A Global Carleman inequality for the Oseen system

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Notation

$$\Omega \text{ smooth bounded domain of } \mathbb{R}^d, \ d = 2, 3,$$

$$\omega \subset \Omega \text{ nonempty open set}, \quad T > 0.$$

Motivation: local exact controllability of the Navier-Stokes system

$$\begin{cases} \partial_t \widetilde{v} - \Delta \widetilde{v} + (\widetilde{v} \cdot \nabla)\widetilde{v} + \nabla \widetilde{p} = u 1_\omega & \text{in } (0,T) \times \Omega, \\ \operatorname{div} \widetilde{v} = 0 & \text{in } (0,T) \times \Omega, \\ \widetilde{v} = 0 & \text{on } (0,T) \times \partial\Omega, \\ \widetilde{v}(0,\cdot) = \widetilde{v}^0 & \text{in } \Omega. \end{cases}$$

Controllability to the trajectories

Motivation

Let us consider a trajectory

$$\begin{cases} \partial_t \overline{v} - \Delta \overline{v} + (\overline{v} \cdot \nabla)\overline{v} + \nabla \overline{p} = 0 & \text{in } (0,T) \times \Omega, \\ \operatorname{div} \overline{v} = 0 & \text{in } (0,T) \times \Omega, \\ \overline{v} = 0 & \text{on } (0,T) \times \partial\Omega, \\ \overline{v}(0,\cdot) = \overline{v}^0 & \text{in } \Omega. \end{cases}$$

We want to find $u \in L^2((0,T) \times \omega)$ such that

$$\widetilde{v}(T,\cdot) = \overline{v}(T,\cdot).$$

Null-controllability

We set

$$v := \widetilde{v} - \overline{v}, \quad p := \widetilde{p} - \overline{p}.$$

The new system is

$$\begin{cases} \partial_t v - \Delta v + (\overline{v} \cdot \nabla)v + (v \cdot \nabla)\overline{v} + (v \cdot \nabla)v + \nabla p = u1_\omega & \text{in } (0,T) \times \Omega, \\ \operatorname{div} v = 0 & \text{in } (0,T) \times \Omega, \\ v = 0 & \text{on } (0,T) \times \partial\Omega, \\ v(0,\cdot) = \widetilde{v}^0 - \overline{v}^0 & \text{in } \Omega. \end{cases}$$

Null-controllability

For $\left\|\widetilde{v}^0 - \overline{v}^0\right\|$ "small", find $u \in L^2((0,T) \times \omega)$ such that

$$v(T,\cdot) = 0.$$

Linearization

Linear system

Null-controllability of the Oseen system

$$\begin{cases} \partial_t v - \Delta v + (\overline{v} \cdot \nabla) v + (v \cdot \nabla)\overline{v} + \nabla p = F + u 1_\omega & \text{in } (0,T) \times \Omega, \\ \operatorname{div} v = 0 & \text{in } (0,T) \times \Omega, \\ v = 0 & \text{on } (0,T) \times \partial\Omega, \\ v(0,\cdot) = v^0 & \text{in } \Omega. \end{cases}$$

Then a fixed-point argument with the map

$$F \mapsto -(v \cdot \nabla) v$$

shows the local controllability of the nonlinear system.

The adjoint system

Definition

The adjoint system of the previous system is

$$\begin{cases} -\partial_t W - \Delta W - (\mathbb{D}W)\,\overline{v} + \nabla \pi_W = G & \text{in } (0,T)\times\Omega, \\ \operatorname{div} W = 0 & \text{in } (0,T)\times\Omega, \\ W = 0 & \text{on } (0,T)\times\partial\Omega, \\ W(T,\cdot) = W^0 & \text{in } \Omega. \end{cases}$$

$\mathbb{D}W$: symmetric gradient.

Change of variables $t \mapsto T - t$:

Our system

$$\begin{cases} \partial_t \widetilde{w} - \Delta \widetilde{w} + \nabla \widetilde{\pi} = \widetilde{f} + (\mathbb{D}\widetilde{w})\, a & \text{in } (0,T)\times\Omega, \\ \operatorname{div} \widetilde{w} = 0 & \text{in } (0,T)\times\Omega, \\ \widetilde{w} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \widetilde{w}(0,\cdot) = \widetilde{w}^0 & \text{in } \Omega. \end{cases}$$

Final state observability

To prove the null-controllability of the Oseen system, we can consider the adjoint system with $\widetilde{f} = 0$:

$$\begin{cases} \partial_t \widetilde{w} - \Delta \widetilde{w} + \nabla \widetilde{\pi} = (\mathbb{D}\widetilde{w})\, a & \text{in } (0,T) \times \Omega, \\ \operatorname{div} \widetilde{w} = 0 & \text{in } (0,T) \times \Omega, \\ \widetilde{w} = 0 & \text{on } (0,T) \times \partial\Omega, \\ \widetilde{w}(0,\cdot) = \widetilde{w}^0 & \text{in } \Omega. \end{cases}$$

Hypotheses on a

$$a \in L^\infty(0,T; L^\infty(\Omega))^d.$$

$$\partial_t a \in L^2(0,T; L^\sigma(\Omega))^d \quad \text{with} \quad \begin{cases} \sigma > 1 & \text{if } d = 2, \\ \sigma > 6/5 & \text{if } d = 3. \end{cases}$$

Final state observability

$$\begin{cases} \partial_t \widetilde{w} - \Delta \widetilde{w} + \nabla \widetilde{\pi} = (\mathbb{D}\widetilde{w})\, a & \text{in } (0,T)\times\Omega, \\ \operatorname{div} \widetilde{w} = 0 & \text{in } (0,T)\times\Omega, \\ \widetilde{w} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \widetilde{w}(0,\cdot) = \widetilde{w}^0 & \text{in } \Omega. \end{cases}$$

Theorem

There exists $C > 0$ such that for any $T > 0$,

$$\int_\Omega |\widetilde{w}(T,y)|^2 \, dy \leqslant C e^{C/T} \iint_{(0,T)\times\omega} |\widetilde{w}|^2 \, dy \, dt.$$

Some references

- E. Fernández-Cara, S. Guerrero, O. Y. Imanuvilov, and J.-P. Puel. "Local exact controllability of the Navier-Stokes system". *J. Math. Pures Appl. (9)* (2004).

Remark

Cost of the control of the form Ce^{C/T^4}.
Method : Carleman estimates but with different weights.

- F. W. Chaves-Silva and G. Lebeau. "Spectral inequality and optimal cost of controllability for the Stokes system". *ESAIM Control Optim. Calc. Var.* (2016).

Remark

Cost of the control of the form $Ce^{C/T}$ *on the Stokes system* ($a = 0$).
Method : spectral inequality.

Our main result: the weight functions

$$\overline{\omega_0} \subset \omega.$$

$$\widetilde{\psi} \in C^\infty(\mathbb{R}^d), \quad \widetilde{\psi} = 0 \text{ on } \partial\Omega, \quad \widetilde{\psi} \in (0,1) \text{ in } \Omega,$$

$$\nabla\widetilde{\psi}(y) = 0 \implies y \in \omega_0.$$

$$\ell(t) := t(T-t) \quad (t \in [0,T]).$$

First parameter: $\lambda \geqslant \lambda_0$.

Definition

We set

$$\widetilde{\zeta}(t,y) := \frac{e^{\lambda\widetilde{\psi}(y)}}{\ell(t)}, \quad \widetilde{\varphi}(t,y) := \frac{e^{\lambda\widetilde{\psi}(y)} - e^{2\lambda}}{\ell(t)} \quad (t \in (0,T),\ y \in \Omega).$$

Our main result: the weight functions

Remark

- *These are the "classical" weight functions for the Carleman estimates of the heat equation.*
- *In FGIP2004, they use*

$$\ell(t) := t^4(T-t)^4 \quad (t \in [0,T]).$$

Second parameter: $\quad s \geqslant s_0(T+T^2).$

We estimate

$$I_1(\widetilde{w}) := \iint_{(0,T)\times\Omega} e^{2s\widetilde{\varphi}} \left[\left(s\widetilde{\zeta}\right)^{-1} \left(|\partial_t \widetilde{w}|^2 + |\Delta \widetilde{w}|^2\right) \right.$$

$$\left. + s\widetilde{\zeta}\, |\nabla \widetilde{w}|^2 + \left(s\widetilde{\zeta}\right)^3 |\widetilde{w}|^2 \right] \, dy \, dt.$$

Our main result: the weight functions

We estimate

$$I_1(\widetilde{w}) := \iint_{(0,T)\times\Omega} e^{2s\widetilde{\varphi}} \left[\left(s\widetilde{\zeta}\right)^{-1} \left(|\partial_t \widetilde{w}|^2 + |\Delta \widetilde{w}|^2\right) \right.$$
$$\left. + s\widetilde{\zeta}\, |\nabla \widetilde{w}|^2 + \left(s\widetilde{\zeta}\right)^3 |\widetilde{w}|^2 \right] \, dy \, dt.$$

Remark

We have

$$\left(s\widetilde{\zeta}\right)^3 e^{2s\widetilde{\varphi}} = \left(\frac{s e^{\lambda \widetilde{\psi}(y)}}{t(T-t)}\right)^3 \exp\left(2s \frac{e^{\lambda\widetilde{\psi}(y)} - e^{2\lambda}}{t(T-t)}\right),$$

with

$$\lambda \geqslant \lambda_0, \quad s \geqslant s_0(T+T^2).$$

Our main result: the weight functions

We estimate

$$I_1(\widetilde{w}) := \iint_{(0,T)\times\Omega} e^{2s\widetilde{\varphi}} \left[\left(s\widetilde{\zeta}\right)^{-1} \left(|\partial_t \widetilde{w}|^2 + |\Delta \widetilde{w}|^2\right) \right. \\ \left. + s\widetilde{\zeta}\, |\nabla \widetilde{w}|^2 + \left(s\widetilde{\zeta}\right)^3 |\widetilde{w}|^2 \right] \, dy \, dt$$

and

$$I_2(\widetilde{\pi}) := \iint_{(0,T)\times\Omega} e^{2s\widetilde{\varphi}} \left[|\nabla \widetilde{\pi}|^2 + \left(s\widetilde{\zeta}\right)^2 |\widetilde{\pi}|^2 \right] \, dy \, dt.$$

Our main result: statement

Theorem

Let $a \in L^\infty((0,T)\times\Omega)^d$.

$$\exists \lambda_0 \gtrsim 1 + \|a\|_{L^\infty((0,T)\times\Omega)^d}, \forall \lambda \geqslant \lambda_0,$$

$$\exists C, s_0 > 0, \forall T > 0, \forall s \geqslant s_0(T+T^2)$$

$$I_1(\widetilde{w}) + I_2(\widetilde{\pi}) \leqslant C \left(\iint_{(0,T)\times\Omega} s\widetilde{\zeta} e^{2s\widetilde{\varphi}} \left|\widetilde{f}\right|^2 \, dy\, dt \right.$$
$$\left. + \iint_{(0,T)\times\omega_1} e^{2s\widetilde{\varphi}} \left[\left(s\widetilde{\zeta}\right)^3 |\widetilde{w}|^2 + \left(s\widetilde{\zeta}\right)^2 |\widetilde{\pi}|^2 \right] \, dy \, dt \right).$$

From our result to the observability

- To get rid of the pressure term in the observation, we need to impose that

$$\partial_t a \in L^2(0,T;L^\sigma(\Omega))^d \quad \text{with} \quad \begin{cases} \sigma > 1 & \text{if } d=2, \\ \sigma > 6/5 & \text{if } d=3. \end{cases}$$

$\longrightarrow$ This follows completely from FGIP2004.

- To obtain the observability of the Oseen system, we cut the integrals in $I_1(\widetilde{w})$ and use energy estimates for the Oseen system.

First step: global Carleman estimates

The Oseen system as a heat equation

We write

$$\begin{cases} \partial_t \widetilde{w} - \Delta \widetilde{w} = -\nabla \widetilde{\pi} + \widetilde{f} + (\mathbb{D}\widetilde{w})\, a & \text{in } (0,T)\times\Omega, \\ \widetilde{w} = 0 & \text{on } (0,T)\times\partial\Omega. \end{cases}$$

Carleman estimate for the heat equation

$$I_1(\widetilde{w}) \lesssim \iint_{(0,T)\times\Omega} e^{2s\widetilde{\varphi}} \left|\nabla\widetilde{\pi}\right|^2 \, dy\, dt + \iint_{(0,T)\times\Omega} e^{2s\widetilde{\varphi}} \left|\widetilde{f}\right|^2 \, dy\, dt$$
$$+ \iint_{(0,T)\times\Omega} e^{2s\widetilde{\varphi}} \left|(\mathbb{D}\widetilde{w})\, a\right|^2 \, dy\, dt$$
$$+ \iint_{(0,T)\times\omega_0} s^3\lambda^4\widetilde{\zeta}^3 e^{2s\widetilde{\varphi}} \left|\widetilde{w}\right|^2 \, dy\, dt.$$

First step: global Carleman estimates

Equation for the pressure

Using that $\operatorname{div} \widetilde{w} = 0$, we have that $\widetilde{\pi}$ satisfies

$$\Delta \widetilde{\pi} = \operatorname{div} \left(\widetilde{f} + (\mathbb{D} \widetilde{w})\, a \right) \quad \text{in } (0, T) \times \Omega.$$

Difficulty: there is no boundary condition!

We apply a Carleman estimate obtained in
O. Y. Imanuvilov and J.-P. Puel. "Global Carleman estimates for weak solutions of elliptic nonhomogeneous Dirichlet problems". *Int. Math. Res. Not.* (2003).

First step: global Carleman estimates

$$\Delta\widetilde{\pi} = \operatorname{div}\left(\widetilde{f} + (\mathbb{D}\widetilde{w})\, a\right) \quad \text{in } (0,T)\times\Omega.$$

We set

$$\varphi_0 := \widetilde{\varphi}_{|\partial\Omega}, \quad \zeta_0 := \widetilde{\zeta}_{|\partial\Omega}.$$

Carleman estimate for the Laplace equation:

$$\begin{aligned} I_2(\widetilde{\pi}) \lesssim \iint_{(0,T)\times\Omega} s\widetilde{\zeta} e^{2s\widetilde{\varphi}} \left|\widetilde{f}\right|^2 \, dy\, dt + \iint_{(0,T)\times\Omega} s\widetilde{\zeta} e^{2s\widetilde{\varphi}} \left|(\mathbb{D}\widetilde{w})\, a\right|^2 \, dy\, dt \\ + \iint_{(0,T)\times\omega_1} s^2\lambda^2\widetilde{\zeta}^2 e^{2s\widetilde{\varphi}} \left|\widetilde{\pi}\right|^2 \, dy\, dt \\ + \int_0^T (s\zeta_0)^{1/2} e^{2s\varphi_0} \left|\widetilde{\pi}\right|^2_{H^{1/2}(\partial\Omega)} \, dt. \end{aligned}$$

First step: global Carleman estimates

Gathering the two Carleman inequalities (heat equation and Laplace equation), we find

Proposition

$$I_1(\widetilde{w}) + I_2(\widetilde{\pi}) \lesssim \iint_{(0,T)\times\Omega} s\widetilde{\zeta} e^{2s\widetilde{\varphi}} \left|\widetilde{f}\right|^2 \, dy\, dt + \iint_{(0,T)\times\omega_1} e^{2s\widetilde{\varphi}} \left[s^3\lambda^4\widetilde{\zeta}^3 \left|\widetilde{w}\right|^2 + s^2\lambda^2\widetilde{\zeta}^2 \left|\widetilde{\pi}\right|^2 \right] \, dy\; dt + \int_0^T (s\zeta_0)^{1/2} e^{2s\varphi_0} \left|\widetilde{\pi}\right|^2_{H^{1/2}(\partial\Omega)} \, dt.$$

$\longrightarrow$ we need to get rid of the boundary term with the pressure.

Truncation and change of variables

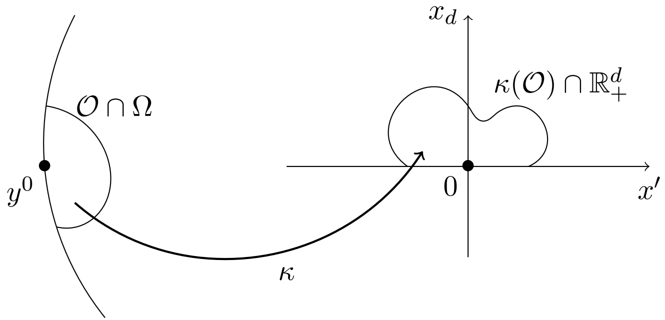


Figure: Local chart $(\mathcal{O}, \kappa)$ in a neighborhood of $y^0 \in \partial\Omega$.

We set

$$\mathbb{R}^d_+ := \left\{ (x', x_d) \in \mathbb{R}^d \ : \ x_d > 0 \right\}.$$

Truncation and change of variables

Truncation in space

$$\theta \in C_0^\infty(\mathbb{R}^d), \quad \operatorname{supp}\theta \subset \mathcal{O}.$$

We set

$$\breve{w} := \theta\widetilde{w}, \quad \breve{\pi} := \theta\widetilde{\pi}.$$

$$\Delta\widetilde{\pi} = \operatorname{div}\left(\widetilde{f} + (\mathbb{D}\widetilde{w})\, a\right) \quad \text{in } (0,T)\times\Omega.$$

becomes

$$-\Delta\breve{\pi} = \breve{H}_0 + \operatorname{div}\breve{H},$$

where

$$\breve{H}_0 := [\theta,\Delta]\widetilde{\pi} + \nabla\theta\cdot\left(\widetilde{f} + (\mathbb{D}\widetilde{w})\, a\right), \quad \breve{H} := -\theta\left(\widetilde{f} + (\mathbb{D}\widetilde{w})\, a\right).$$

Truncation and change of variables

Change of variables

$$\pi := \breve{\pi} \circ \kappa^{-1}$$

$$-\Delta\breve{\pi} = \breve{H}_0 + \operatorname{div} \breve{H} \quad \longrightarrow \quad \mathcal{M}\pi = H_0 + \operatorname{div} H \quad \text{in } (0,T) \times \mathbb{R}^d_+.$$

Lemma

One can choose $(\mathcal{O}, \kappa)$ *such that*

$$\mathcal{M}\pi := -\frac{\partial^2 \pi}{\partial x_d^2} - \sum_{j,k=1}^{d-1} \mathfrak{g}^{j,k} \frac{\partial^2 \pi}{\partial x_j \partial x_k} - \mathfrak{h} \cdot \nabla \pi,$$

with

$$\sum_{j,k=1}^{d-1} \mathfrak{g}^{j,k} \xi_j \xi_k \geqslant C \left|\xi'\right|^2 \quad \left(\xi' = (\xi_1, \ldots, \xi_{d-1}) \in \mathbb{R}^{d-1}\right).$$

Elliptic estimate

Objective

Estimate

$$\int_0^T (s\zeta_0)^{1/2} e^{2s\varphi_0} |\widetilde{\pi}|^2_{H^{1/2}(\partial\Omega)} \, dt.$$

We thus need to consider

$$e^{s\widetilde{\varphi}}\widetilde{\pi} = e^{\frac{s}{\ell}\left[e^{\lambda\widetilde{\psi}}-e^{2\lambda}\right]}\widetilde{\pi} = e^{\tau\widetilde{\phi}}\widetilde{\pi}e^{-\tau e^{2\lambda}},$$

with

$$\tau := \frac{s}{\ell}, \quad \widetilde{\phi} := e^{\lambda\widetilde{\psi}}.$$

Change of variables

Weights on Ω

$$\widetilde{\psi} \in C^\infty(\mathbb{R}^d), \quad \widetilde{\psi} = 0 \text{ on } \partial\Omega,$$

$$\nabla\widetilde{\psi}(y) = 0 \implies y \in \omega_0.$$

$$s, \quad \widetilde{\varphi} := \frac{1}{\ell}\left[e^{\lambda\widetilde{\psi}} - e^{2\lambda}\right].$$

$$e^{s\widetilde{\varphi}}\widetilde{\pi}$$

Weights on $\mathbb{R}^d_+$

$$\psi := \widetilde{\psi} \circ \kappa^{-1},$$

$$\inf_{\kappa(\mathcal{O})} \partial_{x_d}\psi > 0.$$

$$\tau := \frac{s}{\ell}, \quad \phi := e^{\lambda\psi}.$$

$$q := e^{\tau\phi}\pi$$

Conjugaison

Before conjugaison

$$\mathcal{M}\pi = H_0 + \operatorname{div} H \quad \text{in } (0,T) \times \mathbb{R}^d_+,$$

with

$$\mathcal{M}\pi := -\frac{\partial^2 \pi}{\partial x_d^2} - \sum_{j,k=1}^{d-1} \mathfrak{g}^{j,k} \frac{\partial^2 \pi}{\partial x_j \partial x_k} - \mathfrak{h} \cdot \nabla \pi,$$

After conjugaison

$$q := e^{\tau\phi}\pi$$

$$\mathcal{M}_\phi q = G_0 + \operatorname{div}(G) \quad \text{in } (0,T) \times \mathbb{R}^d_+.$$

Conjugaison

$$\mathcal{M}_\phi q = G_0 + \mathrm{div}(G) \quad \text{in } (0,T)\times\mathbb{R}^d_+.$$

where

$$\begin{aligned}\mathcal{M}_\phi q = & -\frac{\partial^2 q}{\partial x_d^2} - \sum_{j,k=1}^{d-1} \mathfrak{g}^{j,k}\frac{\partial^2 q}{\partial x_j \partial x_k} \\ & -\tau^2\left[\left(\frac{\partial\phi}{\partial x_d}\right)^2 + \sum_{j,k=1}^{d-1}\mathfrak{g}^{j,k}\frac{\partial\phi}{\partial x_j}\frac{\partial\phi}{\partial x_k}\right] q \\ & +2\tau\left[\frac{\partial\phi}{\partial x_d}\frac{\partial q}{\partial x_d} + \sum_{j,k=1}^{d-1}\mathfrak{g}^{j,k}\frac{\partial\phi}{\partial x_j}\frac{\partial q}{\partial x_k}\right] + \dots\end{aligned}$$

Ideas of the proof

Simplifications

Assume

$$\mathfrak{g}^{j,k} = \delta_{j,k}, \quad \phi = \phi(x_d), \quad \frac{\partial \phi}{\partial x_d} = 1 > 0,$$

so that

$$\mathcal{M}_\phi q = -\frac{\partial^2 q}{\partial x_d^2} - \sum_{j=1}^{d-1} \frac{\partial^2 q}{\partial x_j^2} - \tau^2 q + 2\tau \frac{\partial q}{\partial x_d} + \ldots$$

Important property of ψ

Recall that

$$\phi := e^{\lambda \psi}, \quad \inf_{\kappa(\mathcal{O})} \partial_{x_d} \psi > 0.$$

Thus

$$\inf_{\kappa(\mathcal{O})} \frac{\partial \phi}{\partial x_d} > 0.$$

Ideas of the proof

Fourier transform

$$q(x', x_d) \longrightarrow \widehat{q}(\xi', x_d).$$

$$\begin{aligned}\widehat{\mathcal{M}_\phi q} &= -\frac{\partial^2 \widehat{q}}{\partial x_d^2} + \left|\xi'\right|^2 \widehat{q} - \tau^2 \widehat{q} + 2\tau \frac{\partial \widehat{q}}{\partial x_d} \\ &= -\left(\frac{\partial}{\partial x_d} - \left(\tau + \left|\xi'\right|\right)\right)\left(\frac{\partial}{\partial x_d} - \left(\tau - \left|\xi'\right|\right)\right)\widehat{q}.\end{aligned}$$

Ideas of the proof

We need to obtain estimates for the equations

$$\frac{\partial}{\partial x_d}\widehat{u} - \left(\tau + \left|\xi'\right|\right)\widehat{u} = \widehat{f}$$

and

$$\frac{\partial}{\partial x_d}\widehat{v} - \left(\tau - \left|\xi'\right|\right)\widehat{v} = \widehat{g}$$

with $\widehat{u}(\xi', x_d) = 0$ and $\widehat{v}(\xi', x_d) = 0$ for $x_d \geqslant R(\mathcal{O})$.

Ideas of the proof

First equation

$$\frac{\partial}{\partial x_d}\widehat{u} - \left(\tau + \left|\xi'\right|\right)\widehat{u} = \widehat{f}$$

We multiply by $(\tau + |\xi'|)\,\widehat{u}$:

$$-\int_{\mathbb{R}^{d-1}} \left(\tau + \left|\xi'\right|\right) \int_0^\infty \frac{1}{2}\frac{\partial}{\partial x_d}\left|\widehat{u}\right|^2 \; dx_d\, d\xi' + \frac{1}{2}\int_{\mathbb{R}^d_+} \left(\tau + \left|\xi'\right|\right)^2 \left|\widehat{u}\right|^2 \; d\xi' \; dx_d \leqslant \frac{1}{2}\int_{\mathbb{R}^d_+} \left|\widehat{f}\right|^2 \; d\xi' \; dx_d$$

Ideas of the proof

First equation

$$\frac{\partial}{\partial x_d}\widehat{u} - \left(\tau + \left|\xi'\right|\right)\widehat{u} = \widehat{f}$$

$$\int_{\mathbb{R}^{d-1}} \left(\tau + \left|\xi'\right|\right) \left|\widehat{u}\left(\xi', 0\right)\right|^2 \, d\xi' + \int_{\mathbb{R}^{d-1}\times\mathbb{R}} \tau^2 \left|\widehat{u}\right|^2 + \left|i\xi'\widehat{u}\right|^2 + \left|\frac{\partial}{\partial x_d}\widehat{u}\right|^2 \, d\xi' \, dx_d \lesssim \int_{\mathbb{R}^{d-1}\times\mathbb{R}} \left|\widehat{f}\right|^2 \, d\xi' \, dx_d$$

Remark

We don't need any boundary condition !

Ideas of the proof

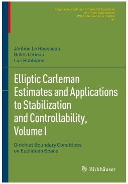
In order to the same for

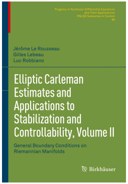
$$\frac{\partial}{\partial x_d}\widehat{v} - \underbrace{\left(\tau - \left|\xi'\right|\right)}_{???}\widehat{v} = \widehat{g}$$

we need to assume

$$\left|\xi'\right| < C\tau \longrightarrow \text{LOW FREQUENCY}$$

Pseudo-differential operators





To justify the above ideas in the case without simplifications, we need to use pseudo-differential operators

Decomposition

$$q \approx \mathrm{Op}_{\top}\left(\chi^{-}\right) q + \mathrm{Op}_{\top}\left(\chi^{+}\right) q.$$

Splitting Low and High frequencies

Decomposition

$$q \approx \mathrm{Op}_{\top}\left(\chi^{-}\right) q + \mathrm{Op}_{\top}\left(\chi^{+}\right) q.$$

Lemma

There exist $\chi^{-}, \chi^{+} \in \mathbf{S}^{0}_{\top,\tau}$ *such that*

$$(x, \xi', \tau) \in \mathrm{supp}(\chi^{+}) \implies |\xi'| \geqslant C\tau \quad \textit{High Frequency}$$

$$(x, \xi', \tau) \in \mathrm{supp}(\chi^{-}) \implies |\xi'| \leqslant C\tau \quad \textit{Low Frequency}$$

Semiclassical norms

Definition

Semiclassical norms: in $\mathbb{R}^d_+$

$$\|u\|_{n,\tau} := \left(\sum_{|\beta|+j\leqslant n} \tau^{2j} \left\| \partial_x^\beta u \right\|^2_{L^2(\mathbb{R}^d_+)} \right)^{1/2}.$$

At the boundary,

$$|u|_{e,\tau} := \left| \mathrm{Op}_\top \left(|\xi'|^2 + \tau^2 \right)^{e/2} u \right|_{L^2(\mathbb{R}^{d-1})}.$$

Estimates of the low frequency part

We assume

$$\mathcal{M}_\phi q = G_0 + \mathrm{div}(G) \quad \text{in } (0,T)\times\mathbb{R}^d_+.$$

Theorem

There exist $\tau^* > 0$ and $C > 0$ such that for any $\tau \geqslant \tau^*$,

$$\tau^2 \left\|\mathrm{Op}_\top\left(\chi^-\right) q\right\|_{1,\tau}^2 + \tau^2 \left|\mathrm{Op}_\top\left(\chi^-\right) q\right|_{1/2,\tau}^2 \leqslant C\left(\|G_0\|_{0,\tau}^2 + \tau^2\|G\|_{0,\tau}^2 + \|q\|_{1,\tau}^2\right).$$

History

$$\begin{tikzcd}
(\widetilde{w},\widetilde{\pi}) \text{ solution of the Oseen system} \arrow[d, "\text{Truncation in space}"] \\
(\breve{w},\breve{\pi}) = \theta(\widetilde{w},\widetilde{\pi}) \arrow[d, "\text{Local chart } (\mathcal{O},\kappa)"'] \\
\pi := \breve{\pi} \circ \kappa^{-1} \arrow[d, "\text{Conjugaison}"'] \\
q := e^{\tau\phi}\pi
\end{tikzcd}$$

Low frequency arguments

Estimates of the low frequency part

$$\Delta\widetilde{\pi} = \operatorname{div}\left(\widetilde{f} + (\mathbb{D}\widetilde{w})\, a\right) \quad \text{in } (0,T)\times\Omega.$$

Corollary

There exists $s_0 > 0$ such that for any $s \geqslant s_0 T^2$,

$$\begin{aligned}
&\left|(s\zeta_0)^{1/4} e^{s\varphi_0} \mathrm{Op}_{\top}\left(\chi^-\right)\left([\theta\widetilde{\pi}]\circ\kappa^{-1}\right)\right|^2_{L^2\left(0,T;H^{1/2}\left(\mathbb{R}^{d-1}\right)\right)} \\
&\quad \lesssim \left\|\left(s\widetilde{\zeta}\right)^{1/4} e^{s\widetilde{\varphi}}\widetilde{f}\right\|^2_{L^2(0,T;L^2(\Omega))^d} + \left\|\left(s\widetilde{\zeta}\right)^{1/4} e^{s\widetilde{\varphi}}\nabla\widetilde{w}\right\|^2_{L^2(0,T;L^2(\Omega))^{d\times d}} \\
&\qquad + \left\|\left(s\widetilde{\zeta}\right)^{1/4} e^{s\widetilde{\varphi}}\widetilde{\pi}\right\|^2_{L^2(0,T;L^2(\Omega))} + \left\|\left(s\widetilde{\zeta}\right)^{-3/4} e^{s\widetilde{\varphi}}\nabla\widetilde{\pi}\right\|^2_{L^2(0,T;L^2(\Omega))^d}.
\end{aligned}$$

High frequency estimates

Objective

Estimate

$$\int_0^T (s\zeta_0)^{1/2}\, e^{2s\varphi_0}\, |\widetilde{\pi}|^2_{H^{1/2}(\partial\Omega)}\, dt.$$

Truncation in space

$$\theta \in C_0^\infty(\mathbb{R}^d), \quad \operatorname{supp}\theta \subset \mathcal{O}.$$

We set

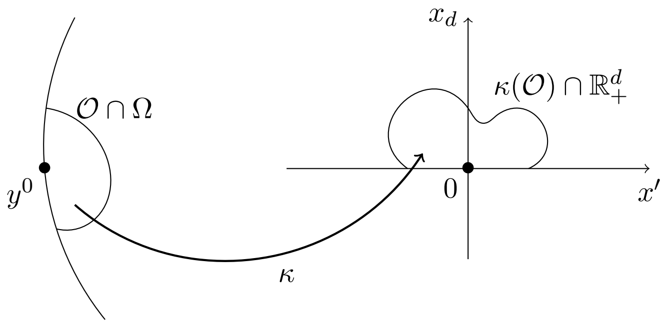
$$\breve{w} := \theta \widetilde{w}, \quad \breve{\pi} := \theta \widetilde{\pi}.$$

Weak conjugaison

We set

$$w^\natural := (s\zeta_0)^{1/4}\, e^{s\varphi_0} \breve{w}, \quad \pi^\natural := (s\zeta_0)^{1/4}\, e^{s\varphi_0} \breve{\pi}.$$

High frequency estimates



Definition

For $f : \Omega \cap \mathcal{O} \to \mathbb{R}$, we set

$$\mathbb{O}^+ f := \left[\mathrm{Op}_\top\left(\chi^+\right)\left(f \circ \kappa^{-1}\right)\right] \circ \kappa.$$

High frequency estimates

With this operator, we define

$$w^\sharp := \mathbb{O}^+ w^\natural, \quad \pi^\sharp := \mathbb{O}^+ \pi^\natural$$

so that

$$\begin{cases} \partial_t w^\sharp - \Delta w^\sharp + \nabla \pi^\sharp = f^\sharp & \text{in } (0,T) \times \Omega, \\ \operatorname{div} w^\sharp = g^\sharp & \text{in } (0,T) \times \Omega, \\ w^\sharp = 0 & \text{on } (0,T) \times \partial\Omega, \\ w^\sharp(0,\cdot) = 0 & \text{in } \Omega. \end{cases}$$

High frequency estimates

Lemma

We have the estimate

$$\left\|w^\sharp\right\|^2_{L^2(0,T;H^2(\Omega))^d} + \left\|\partial_t w^\sharp\right\|^2_{L^2(0,T;L^2(\Omega))^d} + \left\|\pi^\sharp\right\|^2_{L^2(0,T;H^1(\Omega))}$$
$$\lesssim \left\|f^\sharp\right\|^2_{L^2(0,T;L^2(\Omega))^d} + \left\|g^\sharp\right\|^2_{L^2(0,T;H^1(\Omega))}$$
$$+ \left\|\tau^{-1}\partial_t g^\sharp\right\|_{L^2(0,T;L^2(\Omega))} \left\|\tau\pi^\sharp\right\|_{L^2(0,T;L^2(\Omega))}.$$

Question: semiclassical norms?

High frequency estimates

Property of χ^+

$$(x,\xi',\tau) \in \operatorname{supp}(\chi^+) \implies |\xi'| \geqslant C\tau \quad \text{High Frequency}$$

$$\left\| \mathrm{Op}_{\mathsf{T}}\left(\chi^+\right) f \right\|^2_{H^1\left(\mathbb{R}^d_+\right)} \geqslant \left\| \mathrm{Op}_{\mathsf{T}}\left(\chi^+\right) f \right\|^2_{L^2\left(\mathbb{R}_+; H^1\left(\mathbb{R}^{d-1}\right)\right)}.$$

Consequently,

$$\begin{aligned} &\left\| \mathrm{Op}_{\mathsf{T}}\left(\chi^+\right) f \right\|^2_{H^1\left(\mathbb{R}^d_+\right)} \\ &\qquad \gtrsim \left(\mathrm{Op}_{\mathsf{T}}\left(1 + \left|\xi'\right|^2\right) \mathrm{Op}_{\mathsf{T}}\left(\chi^+\right) f, \mathrm{Op}_{\mathsf{T}}\left(\chi^+\right) f \right)_{L^2\left(\mathbb{R}^d_+\right)}. \end{aligned}$$

High frequency estimates

Property of χ^+

$$(x, \xi', \tau) \in \operatorname{supp}(\chi^+) \implies 1 + |\xi'|^2 \geqslant C(\tau^2 + |\xi'|^2) \quad \text{High Frequency}$$

We have obtained

$$\left\| \mathrm{Op}_\top \left(\chi^+\right) f \right\|^2_{H^1\left(\mathbb{R}^d_+\right)} \gtrsim \left(\mathrm{Op}_\top \left(1 + \left|\xi'\right|^2\right) \mathrm{Op}_\top \left(\chi^+\right) f, \mathrm{Op}_\top \left(\chi^+\right) f \right)_{L^2\left(\mathbb{R}^d_+\right)}.$$

Gårding inequality:

$$\left\| \mathrm{Op}_\top \left(\chi^+\right) f \right\|^2_{1,\tau} \lesssim \left\| \mathrm{Op}_\top \left(\chi^+\right) f \right\|^2_{H^1\left(\mathbb{R}^d_+\right)} + \|f\|^2_{L^2\left(\mathbb{R}^d_+\right)}.$$

High frequency estimates

We thus deduce

Lemma

There exist $C > 0$ and $s_0 > 0$ such that for any $T > 0$, for any $s \geqslant s_0(T + T^2)$

$$\left\| \mathrm{Op}_{\top}\left(\chi^{+}\right)\left(w^{\natural} \circ \kappa^{-1}\right) \right\|^2_{L^2(0,T;H^2_\tau)} + \left\| \mathrm{Op}_{\top}\left(\chi^{+}\right)\left(\pi^{\natural} \circ \kappa^{-1}\right) \right\|^2_{L^2(0,T;H^1_\tau)}$$
$$\leqslant C\Big(\left\| f^{\sharp} \right\|^2_{L^2(0,T;L^2(\Omega))^d} + \left\| g^{\sharp} \right\|^2_{L^2(0,T;H^1(\Omega))} + \left\| \tau^{-1}\partial_t g^{\sharp} \right\|^2_{L^2(0,T;L^2(\Omega))}$$
$$+ \left\| w^{\natural} \right\|^2_{L^2(0,T;L^2(\Omega))^d} + \left\| \pi^{\natural} \right\|^2_{L^2(0,T;L^2(\Omega))} \Big).$$

High frequency estimates

Semiclassical trace lemma

$$\left|(s\zeta_0)^{1/4}\, e^{s\varphi_0} \mathrm{Op}_\top\left(\chi^+\right)\left(\theta\widetilde{\pi}\circ\kappa^{-1}\right)\right|^2_{L^2(0,T;H^{1/2}(\mathbb{R}^{d-1}))} \lesssim \left\|\mathrm{Op}_\top\left(\chi^+\right)\left(\pi^\natural\circ\kappa^{-1}\right)\right\|^2_{L^2(0,T;H^1_\tau)}.$$

$$\begin{tikzcd}
 & \boxed{(\widetilde{w},\widetilde{\pi}) \text{ solution of the Oseen system}} \arrow[d, "\text{Truncation in space}"] & \\
 & \boxed{(\breve{w},\breve{\pi}) = \theta(\widetilde{w},\widetilde{\pi})} \arrow[dl, "\text{Local chart } (\mathcal{O},\kappa)"'] \arrow[dr, "\text{Weak conjugaison}"] & \\
\boxed{\pi := \breve{\pi}\circ\kappa^{-1}} \arrow[d, "\text{Conjugaison}"'] & & \boxed{(w^\natural,\pi^\natural) = (s\zeta_0)^{1/4}\, e^{s\varphi_0}\,(\breve{w},\breve{\pi})} \arrow[d, "\text{Localization in high-frequency}"] \\
\boxed{q := e^{\tau\phi}\pi} & & \boxed{(w^\sharp,\pi^\sharp) = \mathbb{O}^+\left(w^\natural,\pi^\natural\right)}
\end{tikzcd}$$

Low frequency arguments

High frequency arguments

End of the proof

- We estimate all the "commutator" terms.
- We add the estimates in low and high frequency modes.
- We consider a finite family $(\mathcal{O}_j, \kappa_j)$, $j = 1, \ldots, J$ such that

$$\partial\Omega \subset \bigcup_{j=1}^{J} \mathcal{O}_j.$$

- We consider $\{\theta_j\}_{j=1}^{J}$ a partition of unity associated with this covering:

$$\theta_j \in C^\infty\left(\mathbb{R}^d\right), \quad \theta_j \geqslant 0, \quad \sum_{j=1}^{J} \theta_j \equiv 1 \text{ on } \partial\Omega, \quad \operatorname{supp} \theta_j \subset \mathcal{O}_j.$$

Our main result: statement

Theorem

Let $a \in L^\infty((0,T)\times\Omega)^d$.

$$\exists \lambda_0 \gtrsim 1 + \|a\|_{L^\infty((0,T)\times\Omega)^d}, \forall \lambda \geqslant \lambda_0,$$

$$\exists C, s_0 > 0, \forall T > 0, \forall s \geqslant s_0(T+T^2)$$

$$I_1(\widetilde{w}) + I_2(\widetilde{\pi}) \leqslant C\left(\iint_{(0,T)\times\Omega} s\widetilde{\zeta}e^{2s\widetilde{\varphi}}\left|\widetilde{f}\right|^2 \, dy\,dt \right.$$
$$\left. + \iint_{(0,T)\times\omega_1} e^{2s\widetilde{\varphi}}\left[\left(s\widetilde{\zeta}\right)^3|\widetilde{w}|^2 + \left(s\widetilde{\zeta}\right)^2|\widetilde{\pi}|^2\right] \, dy \; dt\right).$$