Nodal domains of eigenfunctions of sub-Laplacians

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Control of PDEs and related topics
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Nodal sets

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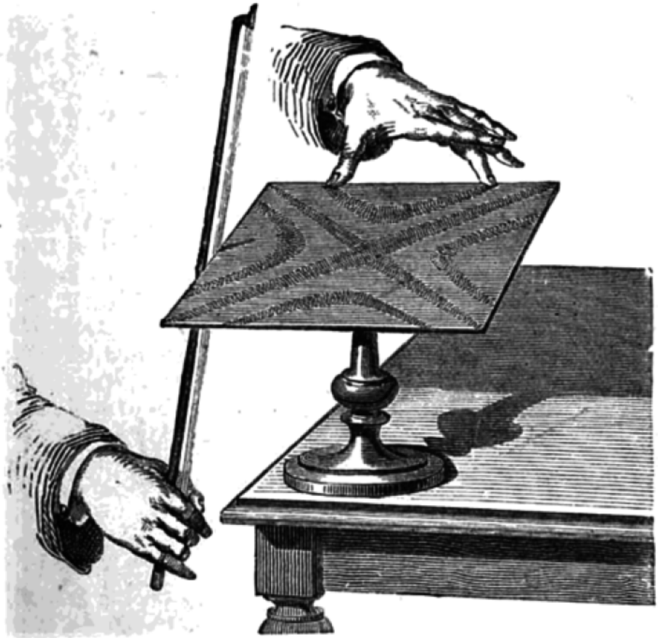
We are interested in

- the **nodal set**

$$Z_{\varphi_\lambda} = \{x \in M \mid \varphi_\lambda(x) = 0\}$$

- the **nodal domains**, i.e., the connected components of $M \setminus Z_{\varphi_\lambda}$. In each nodal domain, the eigenfunction has constant sign.

Nodal portrait of the Gaussian spherical harmonic of degree 40 (figure by A. Barnett).



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- Related to properties of solutions to elliptic PDE (e.g. Carleman inequalities). Remark: $u(x,t) = \varphi_\lambda(x) e^{\sqrt{\lambda} t}$ harmonic in $n+1$ variables.
- Difficulty: nodal sets and nodal domains are very unstable.

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- an eigenfunction with eigenvalue λ cannot vanish at order more than $c\sqrt{\lambda}$ for some c depending only on M.
- for a polynomial of degree N, the Bernstein inequality holds

$$\sup_{x \in (-1,1)} |P_N'(x)| \leqslant N^2 \sup_{x \in (-1,1)} |P_N(x)|$$

  and for eigenfunctions

$$\sup_M |\nabla \varphi_\lambda| \leqslant C\sqrt{\lambda} \sup_M |\varphi_\lambda|.$$

Classical results on nodal sets

Some results:

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- Yau's conjecture: $\exists c, C > 0$ depending only on M such that

$$c\sqrt{\lambda} \leqslant \mathcal{H}^{d-1}(\varphi_\lambda) \leqslant C\sqrt{\lambda}$$

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- Nodal sets of random eigenfunctions, or linear combinations thereof (Nazarov-Sodin, Gayet-Welschinger).

Our goal

Study of nodal sets of eigenfunctions of hypoelliptic Laplacians

- Part of a general program to study eigenfunctions of these operators (Colin de Verdière, Hillairet, Trélat, many others) and PDEs driven by these Laplacians.
- Give new intuitions on nodal sets in the elliptic case. E.g.: Yau's conjecture fails.

Based on a joint work with S. Eswarathasan (2022).

Sub-Laplacians

To define a sub-Laplacian, we need:

- M a **manifold** smooth, compact, connected of dimension d.
- $X_1, \dots, X_m$ smooth **vector fields** on M (not necessarily independent) such that

  $$\forall q \in M, \quad \mathrm{Span}(X_1(q), \dots, X_m(q), [X_i, X_j](q), [X_{i_1}, [X_{i_2}, ...]](q)) = T_q M$$

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- μ a smooth **volume** on M.

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Then the **sub-Laplacian** reads

$$\Delta = -\sum_{i=1}^{m} X_i^* X_i = \sum_{i=1}^{m} X_i^2 + \mathrm{div}_\mu(X_i) X_i,$$

where the star is the transpose in $L^2(M, \mu)$.

Sub-Laplacians are hypoelliptic, i.e., $\Delta u \in C^\infty(V) \Rightarrow u \in C^\infty(V)$.
Natural geometry: sub-Riemannian geometry.

Examples of sub-Laplacians

- **Baouendi-Grushin.** Vector fields $X_1 = \partial_x$ and $X_2 = x\partial_y$ in $(-1,1)_x \times (\mathbb{R}/2\pi\mathbb{Z})_y$ and $\mu = dxdy$:

  $$\Delta = \partial_x^2 + x^2\partial_y^2$$

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- **Heisenberg.** Vector fields $X_1 = \partial_x$ and $X_2 = \partial_y + x\partial_z$ in $\mathbb{R}^3/\Gamma$ and $\mu = dxdydz$:

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If M has a boundary, we put Dirichlet boundary conditions. Sub-Laplacians on compact manifolds have discrete point spectrum

$$0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n \leqslant\rightarrow +\infty$$

(with repetitions according to multiplicities). There exists an orthonormal basis $\{\varphi_n\}_{n\in\mathbb{N}}$ of $L^2(M,\mu)$ such that $-\Delta\varphi_n = \lambda_n\varphi_n$.

Main results

Theorem (Courant)

For any $n \in \mathbb{N}$, any eigenfunction of $-\Delta$ with eigenvalue λ_n has at most:

- $n + \mathrm{mult}(\lambda_n) - 1$ *nodal domains in general;*
- n *nodal domains if $M, \mu, X_1, \ldots, X_m$ are analytic.*

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Theorem (Density of the nodal set)

There exists $C > 0$ depending only on M such that for any eigenpair (φ, λ), the nodal set Z_φ intersects any ***sub-Riemannian ball*** *of radius greater than $C/\sqrt{\lambda}$.*

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- Paper by R. Frank and B. Helffer on the Pleijel bound for sub-Laplacians (2024).

Sub-Riemannian balls.

There is a notion of **metric**:

$$g_q(v) = \inf \left\{ \sum_{j=1}^{m} u_j^2, \quad v = \sum_{j=1}^{m} u_j X_j(q) \right\}.$$

One then naturally defines the **length** of a path $\gamma : [0, T] \to M$ as

$$\ell(\gamma) = \int_0^T g_{\gamma(t)}(\dot{\gamma}(t))^{\frac{1}{2}} \, dt.$$

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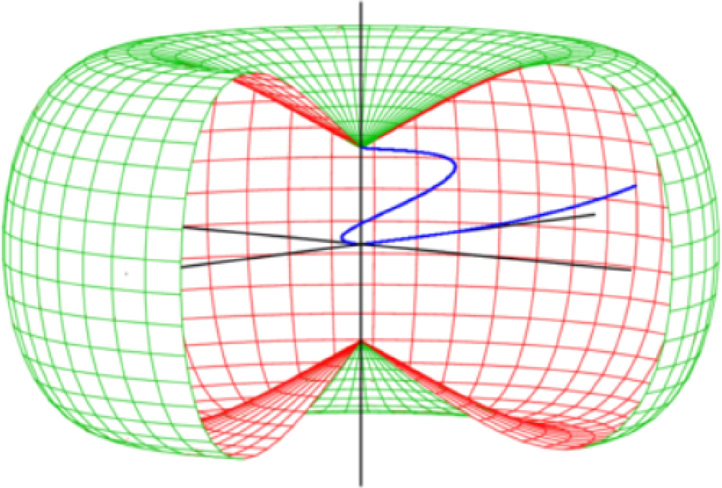
To ensure $\ell(\gamma) < +\infty$, we require $\frac{d}{dt}\gamma \in \mathrm{Span}(X_1(\gamma(t)), \ldots, X_m(\gamma(t)))$. The **sub-Riemannian distance** is then defined as

$$d(q, q') = \inf_{\gamma \in \mathcal{P}_{q,q'}} \ell(\gamma).$$

where $\mathcal{P}_{q,q'}$ is the set of paths from q to q'. The **sR ball** centered at $q \in M$ and with radius $r > 0$ is

$$B_r(q) = \{q' \in M,\ d(q, q') < r\}.$$



A ball in the Heisenberg case. *Source: Mathoverflow.*

Sharpness of Theorem 2 (density of the nodal set)

Theorem 2 is sharp in the following sense. Consider the Baouendi-Grushin sub-Laplacian

$$\Delta = \partial_x^2 + x^2 \partial_y^2$$

on $(-1,1)_x \times (\mathbb{R}/2\pi\mathbb{Z})_y$. If $k \in \mathbb{Z}$ and ψ_k is the normalized ground state (positive over $(-1,1)$) of the 1D operator

$$H_k = -\partial_x^2 + k^2 x^2, \qquad x \in (-1,1),$$

then

$$\Psi_k : (x,y) \mapsto \psi_k(x)\cos(ky)$$

is an eigenfunction of $-\Delta$ associated with eigenvalue $\mu_k \approx |k|$.

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- The nodal set is given by horizontal lines separated by π/k.
- Ball-box theorem: a sR ball of radius ε looks like a box

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- One must choose ε of order $1/\sqrt{k}$, i.e., $\mu_k^{-1/2}$. For c small, the sR ball of radius $c\mu_k^{-\frac{1}{2}}$ does not necessarily intersect the nodal set of Ψ_k.
- Sharpness also in the “elliptic direction" (along the x axis).

Failure of Yau's bound

- A sub-Riemannian manifold M has Hausdorff dimension Q, computed w.r.t. the sR distance. E.g., $Q = 4$ in Heisenberg.
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  fails in Heisenberg (but the failure is much more general).
- Toy example: in the Baouendi-Grushin case, the total “length" of nodal lines if of order k, i.e. μ_k instead of $\sqrt{\mu_k}$.
- However Hausdorff is not well-defined in this case. The actual proof is in the Heisenberg case (which is equiregular, not singular).

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$$c\sqrt{\lambda} \leqslant \mathscr{H}^{Q-1}(Z_{\varphi_\lambda}) \leqslant C\lambda^{s/2}.$$
- "Test" on higher order Baouendi-Grushin operator $\partial_x^2 + x^{2(s-1)}\partial_y^2$.
- The lower bound can be improved to $c\lambda^{s/2}$ for eigenfunctions microlocalized near $\{\sigma(-\Delta) = 0\}$. This is the case of a density one subsequence of eigenfunctions.

Proof of Courant's theorem

- Assume $u \in E_{\lambda_k}$ has at least $(k+1)$ nodal domains $D_1, \ldots, D_{k+1}$. We assume $\lambda_{k-1} < \lambda_k$. Let
$$u_i = u_{|D_i}.$$
- (Not obvious) claim: u_i is the lowest energy eigenfunction in D_i.
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and by the min-max principle f is an eigenfunction for λ_k.
- But f vanishes in the open set D_{k+1}, in contradiction with the unique continuation property of eigenfunctions (due to analyticity).
- Similar argument without unique continuation yields weaker bound.

Proof of the density of nodal sets

Main (standard) idea: if a ball B of radius r does not intersect the zero set of φ_λ, then $B \subset D$ for some nodal domain D of φ_λ, and thus

$$\lambda = \lambda_1(D) \leqslant \lambda_1(B) \leqslant c r^{-2}.$$

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Lemma

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- The constant $c(q)$ can be chosen independent of q if the geometry is locally the same everywhere (e.g. Heisenberg but not Baouendi-Grushin). In this case, the lemma is sufficient to conclude.
- Otherwise, we use a "desingularization procedure" to reduce to this simpler case: we had some variables, and extend eigenfunctions in a way that they have no dependence in these directions. For instance Baouendi-Grushin to Heisenberg.

In the sequel, we focus on the proof of the lemma.

Proof of the fact that $\lambda_1(B_r(q)) \leqslant c(q)r^{-2}$

- The scaling like r^{-2} in the Euclidean case is easy to see. We have

$$\lambda_1(V) = \inf_{u \in C_c^\infty(V)} \frac{\|\nabla u\|^2}{\|u\|^2}$$

and if we rescale functions like $u_r(x) = u(rx)$ we get that

$$\lambda_1(B_r) = \frac{\lambda_1(B_1)}{r^2}. \tag{1}$$

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- Near the origin, we have $X_i = \widehat{X}_i + R_i$ where $\widehat{X}_i$ is **homogeneous** and R_i is a remainder term (can be neglected in the Rayleigh quotient).

Nilpotentization of vector fields

- Given $q \in M$, it is possible to "approximate" the vector fields around q by a (simpler) **nilpotent** structure.
- **Example:** $X_1 = \cos(\theta)\partial_x + \sin(\theta)\partial_y$ and $X_2 = \partial_\theta$ on $\mathbb{R}^2_{x,y} \times \mathbb{T}_\theta$.

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- In which coordinates do we write the vector fields?
- Sub-Riemannian **flag** of $(M, \mathcal{D}, g)$: $\mathcal{D}^0 = \{0\}$, $\mathcal{D}^1 = \mathcal{D}$, and

$$\forall j \geqslant 1, \qquad \mathcal{D}^{j+1} = \mathcal{D}^j + [\mathcal{D}, \mathcal{D}^j].$$

- **Fix** $q \in M$. We have a flag

$$\{0\} = \mathcal{D}^0_q \subset \mathcal{D}^1_q \subset \ldots \subset \mathcal{D}^{r-1}_q \subsetneq \mathcal{D}^{r(q)}_q = T_q M.$$

- **Weights:** $w_i(q) = 1+$ number of brackets needed to generate the i^{th} direction at q (for $1 \leqslant i \leqslant n$). A family $(Z_1, \ldots, Z_n)$ of n vector fields is **adapted** to the flag at q if it is a frame of $T_q M$ at q and if $Z_i(q) \in \mathcal{D}^{w_i(q)}_q$ for $1 \leqslant i \leqslant n$.

Nilpotentization of vector fields

- Given $q \in M$, it is possible to “approximate" the vector fields around q by a (simpler) **nilpotent** structure.
- **Example:** $X_1 = \cos(\theta)\partial_x + \sin(\theta)\partial_y$ and $X_2 = \partial_\theta$ on $\mathbb{R}^2_{x,y} \times \mathbb{T}_\theta$.
- In which coordinates do we write the vector fields?
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- The inverse of the local diffeomorphism

$$(x_1, \ldots, x_n) \mapsto \exp(x_1 Z_1) \circ \cdots \circ \exp(x_n Z_n)(q)$$

defines **exponential coordinates** of the 2$^{\text{nd}}$ kind at q. We now work in these coordinates. They are **privileged coordinates**.

Proof. Nilpotentization, II

Every vector field X_i has a Taylor expansion

$$X_i(x) \sim \sum_{\alpha,j} a_{\alpha,j} x^\alpha \partial_{x_j}.$$

We **group** terms together depending on their weights

$$X_i = X_i^{(-1)} + X_i^{(0)} + X_i^{(1)} + \dots$$

and we set

$$\widehat{X}_i = X_i^{(-1)}.$$

Perspectives, I

- Consider a family of Laplace-Beltrami operators of the form

$$\Delta_{g_\varepsilon} = \Delta_{\mathrm{sR}} + \varepsilon^2 \Delta_h$$

where Δ_{sR} is a fixed analytic sub-Laplacian and Δ_h is a fixed analytic Laplace-Beltrami operator (defined on the same domain of $\mathbb{R}^d$), how do the constants c_ε and C_ε in

$$c_\varepsilon \sqrt{\lambda} \leqslant \mathcal{H}^{d-1}(Z_{\varphi_\lambda^{(\varepsilon)}}) \leqslant C_\varepsilon \sqrt{\lambda}$$

behave as $\varepsilon \to 0$. Follow Donnelly-Fefferman paper: complex extension of eigenfunctions, Cauchy-Crofton, Carleman etc.

- Study Yau-type bound for sub-Laplacians, going beyond our observation.

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- Study Yau-type bound for sub-Laplacians, going beyond our observation.
- Order of vanishing of eigenfunctions (unique continuation - qualitative and quantitative).
- Unique continuation at infinity (Landis conjecture): if $\Delta u + Vu = 0$ with $|V| \leqslant 1$ and $|u(x)|$ tends superexponentially fast to 0 at infinity, then $u \equiv 0$.

Perspectives, II

- Nodal sets of **typical** (or **random**) **eigenfunctions**. For instance on $\mathbb{S}^2 \subset \mathbb{R}^3$, we consider an o.n.b $\{Y_j\}$ of the $(2k+1)$ dimensional space of **spherical harmonics** with eigenvalue $\lambda = (k+1)^2$. We set

$$f = \sum_{j=-k}^{k} \xi_j Y_j$$

where ξ_j are i.i.d. Gaussian random variables with $\mathbb{E}\xi_j^2 = \frac{1}{2k+1}$.
Then [Nazarov-Sodin 2009] proves that $\exists a > 0$ such that $\forall \varepsilon > 0$,

$$\mathbb{P}\left\{ \left| \frac{N(f)}{k^2} - a \right| > \varepsilon \right\} \leqslant C(\varepsilon) e^{-c(\varepsilon) k}.$$

where $N(f)$ is the number of nodal domains.

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- The key quantity to compute is the covariance kernel (of the Gaussian process)

$$K_{[0,\lambda]}(x,y) = \sum_{k\in\mathbb{N}, \lambda_k \leqslant \lambda} \varphi_k(x)\varphi_k(y)$$

when $\lambda \to +\infty$ and for $x, y \in M$, and prove its “universality".

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Thank you for your attention!