Tiling billiards in the wind-tree model

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Translation, Dilation, Affine and other structures on Surfaces
Toulouse, 8 April 2025

Plan

Some context

Tiling billiards: Definition

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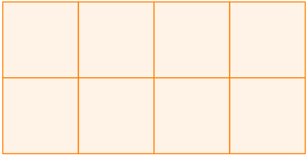
A tiling billiard is a dynamical system whose trajectories are:

- *in a polygonal tiling,*



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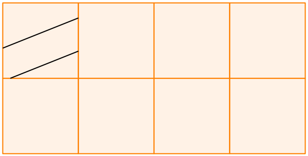

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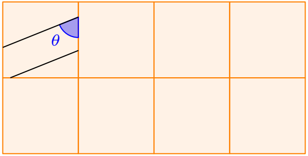

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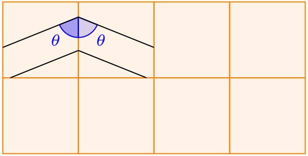

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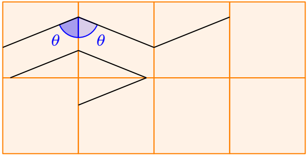

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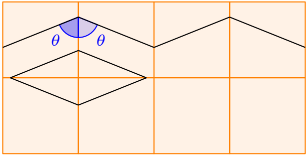

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Fig. 1. Photograph of the left-handed metamaterial (LHM) sample. The LHM sample consists of square copper split ring resonators and copper wire strips on fiber glass circuit board material. The rings and wires are on opposite sides of the boards, and the boards have been cut and assembled into an interlocking lattice.

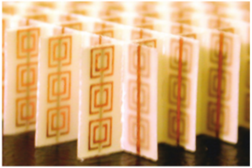
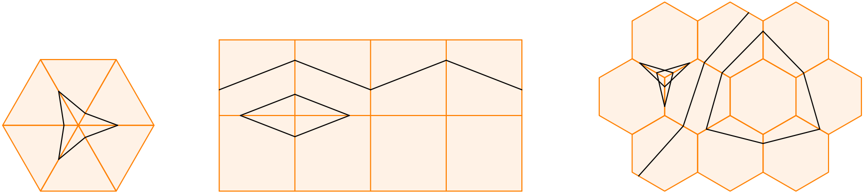


Figure: Metamaterial - From : *Experimental Verification of a Negative Index of Refraction*, R. A. Shelby, D. R. Smith, S. Schultz

Questions



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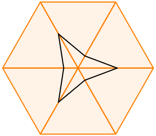

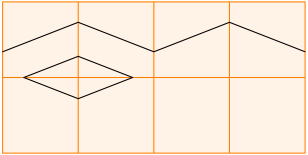

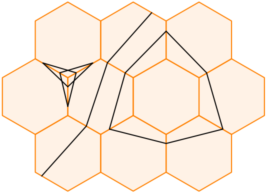

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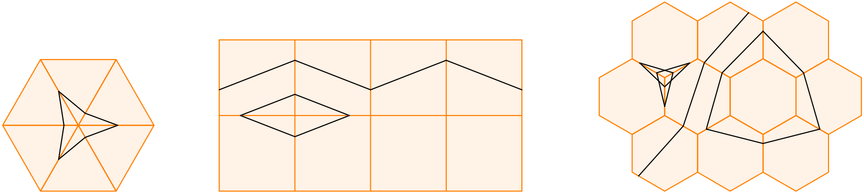

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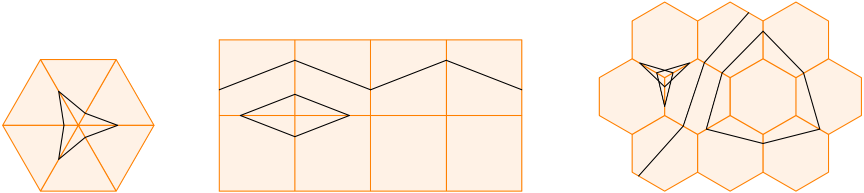


In general, what can be said about trajectories?

- Are they bounded, periodic?
- If unbounded, are they reccurent or do they escape to infinity? How?
- Are they stable under perturbations of the tiling or of the initial point and initial direction?

An example: the triangle tiling billiards

Theorem (Baird-Smith,Davis,Fromm,Iyer - 2018 - and Hubert,Paris-Romaskevich - 2019)

For any triangle, for almost every initial direction, the trajectory is either periodic or at bounded distance from a line.

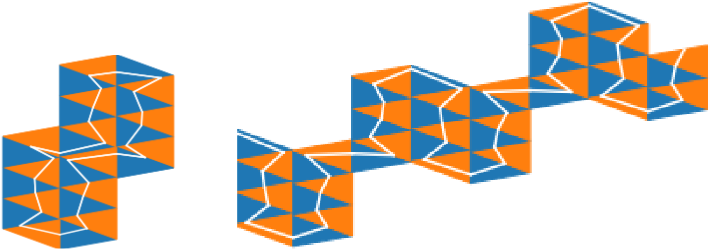
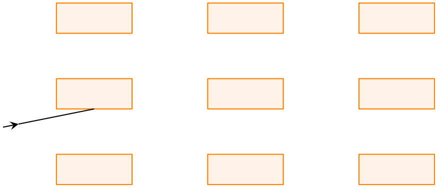


Figure: The two generic types of trajectories

Wind-tree model: Definition



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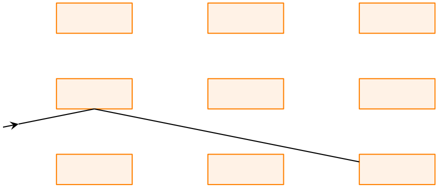

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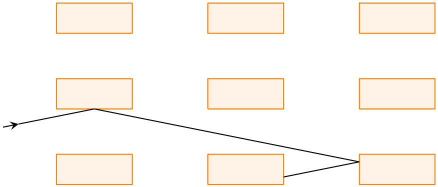

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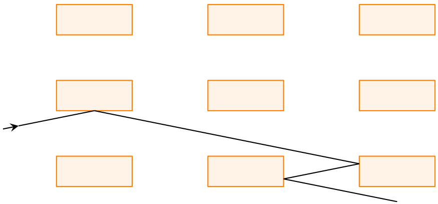

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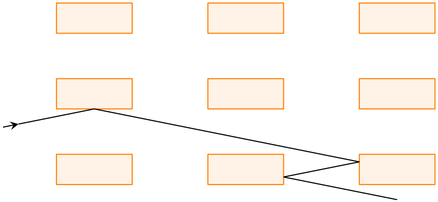

Figure: The wind-tree model

Denote φ_t the flow, i.e. $\varphi_t(x, \theta)$ is the point, at time t, of trajectory that begins in x with angle θ.

Wind-tree model: Recurrence and diffusion rate

Theorem (Avila, Hubert - 2020)

For every $(a, b) \in (0, 1)^2$ *, for almost every initial direction* θ*, for every initial point* x*, the trajectory* $t \mapsto \varphi_t(x, \theta)$ *is recurrent.*

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Theorem (Delecroix, Hubert, Lelièvre - 2017)

For every $(a,b) \in (0,1)^2$, for almost every initial direction θ, for every initial point x having infinite future orbit, the following holds:

$$\limsup_{t\to\infty} \frac{\log d(x,\varphi_t(x,\theta))}{\log t} = \frac{2}{3}.$$

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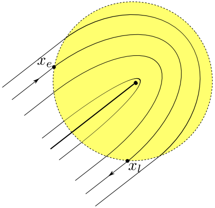
Eaton Lenses: Definition

Let $R > 0$.



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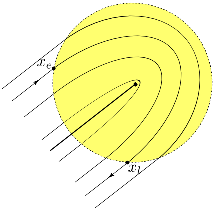
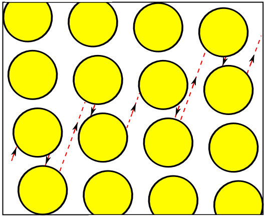
Let $R > 0$. Let Λ be a lattice of $\mathbb{R}^2$.





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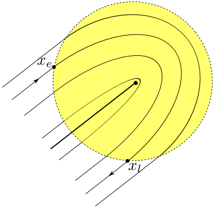

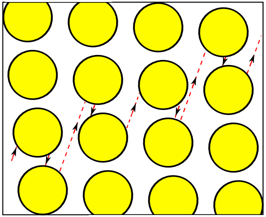

Say that the pair (Λ, R) is *admissible* when the lenses are pairwise disjoint. Denote $L(\Lambda, R)$ this system of Eaton lenses.

Eaton Lenses: trapped trajectories

Theorem (Frączek, Schmoll - 2014)

For almost every admissible pair (Λ, R) *there exist constants* $C = C(\Lambda, R) > 0$ *and* $\Theta = \Theta(\Lambda, R) \in \mathbb{S}^1$*, such that every vertical light ray in* $L(\Lambda, R)$ *is trapped in an infinite band of width* $C > 0$ *in direction* Θ*.*

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Theorem (Frączek, Shi, Ulcigrai - 2018)

*For **every** admissible pair (Λ, R), for **almost every direction** η, there exist constants $C = C(\Lambda, R, \eta) > 0$ and $\Theta = \Theta(\Lambda, R, \eta) \in \mathbb{S}^1$, such that every light ray in direction η in $L(\Lambda, R)$ is trapped in an infinite band of width $C > 0$ in direction Θ.*

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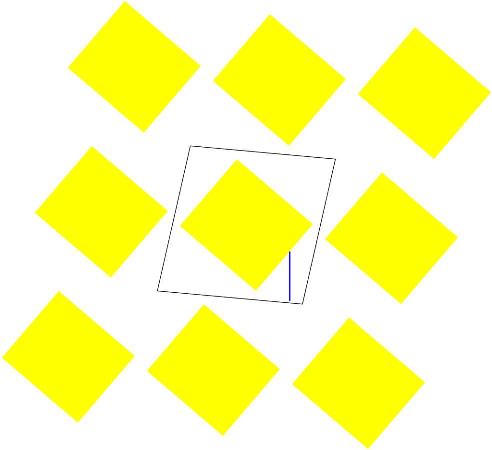
Let $\Lambda \subset \mathbb{R}^2$ be a lattice. Let $a, b > 0$. Let $\eta \in [0, \pi)$



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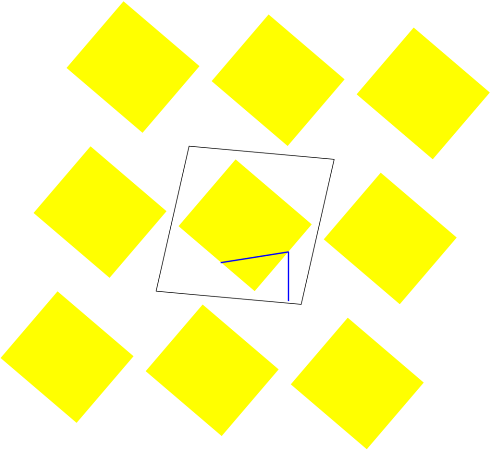

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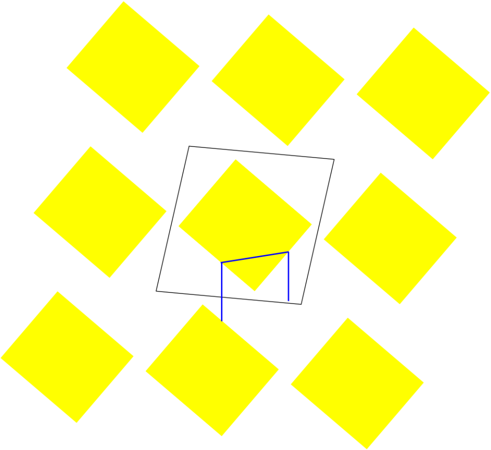

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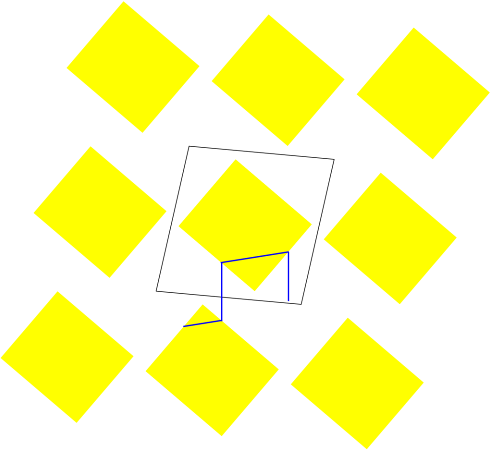

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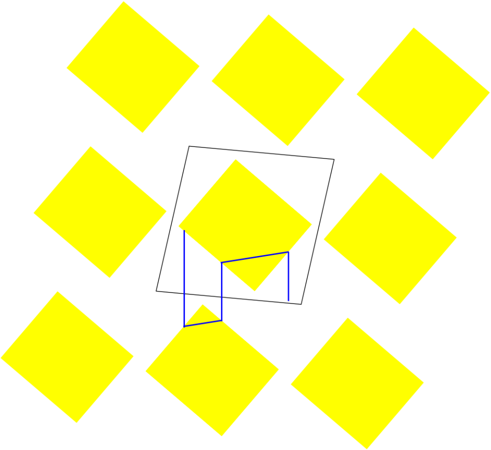

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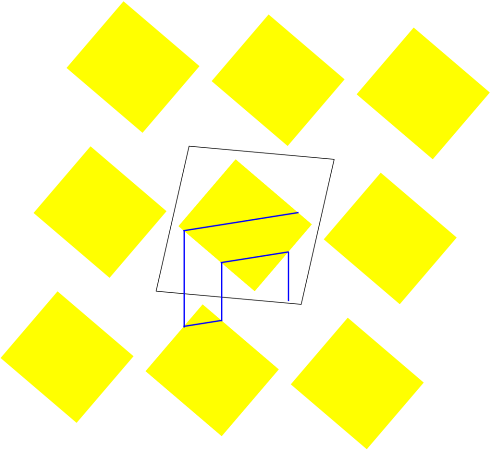

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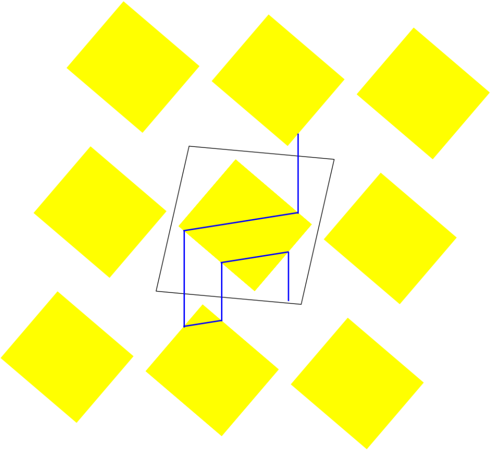

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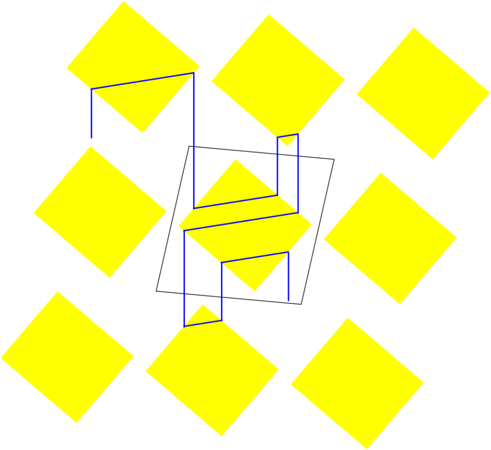

Tiling biliard in wind-tree: the result

Say that the tuple (Λ, a, b, η) is *admissible* when the rectangles are pairwise disjoint. Denote $W(\Lambda, a, b, \eta)$ this system.

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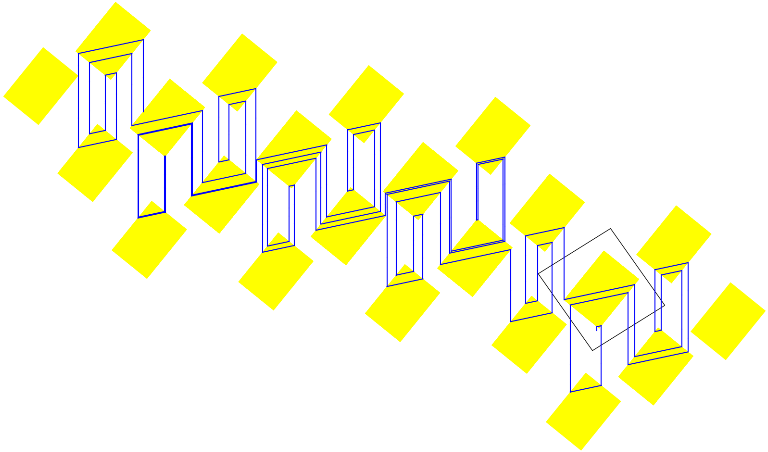
Theorem (J. - 2024)

For almost every admissible tuple (Λ, a, b, η), there exist constants $C = C(\Lambda, R, a, b, \eta) > 0$ and $\Theta = \Theta(\Lambda, R, a, b, \eta) \in \mathbb{S}^1$, such that every vertical trajectory in $W(\Lambda, a, b, \eta)$ is trapped in an infinite band of width $C > 0$ in direction Θ.

Illustration of the result



Quick overview of Half-translation surfaces

Translation surfaces: Definition

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A translation surface is a finite collection of polygons with pairs of parallel sides of same length, that we identify by translation. The polygons are given with a specified direction.

Equivalence up to cut and paste operations.

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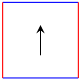

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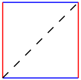

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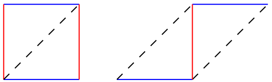

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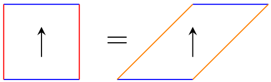

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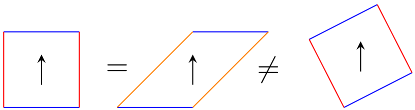

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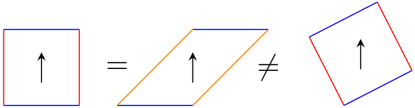

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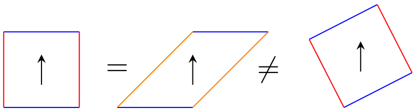

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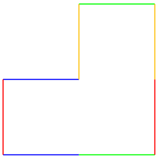

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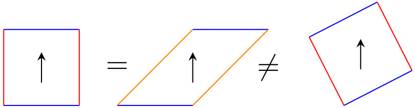

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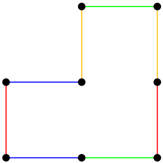

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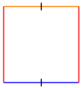

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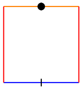

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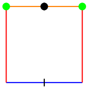

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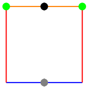

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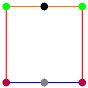

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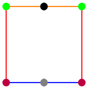

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Moduli space of (half) translation surfaces

- **stratification** of the Moduli space

$$\mathcal{H}_g = \bigsqcup_{\sum m_i = 2g-2} \mathcal{H}(\underline{m}), \qquad \mathcal{Q}_g = \bigsqcup_{\sum d_i = 4g-4} \mathcal{Q}(\underline{d})$$

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The restiction of this action to the diagonal subgroup gives the geodesic flow (also called *Teichmüler flow*):

$$\boldsymbol{g_t} = \begin{pmatrix} \boldsymbol{e^t} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{e^{-t}} \end{pmatrix}.$$

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Each stratum is equipped with a measure μ (Lebesgue measure).

Theorem (Masur–Veech, 1982)

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Theorem (Masur's criterion, 1992)

Let $X \in \mathcal{Q}(\underline{k})$ be a (half) translation surface. If $g_t.X$ does not escape, i.e. if there exists a compact $K \subset \mathcal{Q}(\underline{k})$ such that $\{t > 0 \,|\, g_t.X \in K\}$ is not bounded, then the vertical flow in X is uniquely ergodic.

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We have two dynamical systems, both equipped with probability measure, and linked with each other.

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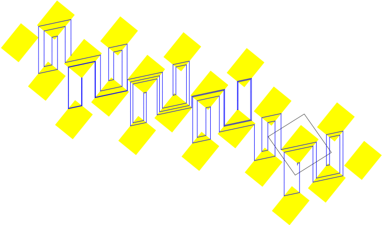
Geodesic flow on a (half) translation surface	$\leftsquigarrow$ (Masur's criterion)	Geodesic flow on the moduli space
probability measure		probability measure, ergodic

Sketch of the proof

Recall the result

Theorem (J. - 2024)

For almost every admissible tuple (Λ, a, b, η), there exist constants $\Theta = \Theta(\Lambda, a, b, \eta) \in \mathbb{S}^1$ and $C = C(\Lambda, a, b, \eta) > 0$, such that every vertical trajectory is trapped in an infinite band of width $C > 0$ in direction Θ.



The importance of renormalization

TB in the
WT model

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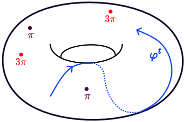
TB in the WT model ⟶ HTS



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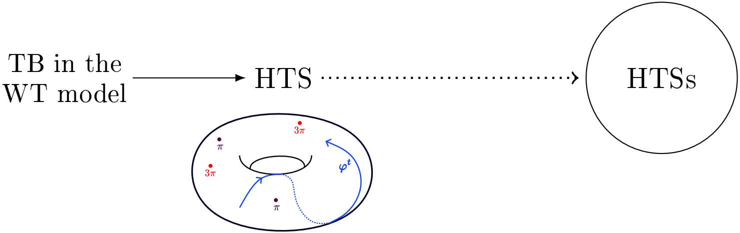

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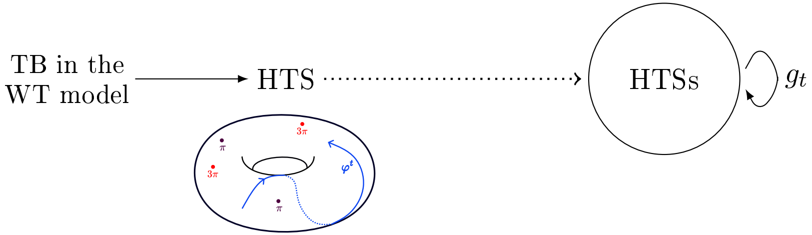

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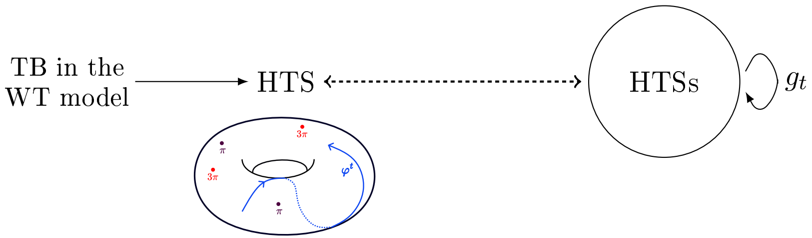
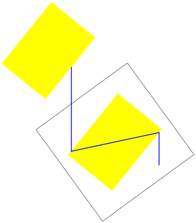


A corresponding surface



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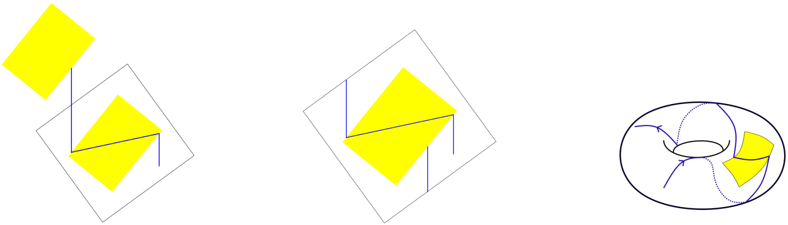

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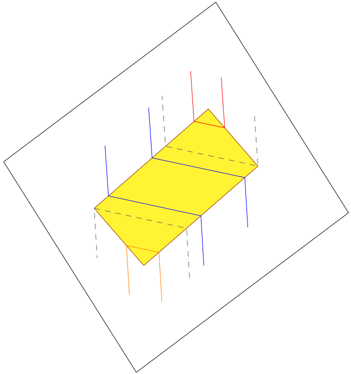

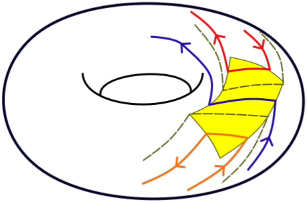

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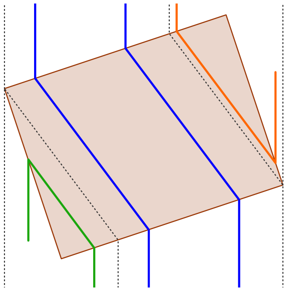
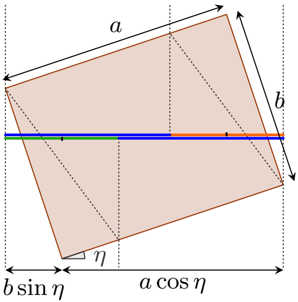
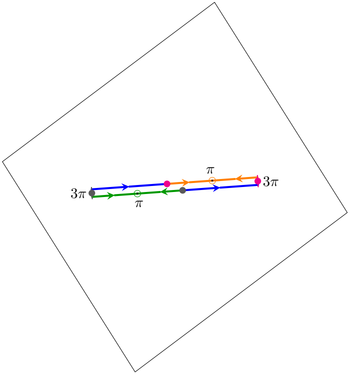




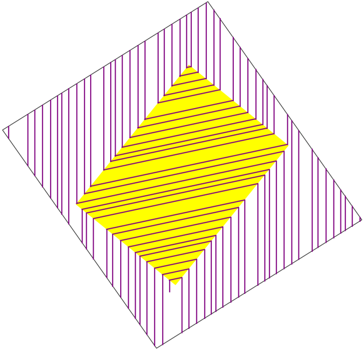
Stratum of the corresponding surface

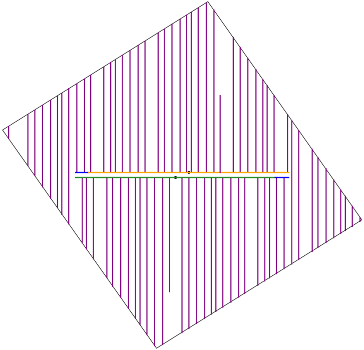


This surface is in $\mathcal{Q}(1^2, -1^2)$.

Trajectories on both surfaces

We get the "same" trajectory on both surfaces.





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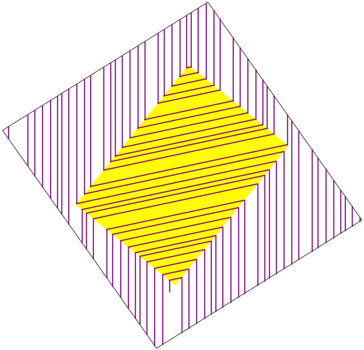
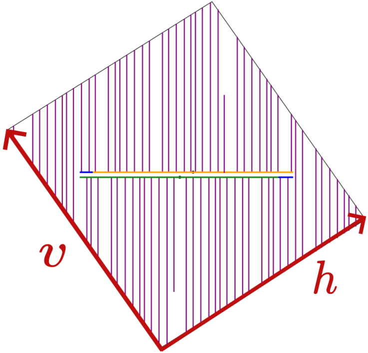
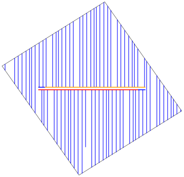




The question is now: how many times does the trajectory intersect the curves v and h?

Teichmüller flow

We renormalize the surface via the Teichmüller flow to get a smaller curve.



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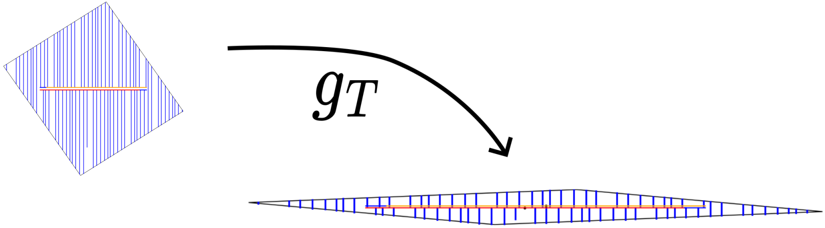
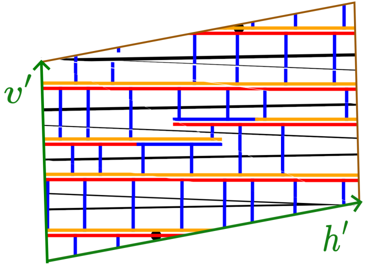


Teichmüller flow

After cutting and pasting, we get new curves, h' and v', that intersect our trajectory only a few times.



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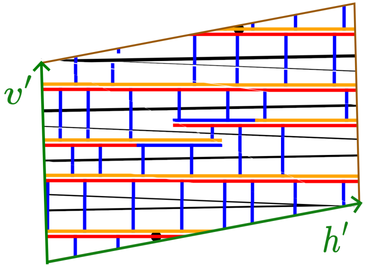
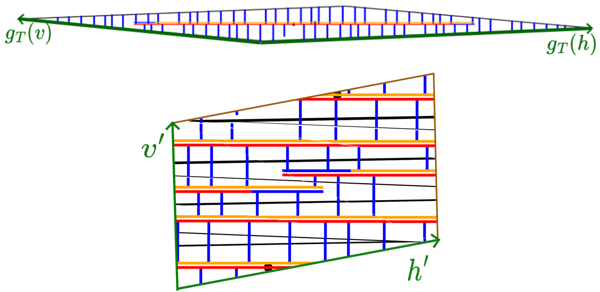


Question: What is the relation between h', v' and $g_T(h)$, $g_T(v)$?

The Kontsevich-Zorich cocycle

Cutting and pasting corresponds to a change of basis of the homology $H_1(g_T(\Sigma))$ of the surface $g_T(\Sigma)$ from "old" basis $(g_T(h), g_T(v))$ to the "new" basis (h', v').



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We denote A_T the transition matrix from the basis (h', v') to the basis $(g_T(h), g_T(v))$, i.e.

$$\begin{cases} g_T(h) & = ah' + bv' \\ g_T(v) & = ch' + dv' \end{cases},$$

where

$$A_T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We are interested in the growth of A_T as T goes to infinity.

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- We have to find a w that is contracted

Oseledets' theorem

The Oseledets theorem is a multiplicative version of the ergodic theorem:

Theorem (Birkhoff, 1931)

Let $(\mathtt{X}, \Phi, \mu)$ be a dynamical system. If Φ is ergodic, then

$$\forall f \in L^1(\mu),\ \mu\text{-a.e. } x,\ \frac{1}{n}\sum_{k=0}^{n-1} f(\Phi^k x) \underset{n\to\infty}{\longrightarrow} \int_{\mathtt{X}} f \mathrm{d}\mu.$$

Oseledets' theorem

Theorem (Oseledets, 1965)

Let $(\mathtt{X}, \Phi, \mu)$ *be a dynamical system. Let* $\mathcal{E}$ *be a vector bundle over* $\mathtt{X}$*.*

Let $A : \begin{array}{ccc} \mathtt{X} & \longrightarrow & \left(\mathcal{E}_x \to \mathcal{E}_{\Phi(x)}\right) \\ x & \mapsto & A_x \end{array}$ *be a cocycle.*

Assume that Φ *is ergodic and* $\int_{\mathtt{X}} \log |||A_x||| \mathrm{d}\mu < \infty$*. Then it exist Lyapunov exponents*

$$\theta_1 > \cdots > \theta_k$$

and, for μ*-almost every* x*, a decomposition*

$$\mathcal{E}_x = E_1^x \supset \cdots \supset E_k^x \supset \{0\}$$

with $A_x E_i^x \subset E_i^{\Phi(x)}$*, dim*$(E_i^x)$ *almost everywhere constant, such that:*

$$\forall v \in E_i^x \setminus E_{i+1}^x, \ \frac{\log |||A_x^{(n)} v|||}{n} \xrightarrow[n \to \infty]{} \theta_i.$$

A contracted direction

For almost every admissible parameters (Λ, a, b, η), the surface is generic (the orbit under the Teichmüller flow is reccurent, one can apply the Osseledets theorem).

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The vector $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ gives the direction of the strip in which the trajectory is trapped.

Thank you!