Estimate of the Stokes system at a boundary

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$$\begin{cases} \partial_t U - \Delta U + \nabla q = F \\ \operatorname{div} U = h \end{cases}$$

on $\mathcal{M}$ open set of $\mathbb{R}^d$
or smooth manifold of dimension d
with or without boundary

+ initial conditions
+ Boundary conditions. general (Lopatinskii-Sapiro type)
precise conditions given below

typical examples: . Dirichlet $U|_\partial = G$

. Navier: $\nu \cdot U = G_\nu$ $\quad \left(({\nabla U} + {}^t\nabla U)\cdot \nu\right)_{\tan} = G_{\tan}$

. Neumann $\left({}^t\nabla U + \kappa \nabla U\right)\nu - q\nu = G$

<u>Observability</u> $T > 0$ $\omega \subset M$ open subset

$$\int_0^T \left(\|U\|^2 + \|q\|^2 \right) dt \lesssim \int_0^T \left(\|U^2\|_{L^2(\omega)} + \|q\|^2_{L^2(\omega)} \right) dt$$

$$+ \int_0^T \left(\|F\| + \|h\| + |G| \right) dt$$

boundary value.

<u>Ref</u> : homogeneous Dirichlet . Fernandez-Cara, Guerrero, Imanuvilov, Puel 04
+ flat Laplace . Buffe, Takahashi 25

Application observability / null controllability of Oseen system
NS (local result)

. Navier : Guerrero 2006
Fernandez-Cara, Gonzalez-Burgos, Imanuvilov, Puel (2006–2008)

Strategy $\partial_t U - \Delta U = F - \nabla q$ + homog. Dirichlet

Heat-type Carleman estimate for U.

Apply divergence and use $\operatorname{div} U = 0$

$\Delta q = \operatorname{div} F$ No boundary condition.

Multiplier + microlocal argument $\longrightarrow$ estimation of $q|_{\partial_y}$

Elliptic Carleman estimate for q + time integration

$\longrightarrow$ combine estimations.

Carleman estimates

Estimations of the form

$$\tau^{3/2}\|e^{\tau\varphi}u\|_{L^2} + \tau^{1/2}\|e^{\tau\varphi}\nabla u\|_{L^2} \lesssim C\,\|e^{\tau\varphi}Pu\|$$

- P order 2, elliptic
- $\varphi = \varphi(x)$ weight function
- $u \in \mathcal{C}_c^\infty$, $\tau > 0$ large

- if u defined at a boundary

boundary condition (order k)

$$\tau^{3/2}\|e^{\tau\varphi}u\|_{L^2} + \tau^{1/2}\|e^{\tau\varphi}\nabla u\|_{L^2} \lesssim C\left(\|e^{\tau\varphi}Pu\|_{L^2} + |e^{\tau\varphi}Bu|_{\frac{3}{2}-k}\right)$$
$$+\ \tau^{3/2}|e^{\tau\varphi}u|_\partial|_{L^2} + \tau^{1/2}|e^{\tau\varphi}\nabla u|_\partial|_{L^2}$$

+ Observation term

Carleman estimates   Parabolic version

$$\| \tilde{\tau}^{3/2} e^{\tau\eta\phi} u \|_{L^2} + \| \tilde{\tau}^{1/2} e^{\tau\eta\phi} \nabla u \|_{L^2} \leq C \, \| e^{\tau\eta\phi} P u \|_{L^2}$$

. $P = \partial_t - \Delta$. $u \in C_c^\infty$, $\tau > 0$ large

. $\varphi = \varphi(x)$ weight function. $\eta = \dfrac{1}{t\,(T-t)}$

$\phi = \varphi - K < 0$

$\tilde{\tau} = \tau \eta \varphi$

Carleman estimates

Elliptic estimate

$$\tau^{3/2} \| e^{\tau\varphi} u \|_{L^2} + \tau^{1/2} \| e^{\tau\varphi} \nabla u \|_{L^2} \leq C \| e^{\tau\varphi} P u \|_{L^2}$$

$$P_\varphi = e^{\tau\varphi} P e^{-\tau\varphi} .$$

$$\tau^{3/2} \| v \| + \tau^{1/2} \| \nabla v \| \lesssim \| P_\varphi v \|$$

principal symbol $\quad p = |\xi|^2 = \sum_j |\xi_j|^2$

$$p_\varphi = \sum_j |\xi_j + i\tau \partial_j \varphi|^2 = \underbrace{|\xi|^2 - |\tau d\varphi|^2}_{p_2} + i \underbrace{2\tau \xi \cdot d\varphi}_{p_1}$$

N.B. p_φ can vanish $\quad p_\varphi$ not elliptic

$$\|P_\varphi v\|^2_{L^2} = \|(Q_2 + iQ_1)v\|^2_{L^2} = \|Q_2 v\|^2_{L^2} + \|Q_1 v\|^2_{L^2} + 2\mathrm{Re}(Q_2 v, iQ_1 v)_{L^2}$$

$$= (Q_2^2 v, v)_{L^2} + (Q_1^2 v, v)_{L^2} + (i[Q_2, Q_1]v, v)$$

q_2^2 order 4 can vanish

q_1^2 order 4 can vanish

$\{q_2, q_1\}$ order 3

$$\geq \Big(\underbrace{\Big(\frac{\mu}{\tau}(Q_2^2 + Q_1^2) + i[Q_2, Q_1]\Big)}_{\text{order } 3} v, v\Big)$$

$\mu > 0$

$\frac{\mu}{\tau} \ll 1$

$$\frac{\mu}{\tau}(q_2^2 + q_1^2) + \{q_2, q_1\}$$

Lemma if $|d\psi| \geq c > 0$ and $\varphi = e^{\gamma\psi}$ then

$$q_2 = q_1 = 0 \implies \{q_2, q_1\} > 0 \quad \text{for } \gamma > 0 \text{ large.}$$

Lemma if $|d\psi| \geq c > 0$ and $\varphi = e^{\gamma\psi}$ then

$$\frac{\mu}{\tau}\left(q_2^2 + q_1^2\right) + \{q_2, q_1\} \gtrsim \frac{1}{\tau}\left(\tau^4 + |\xi|^4\right)$$

for $\mu > 0$ and $\gamma > 0$ large

$$\Rightarrow \quad \|P_\varphi v\|^2 \gtrsim \tau^3 \|v\|^2 + \tau \|\nabla v\|^2 + \tau^{-1} \|D^2 v\|^2$$

Dependence on γ (Carleman second large parameter)

$$\gamma \frac{\mu}{\tilde{\tau}}\left(q_2^2 + q_1^2\right) + \{q_2, q_1\} \gtrsim \frac{\gamma}{\tilde{\tau}}\left(\tilde{\tau}^4 + |\xi|^4\right)$$

$$\|P_\varphi v\| \gtrsim \gamma^{\frac{1}{2}}\left(\|\tilde{\tau}^{3/2} v\| + \|\tilde{\tau}^{\frac{1}{2}} \nabla v\| + \|\tilde{\tau}^{-1/2} D^2 v\|\right)$$

Can be of great help

Back to Stokes — Away from the boundary, metric g

$$\partial_t U - \Delta_g U + \nabla_g q = F \qquad \mathrm{div}_g U = h$$

Apply the divergence $\qquad [\,\mathrm{div}_g, \Delta_g\,] = \underbrace{R_1 \text{ first-order}}_{\text{curvature}}$

- $$\Delta_g q = \mathrm{div}_g F - R_1 U - (\partial_t - \Delta_g) h$$

time-uniform elliptic equation + time integration

① $$\gamma \left(\int_0^T \left(\| \tilde{\tau}^{1/2} e^{\tau\eta\phi} q \|^2 + \| \tilde{\tau}^{-1/2} e^{\tau\eta\phi} \nabla q \|^2 \right) dt \right)$$
$$\lesssim \int_0^T \left(\| e^{\tau\eta\phi} F \|^2 + \| e^{\tau\eta\phi} U \|^2 + \| \tilde{\tau}^{-1} e^{\tau\eta\phi} (\partial_t - \Delta) h \|^2 \right) dt$$

- $$\partial_t U_i - \Delta U_i = F_i - \partial_i q + R_1' U \qquad R_1' \text{ first-order}$$

$$\gamma \int_0^T \left(\| \tilde{\tau}^{3/2} e^{\tau\eta\phi} U \|^2 + \| \tilde{\tau}^{1/2} e^{\tau\eta\phi} \nabla U \|^2 \right) dt$$
$$\lesssim \int_0^T \left(\| e^{\tau\eta\phi} F \|^2 + \| e^{\tau\eta\phi} \nabla q \|^2 + \cancel{\| e^{\tau\eta\phi} R_1' U \|^2} \right) dt$$

Back to Stokes — Away from the boundary, metric g

$$\partial_t U - \Delta_g U + \nabla_g q = F \qquad \mathrm{div}_g U = h$$

Apply the divergence $\quad [\,\mathrm{div}_g, \Delta_g\,] = \underbrace{R_1 \text{ first-order}}_{\text{curvature}}$

- $$\Delta_g q = \mathrm{div}_g F - R_1 U - (\partial_t - \Delta_g) h$$

time-uniform elliptic equation + time integration

1
$$\gamma \left(\int_0^T \left(\| \tilde{\tau}^{1/2} e^{\tau\eta\phi} q \|^2 + \| \tilde{\tau}^{-1/2} e^{\tau\eta\phi} \nabla q \|^2 \right) dt \right.$$
$$\lesssim \int_0^T \left(\| e^{\tau\eta\phi} F \|^2 + \| e^{\tau\eta\phi} U \|^2 + \| \tilde{\tau}^{-1} e^{\tau\eta\phi} (\partial_t - \Delta) h \|^2 \right) dt$$

- $$\partial_t U_i - \Delta U_i = F_i - \partial_i q + R_1' U \qquad R_1' \text{ first-order}$$

$$\gamma \int_0^T \left(\| \tilde{\tau} e^{\tau\eta\phi} U \|^2 + \| e^{\tau\eta\phi} \nabla U \|^2 \right) dt$$
$$\lesssim \int_0^T \left(\| \tilde{\tau}^{-1/2} e^{\tau\eta\phi} F \|^2 + \| \tilde{\tau}^{-1/2} e^{\tau\eta\phi} \nabla q \|^2 \right) dt$$

$\tilde{\tau}^{-1/2}$ shift

① $\gamma \left(\int_0^T \left(\| \tilde{\tau}^{\frac12} e^{\tau\eta\phi} q \|^2 + \| \tilde{\tau}^{-\frac12} e^{\tau\eta\phi} \nabla q \|^2 \right) dt \right)$
$$\lesssim \int_0^T \left(\| e^{\tau\eta\phi} F \|^2 + \| e^{\tau\eta\phi} U \|^2 + \| \tilde{\tau}^{-1} e^{\tau\eta\phi} (\partial_t - \Delta) h \|^2 \right) dt$$

② $\gamma \int_0^T \left(\| \tilde{\tau}\, e^{\tau\eta\phi} U \|^2 + \| e^{\tau\eta\phi} \nabla U \|^2 \right) dt$
$$\lesssim \int_0^T \left(\| \tilde{\tau}^{-\frac12} e^{\tau\eta\phi} F \|^2 + \| \tilde{\tau}^{-\frac12} e^{\tau\eta\phi} \nabla q \|^2 \right) dt$$

① + ε ② and τ large give

$$\gamma \int_0^T \left(\| \tilde{\tau}^{\frac12} e^{\tau\eta\phi} q \|^2 + \| \tilde{\tau}^{-\frac12} e^{\tau\eta\phi} \nabla q \|^2 + \| \tilde{\tau} e^{\tau\eta\phi} U \|^2 + \| e^{\tau\eta\phi} \nabla U \|^2 \right) dt$$
$$\lesssim \int_0^T \left(\| e^{\tau\eta\phi} F \|^2 + \| \tilde{\tau}^{-1} e^{\tau\eta\phi} (\partial_t - \Delta) h \|^2 \right) dt$$

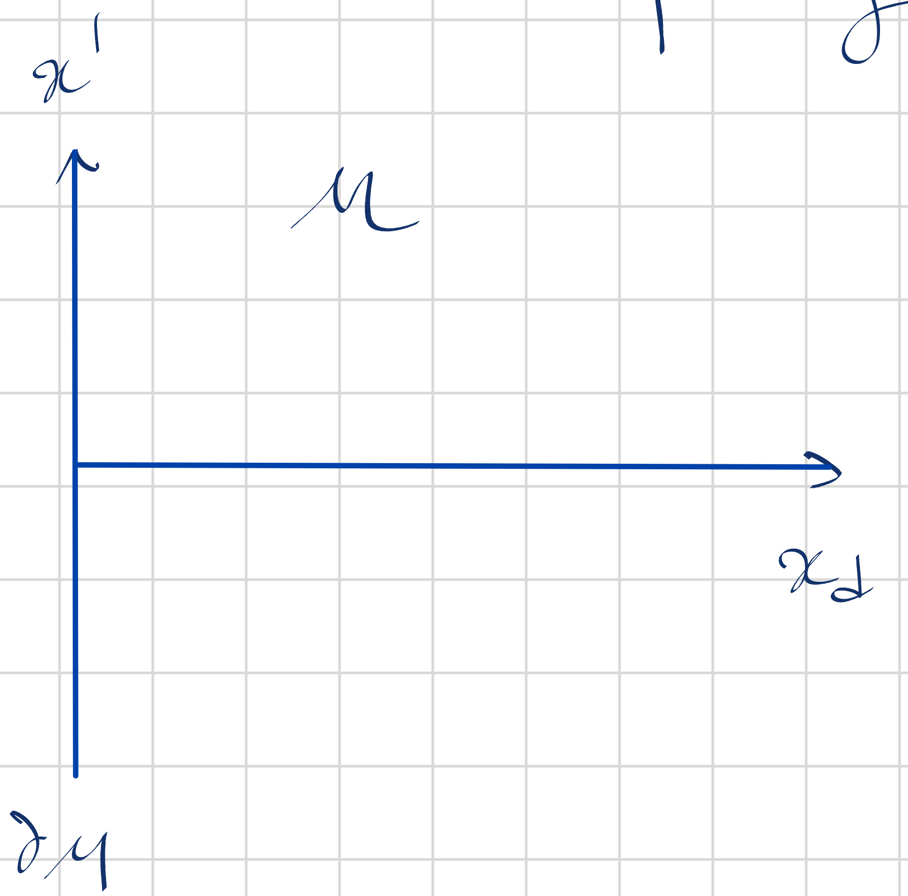
pressure q : $\frac12$ derivative loss ; velocity U full derivative loss

Near a boundary

Recall

$$P = -\Delta_g \qquad P_\varphi = e^{\tau\eta\phi} P e^{-\tau\eta\phi} = Q_2 + i Q_1$$

sub-ellipticity $\quad q_2 = q_1 = 0 \implies \{q_2, q_1\}$.



$$P = D_d^2 + R(x, D') \qquad D = \frac{1}{i}\partial$$

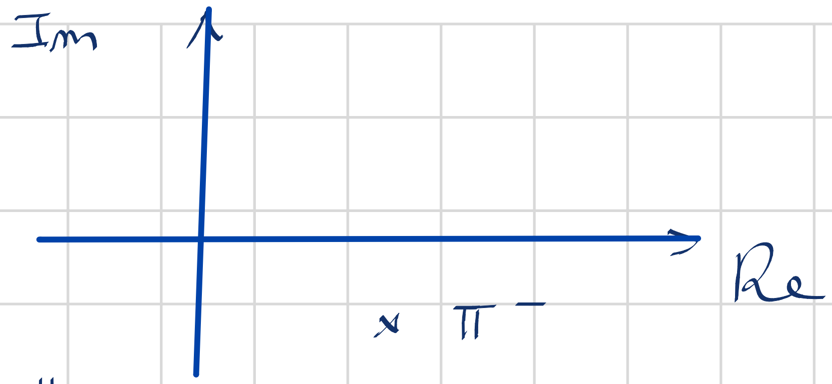
$$P_\varphi = \left(D_d + i\tilde{\tau}\partial_d\varphi\right)^2 + R_\varphi(x, \tau, D')$$

$$p_\varphi = \left(\xi_d + i\tilde{\tau}\partial_d\varphi\right)^2 + r_\varphi(x, \tau, \xi')$$
$$= (\xi_d - \pi^+)(\xi_d - \pi^-)$$

$$r_\varphi(x, \tau, \xi') = r(x, \xi' + i\tilde{\tau} d\varphi') = \beta^2 \quad \text{with } \mathrm{Re}\,\beta \geq 0$$

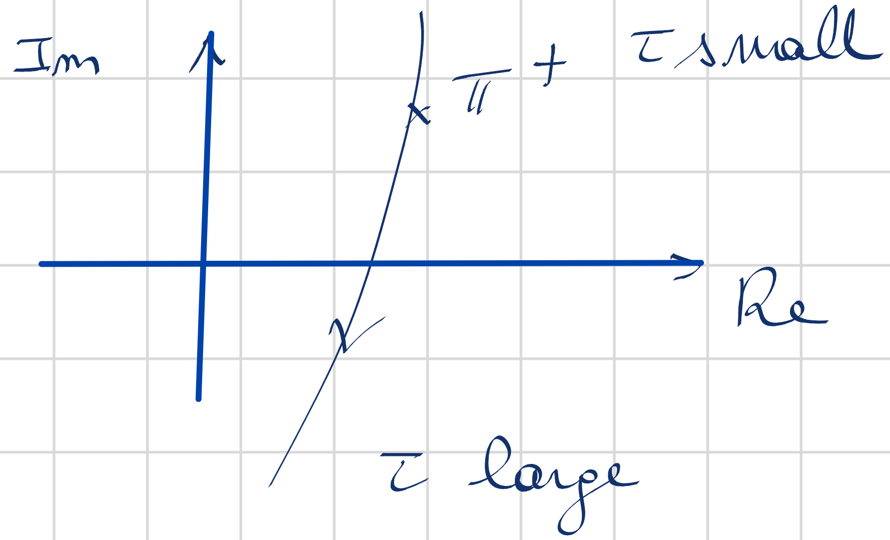
$$\pi^+ = i\beta - i\tilde{\tau}\partial_d\varphi \qquad \pi^- = -i\beta - i\tilde{\tau}\partial_d\varphi$$

- $D_d - Op(\pi^-)$ $\longrightarrow$ elliptic estimate



$$\|\tilde{\tau} v\|_{L^2} + \|\nabla v\|_{L^2} + |Op(\lambda_T^{\frac{1}{2}}) v_{|\partial}|_{L^2} \lesssim \|(D_d - Op(\pi^-)) v\|_{L^2}$$

$\lambda_T^2 = \tilde{\tau}^2 + |\xi'|^2$.



- $D_d - Op(\pi^+)$ $\longrightarrow$ sub-elliptic estimate

$$\gamma^{\frac{1}{2}}\left(\|\tilde{\tau}^{\frac{1}{2}} v\|_{L^2} + \|\tilde{\tau}^{-\frac{1}{2}} \nabla v\|_{L^2}\right) \lesssim \|(D_d - Op(\pi^+)) v\|_{L^2} + |Op(\lambda_T^{\frac{1}{2}}) v_{|\partial}|_{L^2}$$

Parabolic operator

Similar analysis $P = \partial_t - \Delta_q$. Change of time variable

$$s = \tan\left(\frac{\pi t}{T} - \frac{\pi}{2}\right) \qquad \partial_t = \frac{\pi \langle s\rangle^2}{T}\partial_s$$

$$\eta = \frac{T^2}{t(T-t)} = \pi^2\left(\frac{\pi}{2} + \arctan s\right)^{-1}\left(\frac{\pi}{2} - \arctan s\right)^{-1} \simeq \langle s\rangle$$

$$p = \frac{\pi\langle s\rangle^2}{T} i\sigma - r(x,\xi)$$

Conjugation $\quad P_\varphi = e^{\tau\eta\phi} P e^{-\tau\eta\phi} = (D_d + i\tilde{\tau}\partial_d\psi)^2 + M_\varphi$

$$p_\varphi = (\xi_d + i\tilde{\tau}\partial\psi)^2 + m_\varphi \;; \qquad m_\varphi = \frac{\pi\langle s\rangle^2}{T}(i\sigma - \tau\eta'\phi) + \tilde{r}_\varphi(x,\tau,\xi)$$

$$= (\xi_d - \pi^+)(\xi_d - \pi^-) \qquad\qquad = \alpha^2, \text{ with } \operatorname{Re}\alpha \geq 0$$

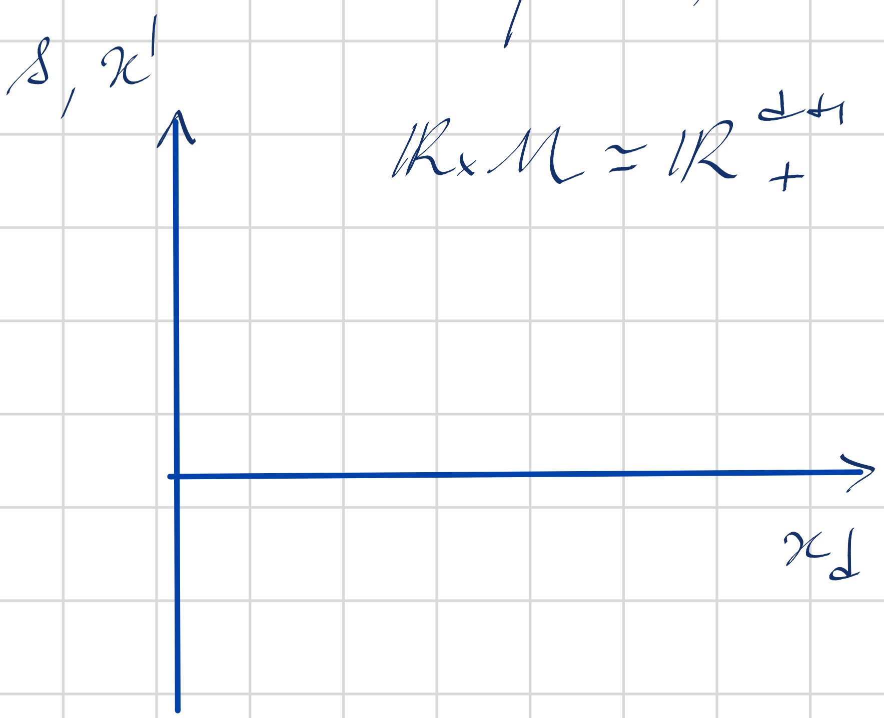
$$\pi^+ = i\alpha - i\tilde{\tau}\partial_d\psi \qquad \pi^- = -i\alpha - i\tilde{\tau}\partial_d\psi$$

Back to Stokes (part 2)

$$(\partial_t - \Delta) U + \nabla q = F \longrightarrow \left(\frac{\pi\langle s\rangle^2}{T}\partial_s - \Delta\right) U + \nabla q = F$$

Conjugaison $e^{\tau \eta \phi}$ as in the parabolic case

$$U^\phi = Op(\chi) e^{\tau\eta\phi} U \qquad q^\phi = i\, Op(\chi) e^{\tau\eta\phi} q$$



$$\Theta = Op_T(\vartheta) \qquad \text{order } 1$$

$$\tilde V = \Theta U^\phi \in \mathbb{C}^d$$

$$\hat V' = D_d^\phi U^{\phi\prime} \qquad D_d^\phi = D_d + i \tilde\tau \partial_d \psi$$

$$\hat V_d = q^\phi \qquad \hat V \in \mathbb{C}^d$$

$$V = \begin{pmatrix} \tilde V \\ \hat V \end{pmatrix} \in \mathbb{C}^{2d}$$

$$\boxed{D_d^\phi V = A V + G}$$

$$D_d^{\phi} V = A V + G \qquad \text{with } A = \begin{pmatrix} 0 & 0 & \Theta & 0 \\ i\,\mathrm{div}^{\phi'} & 0 & 0 & 0 \\ -M_\phi \Theta^{-1} & 0 & 0 & i\nabla^{\phi'} \\ 0 & -M_\phi \Theta^{-1} & -i\,\mathrm{div}^{\phi'} & 0 \end{pmatrix}$$

$$M_\phi = e^{\tau\eta\phi}\left(\frac{\pi}{T}\langle s\rangle^2 \partial_s\right)e^{-\tau\eta\phi} + R_\phi \quad \text{with} \quad R_\phi = e^{\tau\eta\phi} R(x, D') e^{-\tau\eta\phi}$$

$$a = \begin{pmatrix} 0 & 0 & \Theta & 0 \\ -{}^t\zeta' & 0 & 0 & 0 \\ -\Theta^{-1} m_\phi & 0 & 0 & -R\zeta' \\ 0 & -\Theta^{-1} m_\phi & {}^t\zeta' & 0 \end{pmatrix} \quad \text{where } {}^t\zeta' R \zeta' = R(x, \zeta')$$

$$\text{and } \zeta' = \xi' + i\tilde{\tau}\, d\varphi'$$

One seeks the eigenvalues of ${}^t a$ and its eigenvectors

<u>4 regions</u>

1. $m_\phi \neq 0$, $r_\phi \neq 0$, and $m_\phi \neq r_\phi$
2. $m_\phi \neq 0$, $r_\phi \sim 0$
3. $m_\phi \sim 0$
4. $m_\phi \neq 0$, $r_\phi \neq 0$, and $m_\phi \sim r_\phi$

Region 1

$m_\phi \neq 0$, $r_\phi \neq 0$, $m_\phi \neq r_\phi$

$\alpha^2 = m_\phi$ $\quad \mathrm{Re}\,\alpha \geq 0$

$\beta^2 = r_\phi$ $\quad \mathrm{Re}\,\beta \geq 0$

- $$W_v^\pm = \begin{pmatrix} \pm\mu \quad \theta^{-1}\left(({}^t\zeta' R v)\zeta' - m_\phi v\right) \\ -\theta^{-1}\, {}^t\zeta' R v \\ v \\ \mp \mu^{-1}\, {}^t\zeta' R v \end{pmatrix}$$

$v \in \mathbb{C}^{d-1}$, $\mu = i\alpha$

$${}^t a\, W_v^\pm = \pm \mu\, W_v^\pm$$

- $$W_\nu^\pm = \begin{pmatrix} 0 \\ -\theta^{-2} m_\phi \\ \theta^{-1}\zeta' \\ \pm\theta^{-1}\nu \end{pmatrix}$$

$\nu = i\beta$,

$${}^t a\, W_\nu^\pm = \pm \nu\, W_\nu^\pm$$

$$\mathcal{Z} = Op({}^tW_\vartheta^{\pm}) V \qquad D_d^\phi V = AV + G$$

$$Op({}^tW_\vartheta^{\pm}) D_d^\phi V = Op({}^tW_\vartheta^{\pm}) AV + Op({}^tW_\vartheta^{\pm}) G$$

commutators calculus

$$D_d^\phi \mathcal{Z} = \pm Op(\mu) \mathcal{Z} + RHS + \text{remainders}$$

$$\boxed{(D_d - Op(\pi^{\pm}))\mathcal{Z} = RHS + \text{remainders}}$$

with $\pi^{\pm} = \pm \mu - i\tilde{\tau}\partial_d\varphi = \pm i\alpha - i\tilde{\tau}\partial_d\varphi$

- $\underline{\operatorname{Im}\pi^- < 0}$ Elliptic estimate for $\mathcal{Z} = Op(W_\vartheta^-) V$

$$\|\mathcal{Z}\|_{\lambda,1} + |\mathcal{Z}|_0|_{\lambda,\frac{1}{2}} \lesssim \|(D_d - Op(\pi^-))\mathcal{Z}\|$$

- $\underline{\operatorname{Im}\pi^+ \text{ change sign}}$, subelliptic estimate for $\mathcal{Z} = Op(W_\vartheta^+) V$

$$\gamma^{\frac{1}{2}} \|\tilde{\tau}^{-\frac{1}{2}}\mathcal{Z}\|_{\lambda,1} \lesssim \|(D_d - Op(\pi^+))\mathcal{Z}\| + |\mathcal{Z}|_0|_{\lambda,\frac{1}{2}}$$

Similarly

- Elliptic estimate for $Z = Op(W_\nu^-)V$

$$\| Z \|_{\lambda, 1} + | Z|_\partial |_{\lambda, \frac{1}{2}} \lesssim \| (D_d - Op(\pi^-)) Z \|$$

$$\pi^- = -i\nu - i\tilde{\tau}\partial_d\varphi$$

- Subelliptic estimate for $Z = Op(W_\nu^+)V$

$$\gamma^{\frac{1}{2}} \| \tilde{\tau}^{-1/2} Z \|_{\lambda, 1} \lesssim \| (D_d - Op(\pi^+)) Z \| + | Z|_\partial |_{\lambda, 1}$$

$$\pi^+ = +i\nu - i\tilde{\tau}\partial_d\varphi$$

Lopatinskii ellipticity condition

$$\mathrm{Span}\,\{B_\alpha\} \cup \{W_\vartheta^-,\ \vartheta \in \mathbb{C}^{d-1}\} \cup \{W_\nu^-\} = \mathbb{C}^{2d}$$

Dirichlet ✓ Navier ✓ Neumann ✓

Estimate $|V''|_\partial|_{\lambda,\frac{1}{2}} + |\hat{V}_d|_\partial|_{\tilde{\lambda},\frac{1}{2}} \lesssim \sum_\vartheta |Op({}^tW_\vartheta^-)V|_\partial|_{\lambda,\frac{1}{2}} + |Op({}^tW_\nu^-)V|_\partial|_{\lambda,\frac{1}{2}}$

$$+ \sum_\alpha |B_\alpha V|_\partial|_{\lambda,\frac{1}{2}}$$

Result $\gamma^{\frac{1}{2}}\left(\|\tilde{\tau}^{-1/2} V''\|_{\lambda,1} + \|\tilde{\tau}^{-1/2}\hat{V}_d\|_{\tilde{\lambda},1}\right) + |V''|_\partial|_{\lambda,\frac{1}{2}} + |\hat{V}_d|_\partial|_{\tilde{\lambda},\frac{1}{2}}$

$$\lesssim \sum_\alpha |B_\alpha V|_\partial|_{\lambda,\frac{1}{2}} + \|RHS\|$$

Region 2 similar to region 1

Region 3 peculiar, $d-2$ Jordan blocks of size 2

1 Jordan block of size 4

All roots have negative imaginary parts

$\longrightarrow$ Elliptic estimation, no boundary condition needed.

Region 4 $m_\phi \neq 0$, $r_\phi \neq 0$, $m_\phi \sim r_\phi$ $\mu \sim \nu$

2 Jordan blocks of size 2

$W_\nu^\pm$ as in region 1 $\longrightarrow$ $2d-2$ eigenvectors

$W_\nu^\pm$ coincides with $W_{\tilde\nu}^\pm$ for $\tilde\nu = \Theta^{-1}\xi$ if $\mu = \nu$

Set $W_\mu^\pm = \begin{pmatrix} 2\theta^{-1}\xi' \\ \mp\theta^{-1}\mu \\ \pm\mu^{-1}\xi' \\ 0 \end{pmatrix}$ $\quad t_a W_\mu^\pm = \pm\mu W_\mu^\pm + \theta W_\nu^\pm + \mu^{-1}\nu(\nu-\mu)\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Estimate: with loss of $\frac{1}{2}$ a derivative for $W_{2\nu}^+$, W_ν^+

- zero loss for $W_{2\nu}^-$, W_ν^-
- loss of a full derivative for $W_\mu^\pm$

$\longrightarrow$ loss of a full derivative for V'', that is, the velocity U.

important observation $W_\nu^+ - W_\nu^- = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2i\,\theta^{-1}\beta \end{pmatrix}$

$\longrightarrow$ loss of $\frac{1}{2}$ a derivative for $\hat{V}_4$, that is, the pressure.

Final estimate at the boundary

Theorem

$$\gamma\int_0^T\Big(\|\tilde\tau^{\frac12}e^{\tau\eta\phi}q\|^2+\|\tilde\tau^{-\frac12}e^{\tau\eta\phi}\nabla q\|^2+\|\tilde\tau e^{\tau\eta\phi}U\|^2+\|e^{\tau\eta\phi}\nabla U\|^2\Big)dt$$
$$+\int_0^T\Big(|\tilde\tau^{\frac32}e^{\tau\eta\phi}U|_{\partial\mathcal M}|^2+|\tilde\tau^{\frac12}e^{\tau\eta\phi}\nabla U|_{\partial\mathcal M}|^2+|\tilde\tau^{\frac12}e^{\tau\eta\phi}q|_{\partial\mathcal M}|^2\Big)dt$$
$$\lesssim\int_0^T\Big(\|e^{\tau\eta\phi}F\|^2+\|\tilde\tau^{-1}e^{\tau\eta\phi}(\partial_t-\Delta)h\|^2\Big)dt+\sum_j\int_0^T|B_j u|_{\partial\mathcal M}|^2dt\,.$$

Thank you for you attention

Calculus in phase-space

$$\boxed{\tilde{\lambda}^4 = \tilde{\tau}^4 + |\xi|^4 \qquad \lambda^4 = \frac{\langle s\rangle^4}{T^2}\sigma^2 + \tilde{\lambda}^4}$$

metric in phase-space

$$g = \frac{ds^2}{\langle s\rangle^2} + \gamma^2 |dx|^2 + \frac{\langle s\rangle^4 \, d\sigma^2}{T^2 \lambda^4} + \frac{|d\xi|^2}{\lambda^2}$$

$$\partial_x \rho \in \gamma S(\lambda, g) \quad \text{if} \quad \rho \in S(\lambda, g)$$

$$\psi \longrightarrow \psi(\varepsilon x', x_d)$$

$$\boxed{g = \frac{ds}{\langle s\rangle^2} + (1+\varepsilon\gamma)^2 |dx'|^2 + \gamma^2 dx_d^2 + \frac{\langle s\rangle^4 \, d\sigma^2}{T^2 \lambda^4} + \frac{|d\xi|^2}{\lambda^2}}$$

$$\partial_{x'} \rho \in (1+\varepsilon\gamma) S(\lambda, g)$$