Controllability properties of coupled PDEs

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(ongoing) works with Hugo Lhachemi and Christophe Prieur

Toulouse, Conference TRECOS ANR project

Coupled systems

Some (nonexhaustive!) bibliography on control and stabilization of coupled (parabolic / hyperbolic) systems:

Lebeau Zuazua 1998   Zuazua 2001
Zhang Zuazua 2003-2004
Fernandez-Cara de Teresa 2004
Ammar Khodja Benabdallah Dupaix Gonzalez-Burgos de Teresa 2009–2016
Beauchard Zuazua 2011
Badra Takahashi 2014
Wang Krstic 2015   Wang Su Li 2015
Ammar Khodja Chouly Duprez 2016
Kang Guo 2016
Alabau Coron Olive 2017   Chen Vazquez Krstic 2017
Ghousein Witrant 2020
Katz Fridman 2020
Beauchard Koenig Le Balc'h 2020   Benabdallah Boyer Morancey 2020
Xu Liu Krstic Feng 2023
Auriol 2020-2024   Hu Olive 2020–2025   Boyer Olive 2024
Bhandari Boyer 2021
Bhandari Chowdhury Dutta Majumdar Kumbhakar 2023-2024
Tang Wang Kang 2024-2025
Ammar Khodja Benabdallah Gonzalez-Burgos Morancey de Teresa 2026?
Boyer book 2026?

Wave-heat cascade system

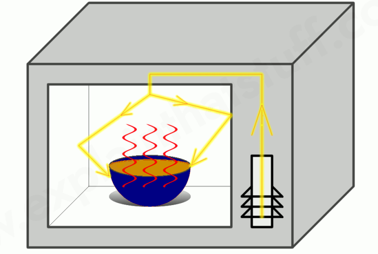
$c \in \mathbb{R}$, $\beta \in L^\infty(0,1)$ coupling function

(1D) Wave-heat cascade

$$\partial_t y = \partial_{xx} y + c\, y + \beta\, z, \qquad y(t,0) = y(t,1) = 0$$

$$\partial_{tt} z = \partial_{xx} z, \qquad z(t,0) = 0, \quad \partial_x z(t,1) = u(t)$$

Simplified model for control of microwave heating



Objectives: controllability properties; output feedback stabilization

Wave-heat cascade system

Seminal paper on controllability of 1D hyperbolic-parabolic coupled systems:

X. Zhang, E. Zuazua, CRAS 2003 and JDE 2004.

$$\partial_t y = \partial_{xx} y$$
$$\partial_{tt} z = \partial_{xx} z$$
$$\partial_x y(t,0) = \partial_x z(t,0)$$
$$y(t,1) = z(t,1) = 0$$

What they did:

- Riesz spectral analysis.
- Establish and use an Ingham-Müntz inequality.

$\Rightarrow$ Controllability in a Hilbert space with exponential weights + polynomial decay.

Ingham-Müntz inequality

(parabolic part: Müntz-Sász) $\lambda_{1,n} \in \mathbb{C}$ $(n \in \mathbb{N}^*)$ satisfying:

$$\exists \alpha > 1 \quad | \quad (\lambda_{1,n}/n^\alpha)_{n\in\mathbb{N}^*} \text{ is bounded}$$

$$\exists n_0 \in \mathbb{N}^* \quad C_1, C_2 > 0 \quad | \quad -\operatorname{Re}(\lambda_{1,n}) \geqslant C_1 \operatorname{Im}(\lambda_{1,n}) \qquad \forall n \geqslant n_0$$

$$|\lambda_{1,n} - \lambda_{1,n'}| \geqslant C_2 |n^\alpha - n'^\alpha| \qquad \forall n, n' \geqslant n_0$$

(hyperbolic part: Ingham) $\lambda_{2,m} \in \mathbb{C}$ $(m \in \mathbb{Z})$ distinct, satisfying:

$$\exists m_0 \in \mathbb{N} \quad \gamma > 0 \quad z_0 \in \mathbb{C} \quad (\mu_m)_{m\in\mathbb{Z}} \in \ell^2(\mathbb{Z}) \quad | \quad \lambda_{2,m} = \gamma\, i\, m + z_0 + \mu_m \quad \forall |m| \geqslant m_0$$

Then: $\forall T > \frac{2\pi}{\gamma}$ $\quad \exists C_T > 0$ s.t.

$$\int_0^T \Big| \sum_{n\in\mathbb{N}^*} a_n e^{\lambda_{1,n} t} + \sum_{m\in\mathbb{Z}} b_m e^{\lambda_{2,m} t} \Big|^2 \mathrm{d}t \geqslant C_T \Big(\sum_{n\in\mathbb{N}^*} |a_n|^2 e^{2\operatorname{Re}(\lambda_{1,n}) T} + \sum_{m\in\mathbb{Z}} |b_m|^2 \Big)$$

$$\forall (a_n)_{n\in\mathbb{N}^*}, (b_m)_{m\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$$

Zhang Zuazua 2003 Komornik Tenenbaum 2015 Bhandari Chowdhury Dutta Kumbhakar 2024

Wave-heat cascade system

$$\partial_t y = \partial_{xx} y + c\, y + \beta\, z, \qquad y(t,0) = y(t,1) = 0$$
$$\partial_{tt} z = \partial_{xx} z, \qquad z(t,0) = 0, \quad \partial_x z(t,1) = u(t)$$

of the form

$$\boxed{\dot{\mathcal{X}}(t) = \mathcal{A}_0 \mathcal{X}(t) + \mathcal{B}_0 u(t)} \qquad \text{with} \qquad \mathcal{X} = \begin{pmatrix} y \\ z \\ \partial_t z \end{pmatrix}$$

Well-posed system (i.e., $\mathcal{B}_0$ admissible) with:

- $\mathcal{A}_0 : D(\mathcal{A}_0) \to \mathcal{H}^0$

  $\mathcal{H}^0 = L^2(0,1) \times H^1_{(0)}(0,1) \times L^2(0,1), \qquad H^1_{(0)}(0,1) = \{ g \in H^1(0,1) \mid g(0) = 0 \}$

$$\mathcal{A}_0 = \begin{pmatrix} \partial_{xx} + c\,\mathrm{id} & \beta\,\mathrm{id} & 0 \\ 0 & 0 & \mathrm{id} \\ 0 & \partial_{xx} & 0 \end{pmatrix}$$

$$D(\mathcal{A}_0) = \Big\{ (f,g,h) \in H^2(0,1) \times H^2(0,1) \times H^1(0,1) \mid f(0) = f(1) = g(0) = g'(1) = h(0) = 0 \Big\}$$

- $\mathcal{B}_0 \in L(\mathbb{R}, D(\mathcal{A}_0^*)')$ given by $\mathcal{B}_0^* \phi = \phi_3(1) \quad \forall \phi = (\phi_1, \phi_2, \phi_3)^\top \in D(\mathcal{A}_0^*)$

Wave-heat cascade system

Spectral properties of $\mathcal{A}_0$:

- Parabolic part ($n \in \mathbb{N}^*$):

$$\boxed{\lambda_{1,n} = c - n^2\pi^2, \qquad \phi_{1,n} = (\phi^1_{1,n}, \phi^2_{1,n}, \phi^3_{1,n})^\top}$$

$$\phi^1_{1,n}(x) = \sqrt{2}\sin(n\pi x), \qquad \phi^2_{1,n}(x) = \phi^3_{1,n}(x) = 0$$

- Hyperbolic part ($m \in \mathbb{Z}$):

$$\boxed{\lambda_{2,m} = (m + 1/2)i\pi, \qquad \phi_{2,m} = (\phi^1_{2,m}, \phi^2_{2,m}, \phi^3_{2,m})^\top}$$

$$\phi^1_{2,m}(x) = \frac{1}{A_m r_m}\int_x^1 \beta(s)\sinh(\lambda_{2,m}s)\sinh(r_m(x-s))\,\mathrm{d}s + \frac{\sinh(r_m(1-x))}{A_m r_m \sinh(r_m)}\int_0^1 \beta(s)\sinh(\lambda_{2,m}s)\sinh(r_m s)\,\mathrm{d}s$$

$$\phi^2_{2,m}(x) = \frac{1}{A_m}\sinh(\lambda_{2,m}x), \qquad \phi^3_{2,m}(x) = \frac{\lambda_{2,m}}{A_m}\sinh(\lambda_{2,m}x)$$

where r_m is a square root of $\lambda_{2,m} - c$ of nonnegative real part, and $A_m = |\lambda_{2,m}| = |m + 1/2|\pi$.

Wave-heat cascade system

Adjoint operator $\mathcal{A}_0^* : D(\mathcal{A}_0^*) \to \mathcal{H}^0$

$$\mathcal{A}_0^* = \begin{pmatrix} \partial_{xx} + c\,\mathrm{id} & 0 & 0 \\ P_\beta & 0 & -\mathrm{id} \\ 0 & -\partial_{xx} & 0 \end{pmatrix}$$

where $P_\beta f = \int_0^{(\cdot)} \int_\tau^1 \beta(s) f(s)\,\mathrm{d}s\,\mathrm{d}\tau$ and

$$D(\mathcal{A}_0^*) = \Big\{ (f, g, h) \in H^2(0,1) \times H^2(0,1) \times H^1(0,1) \;\; | \\ f(0) = f(1) = g(0) = g'(1) = h(0) = 0 \Big\}$$

Wave-heat cascade system

Spectral properties of $\mathcal{A}_0^*$:

- Parabolic part ($n \in \mathbb{N}^*$): $\boxed{\lambda_{1,n} = c - n^2\pi^2, \qquad \psi_{1,n} = (\psi_{1,n}^1, \psi_{1,n}^2, \psi_{1,n}^3)^\top}$

$$\psi_{1,n}^1(x) = \sqrt{2}\sin(n\pi x)$$

$$\psi_{1,n}^2(x) = -\frac{\gamma_n}{\lambda_{1,n}^2\cosh(\lambda_{1,n})}\sqrt{2}\cosh(\lambda_{1,n}(x-1)) - \frac{1}{\lambda_{1,n}^2}\sqrt{2}\int_x^1 \beta(s)\sin(n\pi s)\sinh(\lambda_{1,n}(x-s))\,\mathrm{d}s - \frac{1}{\lambda_{1,n}}\sqrt{2}\int_0^x\int_1^\tau \beta(s)\sin(n\pi s)\,\mathrm{d}s\,\mathrm{d}\tau$$

$$\psi_{1,n}^3(x) = \frac{\gamma_n}{\lambda_{1,n}\cosh(\lambda_{1,n})}\sqrt{2}\cosh(\lambda_{1,n}(x-1)) + \frac{1}{\lambda_{1,n}}\sqrt{2}\int_x^1 \beta(s)\sin(n\pi s)\sinh(\lambda_{1,n}(x-s))\,\mathrm{d}s$$

with $\gamma_n = \int_0^1 \beta(s)\sin(n\pi s)\sinh(\lambda_{1,n}s)\,\mathrm{d}s$ (important role in what follows)

- Hyperbolic part ($m \in \mathbb{Z}$): $\boxed{\lambda_{2,m} = (m+1/2)i\pi, \qquad \psi_{2,m} = (\psi_{2,m}^1, \psi_{2,m}^2, \psi_{2,m}^3)^\top}$

$$\psi_{2,m}^1(x) = 0, \quad \psi_{2,m}^2(x) = \frac{A_m}{|\lambda_{2,m}|^2}\sinh(\lambda_{2,m}x), \quad \psi_{2,m}^3(x) = \frac{A_m\lambda_{2,m}}{|\lambda_{2,m}|^2}\sinh(\lambda_{2,m}x).$$

Wave-heat cascade system

$\Phi = \{\phi_{1,n} \mid n \in \mathbb{N}^*\} \cup \{\phi_{2,m} \mid m \in \mathbb{Z}\}$ is a Riesz basis of $\mathcal{H}^0 = L^2 \times H^1_{(0)} \times L^2$, of dual Riesz basis $\Psi = \{\psi_{1,n} \mid n \in \mathbb{N}^*\} \cup \{\psi_{2,m} \mid m \in \mathbb{Z}\}$.

Hence, $\mathcal{A}_0$ is a Riesz spectral operator and

$$e^{t\mathcal{A}_0}\mathcal{X} = \sum_{n\in\mathbb{N}^*} e^{\lambda_{1,n}t} \langle \mathcal{X}, \psi_{1,n} \rangle \phi_{1,n} + \sum_{m\in\mathbb{Z}} e^{\lambda_{2,m}t} \langle \mathcal{X}, \psi_{2,m} \rangle \phi_{2,m} \qquad \forall \mathcal{X} \in \mathcal{H}^0.$$

Similarly for $\mathcal{A}_0^*$.

Wave-heat cascade system

Exact (null) controllability for $\dot{\mathcal{X}}(t) = \mathcal{A}_0\mathcal{X}(t) + \mathcal{B}_0 u(t)$ **in some Hilbert space** V (V_0) is equivalent, by duality, to the observability inequality

$$\int_0^T |\mathcal{X}^3(t,1)|^2\,\mathrm{d}t \geqslant C_T\|\mathcal{X}(0,\cdot)\|^2_{V'}, \qquad \left(\|\mathcal{X}(T,\cdot)\|^2_{V_0'}\right)$$

for any solution $\mathcal{X} = \begin{pmatrix}\mathcal{X}^1\\ \mathcal{X}^2\\ \mathcal{X}^3\end{pmatrix}$ of the dual system $\dot{\mathcal{X}}(t) = \mathcal{A}_0^*\mathcal{X}(t)$. (pivot space $\mathcal{H}^0$)

Wave-heat cascade system

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Since $\mathcal{X}^3(t,1) = \sum_{n\in\mathbf{N}^*} e^{\lambda_{1,n}t}\,\langle\mathcal{X}(0),\phi_{1,n}\rangle\,\psi^3_{1,n}(1) + \sum_{m\in\mathbb{Z}} e^{\bar{\lambda}_{2,m}t}\,\langle\mathcal{X}(0),\phi_{2,m}\rangle\,\psi^3_{2,m}(1)$,

it follows from the Ingham-Müntz inequality that $\forall T>2 \quad \exists\,\mathrm{Cst}>0$ s.t.

$$\int_0^T |\mathcal{X}^3(t,1)|^2\,\mathrm{d}t \geqslant \mathrm{Cst}\sum_{n\in\mathbf{N}^*}|\langle\mathcal{X}(0),\phi_{1,n}\rangle|^2 \underbrace{|\psi^3_{1,n}(1)|^2}_{\sim\,\mathrm{Cst}\,\frac{\gamma_n^2}{n^4}e^{-2n^2\pi^2}} \underbrace{e^{2\lambda_{1,n}T}}_{\sim\,\mathrm{Cst}\,e^{-2n^2\pi^2T}\ \text{as}\ n\gg1} + \mathrm{Cst}\sum_{m\in\mathbb{Z}}|\langle\mathcal{X}(0),\phi_{2,m}\rangle|^2 \underbrace{|\psi^3_{2,m}(1)|^2}_{\sim\,\mathrm{Cst}\ \text{as}\ |m|\gg1}$$

$$\geqslant \mathrm{Cst}\underbrace{\left(\sum_{n\in\mathbf{N}^*}|\langle\mathcal{X}(0),\phi_{1,n}\rangle|^2\,\frac{\gamma_n^2}{n^4}\,e^{-\nu n^2} + \sum_{m\in\mathbb{Z}}|\langle\mathcal{X}(0),\phi_{2,m}\rangle|^2\right)}_{\text{this defines }\|\mathcal{X}(0,\cdot)\|^2_{V'}} \quad \text{with} \quad \nu = 2\pi^2(1+T)$$

Wave-heat cascade system

Exact (null) controllability for $\dot{\mathcal{X}}(t) = \mathcal{A}_0\mathcal{X}(t) + \mathcal{B}_0 u(t)$ **in some Hilbert space** V (V_0) is equivalent, by duality, to the observability inequality

$$\int_0^T |\mathcal{X}^3(t,1)|^2\,\mathrm{d}t \geqslant C_T \|\mathcal{X}(0,\cdot)\|^2_{V'} \qquad \left(\|\mathcal{X}(T,\cdot)\|^2_{V_0'}\right)$$

for any solution $\mathcal{X} = \begin{pmatrix}\mathcal{X}^1\\ \mathcal{X}^2\\ \mathcal{X}^3\end{pmatrix}$ of the dual system $\dot{\mathcal{X}}(t) = \mathcal{A}_0^*\mathcal{X}(t)$. (pivot space $\mathcal{H}^0$)

This leads to define

$$V' = \left\{ \sum_{n\in\mathbb{N}^*} a_n\psi_{1,n} + \sum_{m\in\mathbb{Z}} b_m\psi_{2,m} \quad \middle| \quad \sum_{n\in\mathbb{N}^*} |a_n|^2\,\frac{\gamma_n^2}{n^4}\,e^{-\nu n^2} + \sum_{m\in\mathbb{Z}} |b_m|^2 < +\infty \right\}$$

where $\nu = 2\pi^2(1+T)$. Endowed with the norm

$$\|\mathcal{X}\|^2_{V'} = \sum_{n\in\mathbb{N}^*} |\langle \mathcal{X}, \phi_{1,n}\rangle|^2\,\frac{\gamma_n^2}{n^4}\,e^{-\nu n^2} + \sum_{m\in\mathbb{Z}} |\langle \mathcal{X}, \phi_{2,m}\rangle|^2$$

V' is a Hilbert space under the assumption that $\gamma_n \neq 0$ for any $n \in \mathbb{N}^*$.

$\Rightarrow$ We have observability in this space V'.

Wave-heat cascade system

Exact (null) controllability for $\dot{\mathcal{X}}(t) = \mathcal{A}_0\mathcal{X}(t) + \mathcal{B}_0 u(t)$ **in some Hilbert space** V (V_0) is equivalent, by duality, to the observability inequality

$$\int_0^T |\mathcal{X}^3(t,1)|^2\,\mathrm{d}t \geqslant C_T\|\mathcal{X}(0,\cdot)\|^2_{V'} \qquad \left(\|\mathcal{X}(T,\cdot)\|^2_{V_0'}\right)$$

for any solution $\mathcal{X} = \begin{pmatrix}\mathcal{X}^1\\ \mathcal{X}^2\\ \mathcal{X}^3\end{pmatrix}$ of the dual system $\dot{\mathcal{X}}(t) = \mathcal{A}_0^*\mathcal{X}(t)$. (pivot space $\mathcal{H}^0$)

By duality, if $\gamma_n \neq 0$ for any $n \in \mathbb{N}^*$, we have exact controllability in

$$V = \left\{ \sum_{n\in\mathbb{N}^*} a_n\phi_{1,n} + \sum_{m\in\mathbb{Z}} b_m\phi_{2,m} \quad \middle| \quad \sum_{n\in\mathbb{N}^*} |a_n|^2\,\frac{n^4}{\gamma_n^2}\,e^{\nu n^2} + \sum_{m\in\mathbb{Z}} |b_m|^2 < +\infty \right\}$$

where $\nu = 2\pi^2(1+T)$, endowed with the norm

$$\|\mathcal{X}\|_V^2 = \sum_{n\in\mathbb{N}^*} |\langle \mathcal{X}, \psi_{1,n}\rangle|^2\,\frac{n^4}{\gamma_n^2}\,e^{\nu n^2} + \sum_{m\in\mathbb{Z}} |\langle \mathcal{X}, \psi_{2,m}\rangle|^2$$

Recall that $\gamma_n = \int_0^1 \beta(s)\sin(n\pi s)\sinh(\lambda_{1,n}s)\,\mathrm{d}s$ (different expression when $\lambda_{1,n} = 0$).

Similarly, we have exact null controllability in V_0 defined like V but with $\nu = 2\pi^2$.

Wave-heat cascade system

$$\partial_t y = \partial_{xx} y + c\, y + \beta\, z, \qquad y(t,0) = y(t,1) = 0$$

$$\partial_{tt} z = \partial_{xx} z, \qquad z(t,0) = 0, \quad \partial_x z(t,1) = u(t)$$

$$V \subset V_0 \subset \mathcal{H}^0 \subset V_0' \subset V'$$

(continuous and dense embeddings)

$$\gamma_n = \int_0^1 \beta(s) \sin(n\pi s) \sinh(\lambda_{1,n} s)\, \mathrm{d}s$$

Theorem (Lhachemi Prieur Trélat 2025)

If $T > 2$ and $\gamma_n \neq 0 \quad \forall n \in \mathbb{N}^*$ then:

1. Exact controllability in time T in V. (but solutions live in $\mathcal{H}^0$)
2. Exact null controllability in time T in V_0.
3. Approximate controllability in time T in $\mathcal{H}^0$ (or in any other Hilbert space s.t. $H \subset \mathcal{H}^0$ or $\mathcal{H}^0 \subset H$ with continuous and dense embeddings).

If $T < 2$ or if $\gamma_n = 0$ for some $n \in \mathbb{N}^*$ then no exact / exact null / approximate controllability in any time $T > 0$, in any Hilbert space s.t. $H \subset \mathcal{H}^0$ or $\mathcal{H}^0 \subset H$ with continuous and dense embeddings.

Wave-heat cascade system

$$V = \left\{ \sum_{n\in\mathbb{N}^*} a_n \phi_{1,n} + \sum_{m\in\mathbb{Z}} b_m \phi_{2,m} \;\middle|\; \sum_{n\in\mathbf{N}^*} |a_n|^2 \frac{n^4}{\gamma_n^2} e^{\nu n^2} + \sum_{m\in\mathbb{Z}} |b_m|^2 < +\infty \right\} \qquad \gamma_n = \int_0^1 \beta(s) \sin(n\pi s) \sinh(\lambda_{1,n} s)\, ds$$

Comments:

- Compared with $\mathcal{H}^0 = L^2(0,1) \times H^1_{(0)}(0,1) \times L^2(0,1)$, the spaces V and V_0 are much smaller than $\mathcal{H}^0$ in the a_n components, but they coincide with $\mathcal{H}^0$ in the b_m components.
- The controllability spaces V and V_0 are "almost sharp".
- These are non-conventional Hilbert spaces: depending on the coupling function β, the coefficient γ_n may wildly oscillate.

Example: $(0 \leqslant a < b \leqslant 1)$ $\quad \beta(x) = \beta_0 \mathbb{1}_{[a,b]}(x) \quad \forall x \in (0,1)$

$$\Rightarrow \gamma_n = \frac{\beta_0}{\lambda_{1,n}^2 + n^2\pi^2} \Big(-n\pi \sinh(\lambda_{1,n} b)\cos(n\pi b) + n\pi \sinh(\lambda_{1,n} a)\cos(n\pi a) + \lambda_{1,n}\cosh(\lambda_{1,n} b)\sin(n\pi b) - \lambda_{1,n}\cosh(\lambda_{1,n} a)\sin(n\pi a)\Big)$$

The subset $\hat{S}$ of $S = \{(a,b) \mid 0 \leqslant a < b \leqslant 1\}$ such that $\gamma_n \neq 0$ for any $n \in \mathbb{N}^*$ is dense and of full Lebesgue measure in S.

Wave-heat cascade system

Feedback stabilization:

Exact null controllability $\Rightarrow$ (complete) stabilizability in V_0, with a linear feedback

BUT how to define such a feedback in practice?

Actually, thanks to the spectral analysis done, we obtain:

Theorem (Lhachemi Prieur Trélat 2025)

Given any $\delta > 0$, there exists an output-feedback control

- based on the two measurements

$$y_o(t) = \int_0^1 c_o(x) y(t,x)\,\mathrm{d}x \qquad \text{for some (generic) } c_o \in L^2(0,1)$$

$$z_o(t) = \partial_t z(t,1)$$

- explicitly built from a finite number of spectral modes (depending on δ)

making the system exponentially stable in $H_0^1(0,1) \times H_{(0)}^1(0,1) \times L^2(0,1)$ with the decay rate δ.

And for heat-wave?

We have obtained a sharp controllability result for:

(1D) Wave-heat cascade

$$\partial_t y = \partial_{xx} y + c\, y + \beta\, z, \qquad y(t,0) = y(t,1) = 0$$

$$\partial_{tt} z = \partial_{xx} z, \qquad z(t,0) = 0, \quad \partial_x z(t,1) = u(t)$$

Now let us "switch":

(1D) Heat-wave cascade

$$\partial_t y = \partial_{xx} y + c\, y, \qquad y(t,0) = y(t,1) = u(t)$$

$$\partial_{tt} z = \partial_{xx} z + \beta\, y, \qquad z(t,0) = 0, \quad \partial_x z(t,1) = 0$$

Similar eigenelements computations
$\Rightarrow \forall T > 2$ controllability in some spectrally defined (unconventional) Hilbert space

BUT when $\mathrm{supp}(\beta) = [0, 1]$ (e.g., $\beta = 1$) we may expect to have controllability $\forall T > 0$!!
How to do?

And for heat-wave?

Let me mention:

(Multi-D) Heat-wave cascade in Ω

$$\partial_t y = \triangle y + \chi_{\omega_1} u \qquad y(t,\cdot)_{|\partial\Omega} = 0$$

$$\partial_{tt} z = \partial_{xx} z + \chi_{\omega_2} y \qquad z(t,\cdot)_{|\partial\Omega} = 0$$

Fernandez-Cara, de Teresa, DCDS 2004

If $\omega_1 \cap \omega_2$ satisfies “the multiplier domain condition” (stronger than GCC), then exact null controllability in $H^{-1}(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$ in time $T > T_0$.

- Expected because on $\omega_1 \cap \omega_2$ we can choose y as we want and control the wave equation in time $T > T_0$.
- The difficult case would be when $\omega_1 \cap \omega_2 = \emptyset$...

  (or when the heat equation is controlled at the boundary)

  Conjecture:

  Controllability in some unconventional Hilbert space, at least if $T > T_0$.
  For $T < T_0$, I do not know.

Heat-heat cascade system

Heat-heat cascade

$$
\begin{aligned}
&\partial_t y = \partial_{xx} y + ay && a, b \in \mathbb{R}, \quad s \neq 0 \\
&\partial_t z = \partial_{xx} z + bz \\
&\partial_x y(t,0) = sz(t,0), \qquad y(t,1) = \partial_x z(t,1) = 0 \\
&\partial_x z(t,0) = u(t)
\end{aligned}
$$

studied by Tang Wang Kang (SCL 2024 and Automatica 2025)

Theorem (Lhachemi Prieur Trélat 2025)

Exact null controllability in any time $T > 0$ in

$$
V_0 \,/\, V = \Big\{ \sum_{n\in\mathbb{N}} a_n \phi_{1,n} + \sum_{m\in\mathbb{N}} b_m \phi_{2,m} \quad \Big| \quad a_n, b_n \in \mathbb{R}, \\
\sum_{n\in\mathbb{N}} \left(1 + \frac{n^4}{s^2(a-b)^2}\right) e^{2T(n+1/2)^2\pi^2} |a_n|^2 + \sum_{m\in\mathbb{N}} |b_m|^2 e^{2Tm^2\pi^2} < +\infty \Big\}.
$$

Under (explicit, necessary and sufficient) controllability and observability conditions, the system can be exponentially stabilized at any decay rate in $H^1(0,1) \times H^1(0,1)$, with an explicit output-feedback control.

N-heat cascade system

Generalization to:

N-heat cascade

$$
\begin{array}{lll}
y^j_t = y^j_{xx} + a_j y^j & 1 \leqslant j \leqslant N & a_1, \dots, a_N \in \mathbb{R} \\
y^j_x(t,0) = y^{j+1}(t,1), & y^j_x(t,1) = 0 & 1 \leqslant j \leqslant N-1 \\
y^N_x(t,0) = u(t), & y^N_x(t,1) = 0 &
\end{array}
$$

either with two-by-two distinct eigenvalues, or with $a_1 = \cdots = a_N$.

Minimal time for parabolic systems

Coupled parabolic system

$$\partial_t y = \partial_{xx} y + \beta\, z \qquad y(t,0) = y(t,1) = 0$$

$$\partial_t z = \partial_{xx} z + \chi_\omega u \qquad z(t,0) = z(t,1) = 0$$

with

$$\boxed{\operatorname{supp}(\beta) \cap \omega = \emptyset}$$

Well-posed in $L^2(0,1) \times L^2(0,1)$ (with $u \in L^2(0,T)$).

Ammar Khodja, Benabdallah, Gonzalez-Burgos, de Teresa (JMAA 2016)

Under a spectral condition on β, there exists $T_0(\beta)$ (explicit) such that the system is

- exactly null controllable in time $T > T_0(\beta)$,
- not exactly null controllable in time $T < T_0(\beta)$.

(proof by moment method)

Minimal time for parabolic systems

Coupled parabolic system

$$\partial_t y = \partial_{xx} y + \beta\, z \qquad y(t,0) = y(t,1) = 0$$
$$\partial_t z = \partial_{xx} z + \chi_\omega u \qquad z(t,0) = z(t,1) = 0$$

with

$$\boxed{\mathrm{supp}(\beta) \cap \omega = \emptyset}$$

Well-posed in $L^2(0,1) \times L^2(0,1)$ (with $u \in L^2(0,T)$).

We make their statement more precise:

Lhachemi Prieur Trélat, ongoing

Under a spectral condition on β, the system is **always** exactly null controllable, but the controllability space V_T depends on T:

$$V_T = \begin{cases} L^2(0,1) \times L^2(0,1) & \text{if } T > T_0(\beta) \\ \text{much smaller, spectrally defined} & \text{if } T < T_0(\beta) \\ \text{(thanks to Müntz-Sász)} & \end{cases}$$

Generalization to a larger number of coupled parabolic equations, with more "bifurcation" times (ongoing work).

Conclusion

- Ingham-Müntz as an instrumental tool for coupled 1D PDEs.
- Unconventional controllability spaces for internal couplings.

Many questions:

- Coupled 1D PDEs with time-varying coefficients?
- Multi-D case?
  $\to$ combine Carleman with microlocal analysis?
  $\to$ "Ingham-Lebeau-Robbiano" inequalities?
- Plenty of other cascade or coupled systems.
- Semilinear case.