant Lebesgue class measures on $\mathcal{D}(2g-2)$

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Theorem (A-, Bainbridge, Wang)

This map is a local diffeomorphism away from the locus of translation surfaces.

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Goal: Find a $\Gamma \times \mathrm{SL}(2, \mathbb{R})$-invariant Lebesgue class measure on $\mathrm{Hom}(\pi_1(X, p), G_{\mathbb{C}})$ to pull back for Γ the framed mapping class group.

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Goal: Compute $H^1(\mathrm{Mod}_{g,1}, H_1(X; \mathbb{R}))$.

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All elements of $H^1(\mathrm{Mod}_{g,1}, H_1(X; \mathbb{R}))$ are multiples of this cocycle.

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An invariant Lebesgue class measure

Corollary

The restriction of any additive cocycle $A : \mathrm{Mod}_{g,1} \times H^1(X, \mathbb{R}) \to \mathbb{R}$ *to the framed mapping class group is a coboundary.*

Theorem (A-, Salter)

$\mathcal{D}(2g-2) - \mathcal{H}(2g-2)$ admits an $\mathrm{SL}(2, \mathbb{R})$-invariant Lebesgue class measure.

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Theorem (Ghazouani; A-,Salter)

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We notice that $\mathrm{Hom}(\pi_1(X), \mathrm{Aff}(\mathbb{R}))$ has an invariant line bundle upon which the MCG acts trivially. Let $\mathcal{V}$ be the quotient bundle.

Theorem (Ghazouani; A-,Salter)

Let $\Gamma \leq \mathrm{Mod}_g^1$ be reasonably big. The invariant measurable differential forms on $\mathcal{V}$ are generated by the invariant 2-form on the base if Γ is not contained in a framed mapping class group.

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Extensions to other strata

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When there are dilation factors around zeros an explicit cover of $\mathcal{D}(\kappa)$ still has an invariant measure provided that at least one point in κ is a cone point with no dilation factor to its monodromy. This requires a mapping class group group cohomology computation generalizing Morita's work.

An invariant measure on $\mathcal{D}(\kappa)/\mathbb{R}_{>0}$

If we're interested in the moduli space of $\mathcal{D}(2g-2)$ as a moduli space of (G, X) structures we consider $\mathcal{D}(2g-2)/\mathbb{R}_{>0}$.

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Theorem (A-,Salter)

There is an $\mathrm{SL}(2,\mathbb{R})$ invariant measure on the triangulable locus of $\mathcal{D}(2g-2)/\mathbb{R}_{>0}$ if and only if there is an area function on the triangulable locus of $\mathcal{D}(2g-2)$.

Thanks!

Thanks for listening!