Controlled flow of geometric maps

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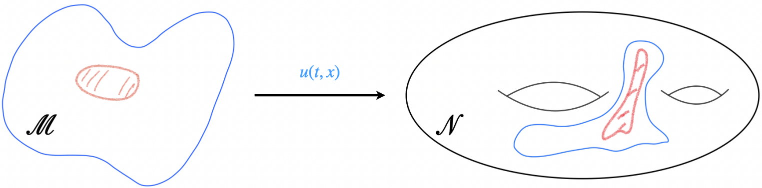
joint with J.-M. Coron (Paris) and J. Krieger (Lausanne)

Control of PDEs and related topics, 2025 Toulouse

北京大學

Controlled maps between manifolds/domains:

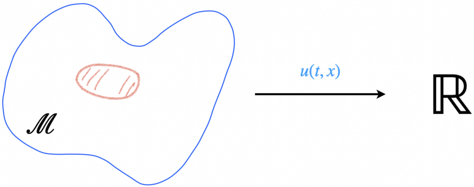


- Motivation: control of singularity formation, cosmology, SPDE and ergodicity, liquid crystal
- Methodology: Analysis, Dynamics, Geometry

Outline of the presentation

Control of the heat equation

$$\partial_t u - \Delta u = \chi_\omega f, \quad (t,x) \in (0,T) \times \mathcal{M}$$



Null controllability: for any u_0, find control f such that $u(0) = u_0, u(T) = 0$?

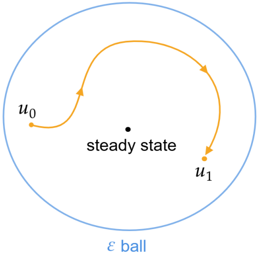
- Fattorini–Russell (1971, 1D case), Lebeau–Robbiano (1995) Fursikov–Imanuvilov (1996) etc.
- Rough control: Burq's talk

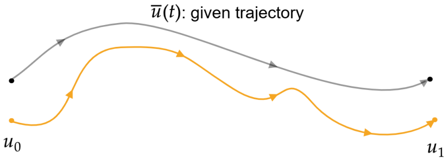
Control of nonlinear equations

$$\partial_t u - \Delta u + g(u) = \chi_\omega f$$

Natural property: local controllability via linearization

- around steady states
- around given trajectories





$$\partial_t u - \Delta u + g(u) = \chi_\omega f$$

Things become completely nonlinear:

global controllability,

1d case: Coron-Trélat (2004)

small-time global controllability

Two open problems

1) (Coron, 2007) Consider nonlinear heat equation

$$u_t - u_{xx} - u^3 = \chi_\omega f, \; x \in \mathbb{T}$$

Does the small-time global controllability between steady states hold?

2) (Dehman–Lebeau–Zuazua, 2003) Consider NLW

$$\partial_t^2 u - \Delta u + u^3 = \chi_\omega f$$

Does the “uniform-time" global controllability hold?

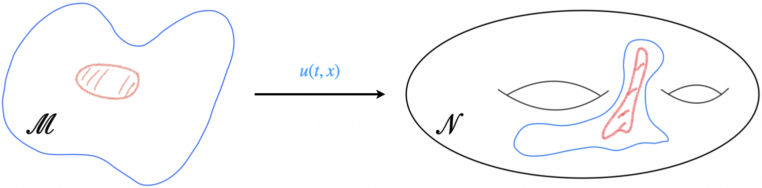
Heat flow

Let u be a map from $\mathbb{R} \times \mathcal{M}$ to $\mathcal{N}$.

$$\partial_t u - \Delta u \perp T_u \mathcal{N}$$

$$\text{or } \partial_t u - \Delta u + S_{jk}(u) \partial^\alpha u^j \partial_\alpha u^k = 0$$

If $\mathcal{N} = \mathbb{S}^n$, then $\partial_t u - \Delta u - u|\nabla u|^2 = 0$ with $u \in \mathbb{R}^{n+1}$.



- Harmonic maps (HM): steady states
- Well-posedness, Singularity formation
- Evolution with extra forces (control)?

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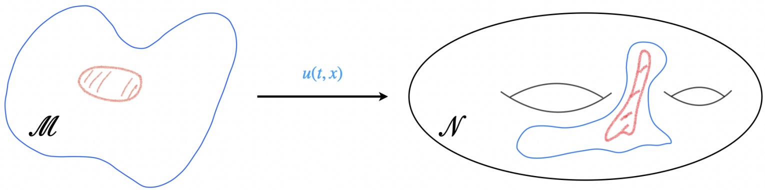
$$\partial_t^2 u - \Delta u \perp T_u \mathcal{N},$$
$$\text{or } \partial_t^2 u - \Delta u + S_{jk}(u) \partial^\alpha u^j \partial_\alpha u^k = 0,$$

where $(u, u_t)(t, x) \in T\mathcal{N}$.



- Harmonic maps (HM): steady states
- Well-posedness, Singularity formation
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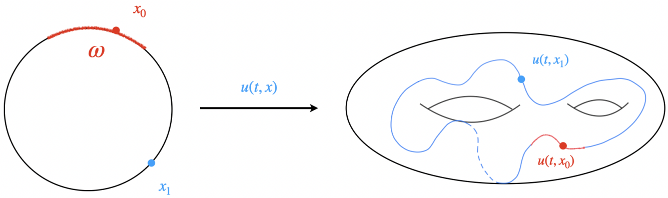
Control problems on geometric equations?

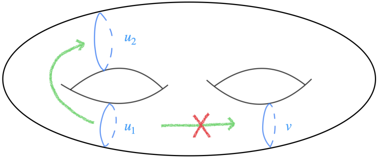
First investigate the case $\mathcal{M} = \mathbb{T}$. Let $\mathcal{N} \subset \mathbb{R}^N$ of dimension d.



$$\partial_t^2 u - \Delta u + S_{jk}(u)\partial^\alpha u^j \partial_\alpha u^k = \chi_\omega f^\perp \quad \text{where} \quad u \in \mathcal{N} \subset \mathbb{R}^N$$

- Coupled system: N components, d controls (tangent space)
- Goal: global controllability and stabilization

Main result 1: global controllability equals homotopy



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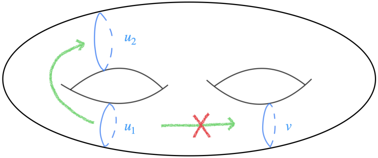


$$\partial_t^2 u - \Delta u + S_{jk}(u)\partial^\alpha u^j \partial_\alpha u^k = \chi_\omega f^\perp$$

Theorem (Coron–Krieger–X., to appear soon thus = 2025)

For wave maps: $\mathbb{T} \to \mathcal{N}$,

Global controllability $\Leftrightarrow$ *Homotopy*

- wave maps $\mathbb{T} \to \mathbb{S}^n$: Krieger–X. (2022), case energy $< 2\pi$
  Coron–Krieger–X. (2023), global controllability
- heat flow $\mathbb{T} \to \mathbb{S}^n$: Coron–X. (2024)

Main result 2: small-time global control between steady states

Open problem (Coron, 2007)

Consider the nonlinear heat equation

$$u_t - u_{xx} - u^3 = \chi_\omega f.$$

The small-time global controllability between steady states?

Theorem (Coron-X. 2024: a positive answer)

Consider heat flow: $\mathbb{T} \to \mathcal{N}$

$$\partial_t u - \Delta u + S_{jk}(u)\partial^\alpha u^j \partial_\alpha u^k = \chi_\omega f$$

For any steady states (including HM) u_0 *and* u_1,

$$\textit{small-time global controllability} \iff u_0 \textit{ is homotopic to } u_1$$

Main result 2: small-time global control between steady states

Open problem (Dehman–Lebeau–Zuazua, 2003)

Consider the controlled NLW

$$\partial_t^2 u - \Delta u + g(u) = \chi_\omega f.$$

The “uniform-time" global controllability?

Theorem (Coron–Krieger–X. 2025)

Let $T = 6\pi$. Consider wave maps: $\mathbb{T} \to \mathcal{N}$

$$\partial_t^2 u - \Delta u + S_{jk}(u)\partial^\alpha u^j \partial_\alpha u^k = \chi_\omega f$$

For any steady states (including HM) u_0 and u_1,

$$\text{“uniform-time" global controllability} \iff u_0 \text{ is homotopic to } u_1$$

Outline of the presentation

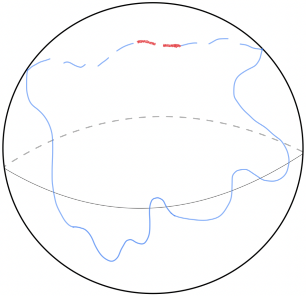
Theorem (Coron–X. 2024)

Consider the heat flow $\mathbb{T} \to \mathbb{S}^n$.
For any initial state $u_0 \in H^1$ *and any point* $p \in \mathbb{S}^n$, *there exists a control* $f \in L^\infty(0, T; L^2)$ *such that* $u(T) = p$.

Since $\mathcal{M} = \mathbb{T}, \mathcal{N} = \mathbb{S}^k$:

$$\partial_t u - \Delta u - u|\partial_x u|^2 = \chi_\omega f^\perp$$

$$E(u) := \int_{\mathbb{T}} |\partial_x u|^2 dx$$



Steady states Φ: HM

$$\Delta \Phi + \Phi|\Phi_x|^2 = 0$$

with energy $2\pi n^2$

Define "ε-approximate HM":

$$\mathcal{Q}_\varepsilon := \bigcup_{\Phi:\,\mathrm{HM}} \left\{ u \in H^1(\mathbb{T}; \mathbb{S}^k) : \|u - \Phi\|_{H^1_x} \leq \varepsilon \right\}$$

Idea: play with energy and HM

Step 1 convergence towards HM
E-S type argument

Step 2 cross HM
nonlinear control

Step 3 local control around p
linearization

Step 1: converge towards HM

Energy dissipates: let control $f = 0$,

$$\frac{1}{2}\frac{d}{dt}E(u(t)) = -\int_{\mathbb{T}^1} |u_t|^2(t,x)dx$$

Eells–Sampson type argument:

- the flow converges to HM
- introduced to study the homotopy problem

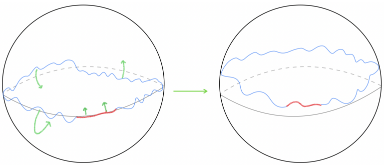
Proposition (Coron–X., 2024)

Let $M, \varepsilon > 0$. There exists T such that, for any initial state with energy smaller than M, the solution becomes a ε-approx. HM at some time $t \in (0, T)$.

Step 2: cross HM

Now the state is close to a HM. Thus $E(u) = 2\pi N^2 + \varepsilon$.

Question: construct control to decrease the energy; $E(u) = 2\pi N^2 - \varepsilon$



$$\partial_t u - \Delta u - |u_x|^2 u = \chi_\omega f^\perp$$

Linearization does not work:

$$\partial_t v - \Delta v = 2(\Phi_x \cdot v_x)\Phi + |\Phi_x|^2 v + f - (f \cdot \Phi)\Phi$$

exist uncontrollable directions

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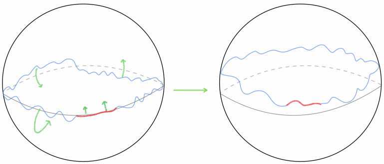

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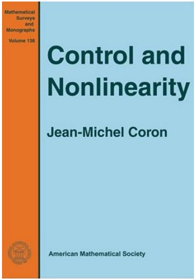
Nonlinearity helps

Power series expansion: Coron-Crépeau (2004)



Power series expansion

Consider

$$\partial_t \bar{u} - \Delta \bar{u} = |\partial_x \bar{u}|^2 \bar{u} + \chi_\omega f^\perp$$

and decompose

$$\bar{u} := \bar{u}_0 + \varepsilon \bar{u}_1 + \varepsilon^2 \bar{u}_2 + \dots$$
$$f := \varepsilon f_1 + \varepsilon^2 f_2 + \dots$$

Cascade nonlinear systems

$$\begin{cases} \partial_t \bar{u}_0 - \Delta \bar{u}_0 = |\partial_x \bar{u}_0|^2 \bar{u}_0, \\ \partial_t \bar{u}_1 - \Delta \bar{u}_1 = 2(\bar{u}_{0x} \cdot \bar{u}_{1x}) \bar{u}_0 + |\bar{u}_{0x}|^2 \bar{u}_1 + f_1 - (f_1 \cdot \bar{u}_0) \bar{u}_0 \\ \partial_t \bar{u}_2 - \Delta \bar{u}_2 = \dots \end{cases}$$

go to second order we obtain:

$$E(T) - E(0) = -2\pi N^2 \varepsilon^2 + O(\varepsilon^3)$$

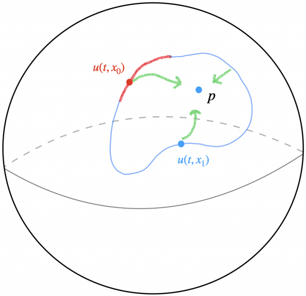
Step 3: local control around a point

A local control problem with geometric constraint

$$\partial_t u - \Delta u = |\partial_x u|^2 u + \chi_\omega f^{\perp}$$

via stereographic projection:

$$\partial_t v - \Delta v + \frac{2v \cdot \partial_x v}{4 + |v^2|} \partial_x v - \frac{2|\partial_x v|^2}{4 + |v|^2} v = \chi_\omega g$$



- a nonlinear control problem without constraint
- nonlinear heat: Liu–Takahashi–Tucsnak, Liu (2018)
- we use quantitative rapid stabilization approach

Quantitative rapid stabilization

Theorem (X., 2020)

For $\lambda > 0$ the heat equation

$$u_t - \Delta u = -\gamma_\lambda 1_\omega P_\lambda u$$

satisfies

$$\|u(t)\|_{L^2} \leq e^{C\sqrt{\lambda}} e^{-\lambda t} \|u(0)\|_{L^2}$$

- Combines Lebeau–Robbiano spectral inequality and constructive feedback control
- Apply to Navier–Stokes (X. 2023), heat flow (Coron–X. 2024)
- $e^{C\sqrt{\lambda}}$ estimate is essential to obtain finite time stabilization (for iteration)

Proposition (Coron–X., 2024)

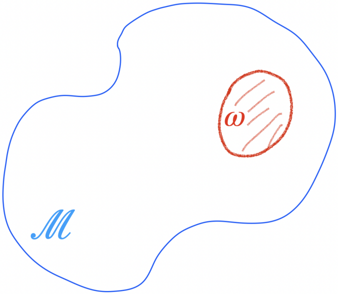
There exists $C > 0$ *s.t. for any* $\lambda > 0$

$$\partial_t v - \Delta v + \frac{2v \cdot v_x}{4 + |v^2|} v_x - \frac{2|v_x|^2}{4 + |v|^2} v = -\lambda e^{C_0 \sqrt{\lambda}} \mathbf{1}_\omega P_\lambda v$$

satisfies

$$\|u(t)\|_{H^1} \leq e^{C\sqrt{\lambda}} e^{-\lambda t} \|u(0)\|_{H^1}$$

provided $\|u(0)\|_{H^1} \leq (e^{C\sqrt{\lambda}})^{-1}$



- Combines Lebeau–Robbiano spectral inequality and constructive feedback control
- Apply to Navier–Stokes (X. 2023), heat flow (Coron–X. 2024)
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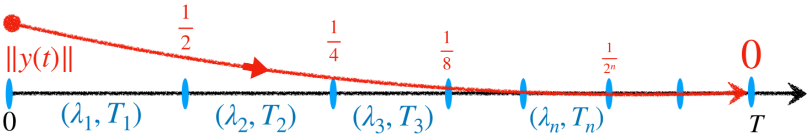
Finite time stabilization

Find a sequence $\{(\lambda_k, T_k)\}$, on each interval we perform quantitative rapid stabilization



Proposition (Coron–X., 2024)

There exists C such that for any T, any initial state $u_0 \in H^1(\mathbb{T})$, and any $p \in \mathbb{S}^k$ with

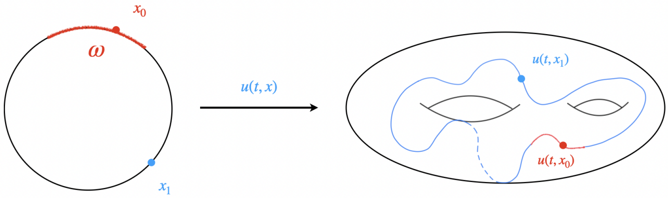
$$\|u_0 - p\|_{H^1(\mathbb{T}^1)} \leq e^{-\frac{C}{T}}$$

there is a control f satisfying

$$\|f\|_{L^\infty(0,T;L^2)} \leq e^{\frac{C}{T}} \|u_0 - p\|_{H^1}$$

such that the solution of the heat flow satisfies $u(T, \cdot) = p$.

Outline of the presentation



$$\partial_t^2 u - \Delta u + S_{jk}(u)\partial^\alpha u^j \partial_\alpha u^k = \chi_\omega f^\perp$$

Theorem (Coron–Krieger–X. 2025)

Wave maps: $\mathbb{T} \to \mathcal{N}$. Global controllability equals homotopy.

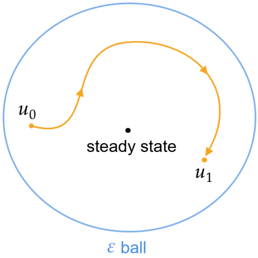
Case $\mathcal{N} = \mathbb{S}^n$: Krieger–X. (2022), Coron–Krieger–X. (2023)

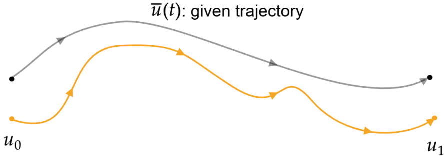
Compared to $\mathbb{S}^n$ cases:

- topology is more complex
- intrinsic geometric constraint: N components, d controls
- no explicite formula (HM, solutions etc.)

Two reductions

(1) global controllability $\Longleftarrow$ local controllability

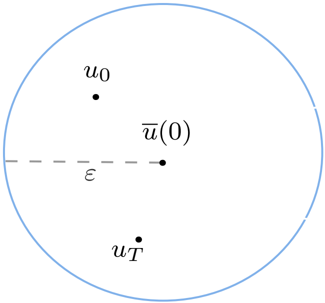




However, the trajectory is not characterizable

(2) local controllability $\Longleftarrow$ the return method

The return method (Coron)



local control

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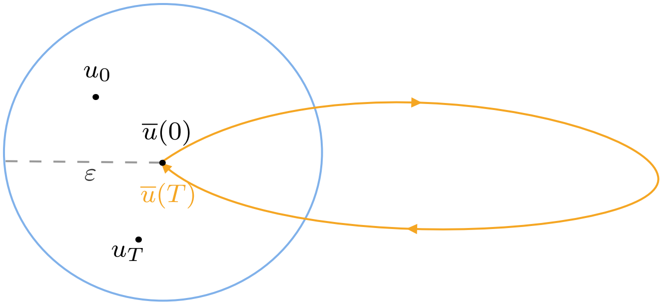

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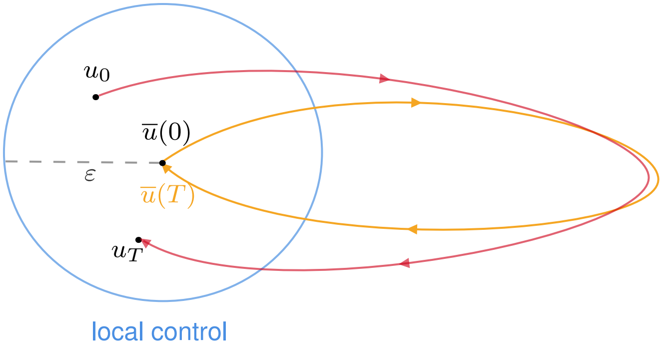


How to construct the return trajectory?

Two intuitions:

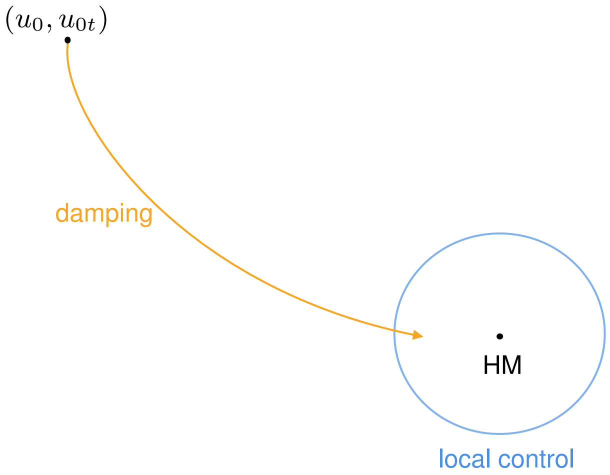
(1) Can we make the solution converge to HM?
    Inspired by Eells-Sampson argument for heat flow

(2) One can expect local controllability around HM
    Though the analysis ought to be more delicate

Construct the return trajectory

(u_0, u_{0t})

Construct the return trajectory



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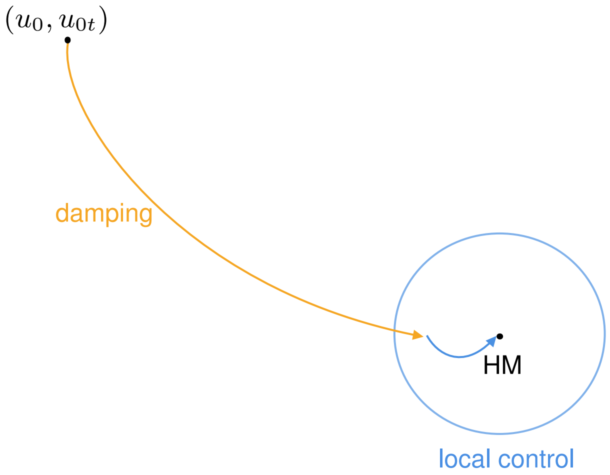

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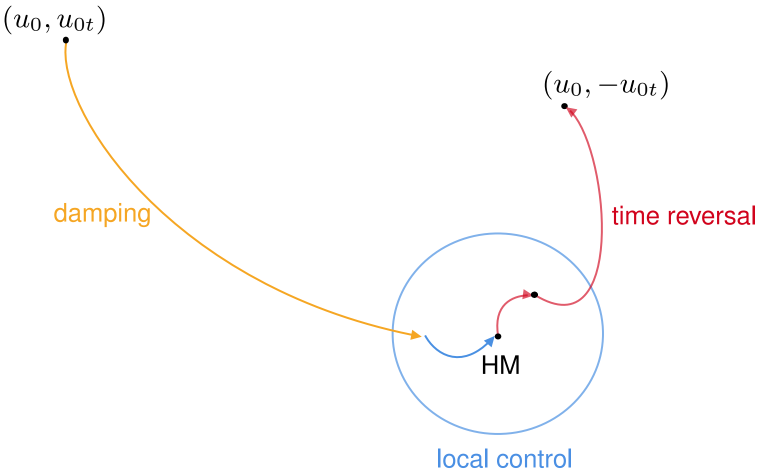

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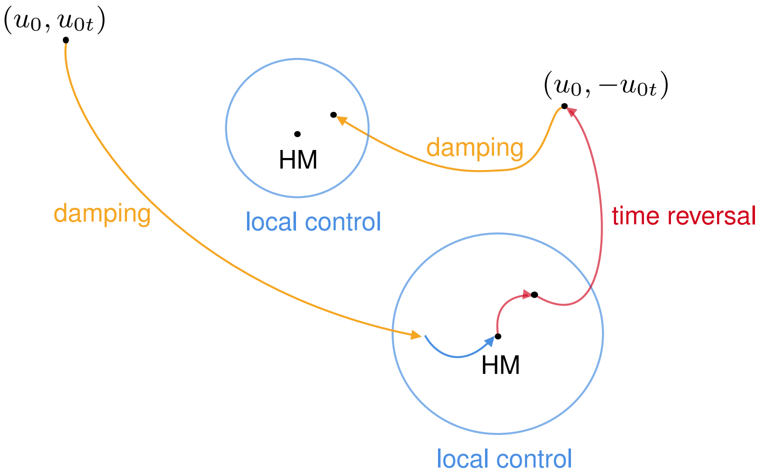

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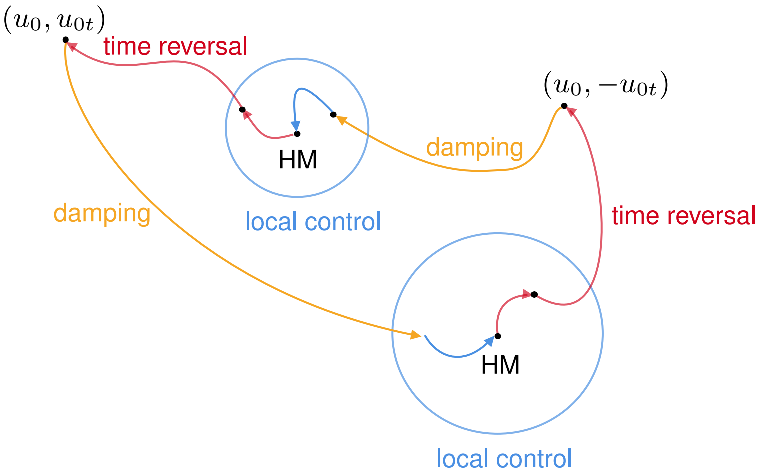


Sketch of the proof

$$\text{global control} \Longleftarrow \text{local control} \Longleftarrow \text{return method}$$

Construct the return trajectory:

Part 1 Stabilize to HM via damping

Part 2 Local controllability around HM (omitted)

Part 3 “Small-time" global controllability between HM (Main result 2)

Part 1. global stabilization to HM

First consider $\mathcal{N} = \mathbb{S}^k$:

$$\partial_t^2 u - \Delta u + (|\partial_t u|^2 - |\partial_x u|^2) u = \chi_\omega f^\perp$$

Question: how to stabilize this system?

Idea: localized damping

$$\partial_t^2 u - \Delta u + (|\partial_t u|^2 - |\partial_x u|^2) u = -a(x)\partial_t u, x \in \mathbb{T}^1,$$

where $a(x)$ $\mathrm{supp}\, \omega$.

Damping stabilization

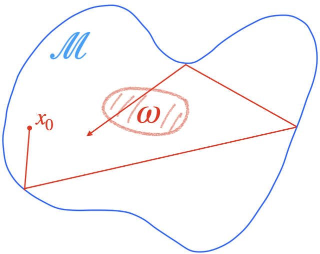
Consider the wave equation, it is known that damping dissipates the energy:

$$\partial_t^2 u - \Delta u = -a(x)\partial_t u$$

then

$$E(T) + \int_0^T \int_\Omega a(x)|u_t|^2 dxdt = E(0) \xrightarrow{(???)} E(t) \lesssim e^{-\varepsilon t} E(0)$$

- Multiplier/Carleman method:
  Lions, Komornik, Zuazua, Ervedoza, Puel, Zhang et. al.
- Microlocal analysis:
  Bardos, Lebeau, Rauch, Burq, Gérard, Dehman, Trélat, Zworski, Laurent et. al.
- Some defocusing equations



Stabilization towards harmonic maps

$$\partial_t^2 u - \Delta u + (|\partial_t u|^2 - |\partial_x u|^2)u = -a(x)\partial_t u, x \in \mathbb{T}$$

- (Krieger–X. 2022) exponential stabilization below 2π energy level set

  Let $\nu > 0$.

  $$E(t) \lesssim e^{-ct} E(0), \ \forall u[0] \in \mathbf{H}(2\pi - \nu)$$

  This is sharp: because HM are steady states.

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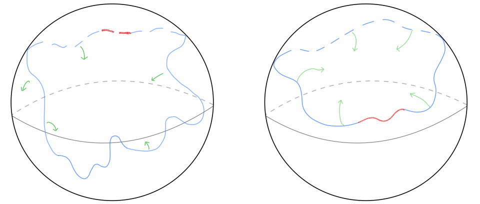


- (Coron–Krieger–X. 2023) for $\mathbb{S}^k$ target, stabilization towards HM
- (Coron–Krieger–X. 2025) for general $\mathcal{N}$ target, stabilization towards HM

Quantitative propagation of smallness

Proposition

Consider the damped WM from $\mathbb{T}$ into $\mathbb{S}^n$ (or general $\mathcal{N}$), and assume $E(0) \leq M$. For any $\varepsilon > 0$, we have either

$$\int_0^{32\pi} \int_{\mathbb{T}} a(x)|u_t|^2 dxdt \geq C\varepsilon^q$$

or

$$\exists t \in [0, 32\pi] \textit{ s.t. is } \varepsilon\textit{-approx. HM}$$

Two ingredients:

- A quantitative "propagation of smallness" result
- An averaging technique to extract the approximate HM

Obstruction to global stabilization

Theorem (Coron–Krieger–X., 2023)

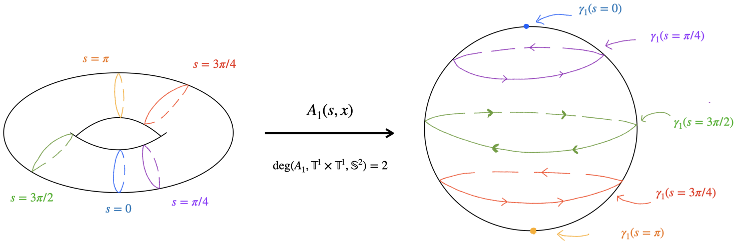
Consider the damped WM from $\mathbb{T}$ to $\mathbb{S}^n$, there is no feedback control such that

$$E(u[t]) \leq h(t)E(u[0]), \ \forall u[0] \in \mathbf{H}(2\pi)$$

for any h satisfying $h(+\infty) = 0$.

Key: homology group $H_2(\mathbb{S}^2; \mathbb{Z})$ and $\Pi^1(C^0(\mathbb{T}^1; \mathbb{S}^2))$ are non-trivial

Topological obstruction



Define $A : \mathbb{T}^1_s \times \mathbb{T}^1_x \to \mathbb{S}^2$ as

$$A(s,x) := \begin{cases} (\sin s \cos x, \sin s \sin x, \cos s)^T, & \forall s \in [0, \pi], \\ (-\sin s \cos x, \sin s \sin x, \cos s)^T, & \forall s \in (\pi, 2\pi). \end{cases}$$

This is a closed curve of initial states with energy $\leq 2\pi$: $\gamma(s) = A(s, \cdot)$

$$E((\gamma(s), 0)) = 2\pi (\sin s)^2 \leq 2\pi$$

Lemma. deg $(A) = 2$

Topological obstruction

- If uniform asymptotic stabilization holds,

$$E(u(t)) \leq p(t)E(u(0)),\ \forall u[0] \in \mathbf{H}(2\pi)$$

  then $\exists T > 0$, the flow with initial state $(\gamma(s), 0)$ satisfies

$$|\Phi(T; (\gamma(s), 0))(x) - \Phi(T; (\gamma(s), 0))(0)| \leq 1/2\ \ \forall s \in \mathbb{S}_s\ \forall x \in \mathbb{S}_x$$

- One can further deform the map $(s, x) \mapsto \Phi(T; (\gamma(s), 0))(x)$ to a one dimensional closed curve. Thus

$$\deg\, \Phi(T; (\gamma(s), 0))(x) = 0.$$

- This contradicts the fact that

$$\deg\, \Phi(0; (\gamma(s), 0))(x) = \deg\, A = 2.$$

Curvature and stabilization around HM

Exponential stabilization around HM?

$$\partial_t^2 u - \Delta u + S_{jk}(u)\partial^\alpha u^j \partial_\alpha u^k = -a(x)\partial_t u, a(x) \text{ supp } \omega$$

Theorem (Coron–Krieger–X. 2025)

Let $\mathcal{N}$ has trivial normal bundle. Let γ be HM: $\mathbb{T} \to \mathcal{N}$.

- *If the sectional curvature is negative on γ, then expo. stabilization*
- *Otherwise, not stable.*

Part 3. Small-time global control between steady states (Main result 2)

Theorem (Coron–Krieger–X. 2025: a positive answer)

Let $T = 6\pi$. Consider wave maps: $\mathbb{T} \to \mathcal{N}$. For any steady states (including HM) u_0 and u_1,

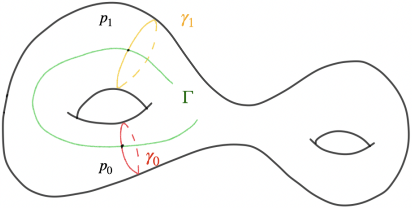
$$\textit{“uniform-time" global controllability} \iff u_0 \textit{ is homotopic to } u_1$$

Deformation on geodesics

$$\Box u + S_{jk}(u)\partial^\alpha u^j \partial_\alpha u^k = \chi_\omega f^\perp$$

Let $\Gamma \subset \mathcal{N}$ be a geodesic.

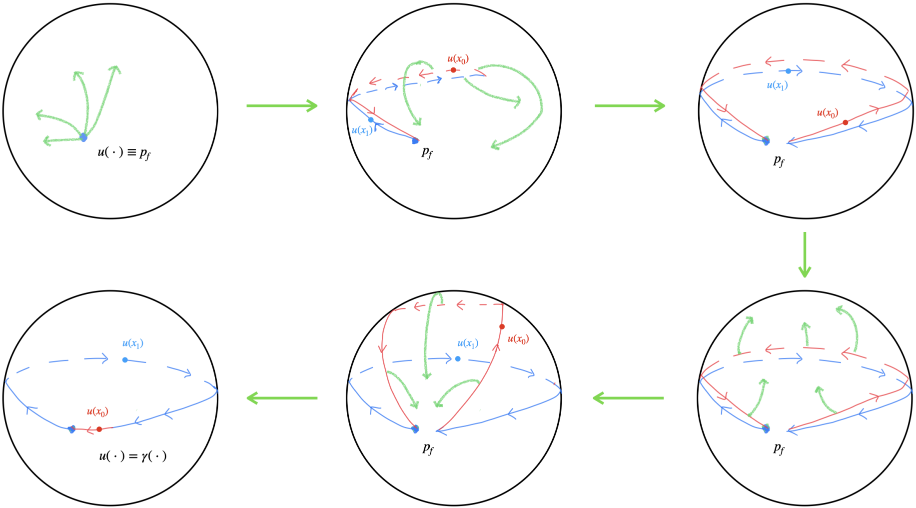
$$\Gamma = \{\bar{u}(x) : x \in \mathbb{T}\}$$

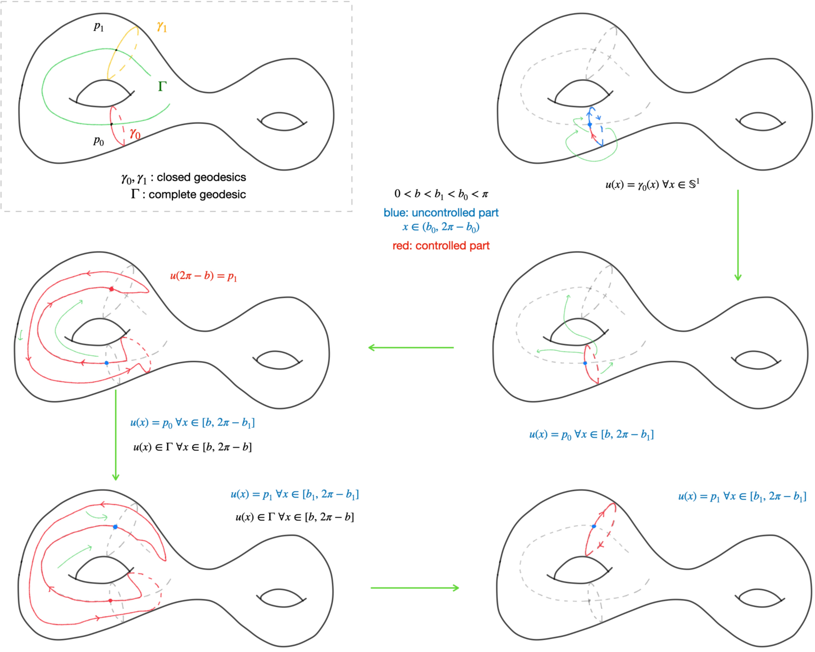


γ_0, γ_1 : closed geodesics
Γ : complete geodesic

- If both initial state $u_0 \in \Gamma$ and $f \in T\Gamma$, then u stays in Γ.
- Let $u(t,x) = \bar{u}(\varphi(t,x))$. Then $\Box\varphi = \chi_\omega g$.
- Gluing: inner part by linear heat equatin on geodescis
  outer part by control to ensure homotopy

Deformation on geodesics





p_1
γ_1
Γ
p_0
γ_0
γ_0, γ_1 : closed geodesics
Γ : complete geodesic
$u(x) = \gamma_0(x)\ \forall x \in \mathbb{S}^1$
$0 < b < b_1 < b_0 < \pi$
blue: uncontrolled part
$x \in (b_0, 2\pi - b_0)$
red: controlled part
$u(2\pi - b) = p_1$
$u(x) = p_0\ \forall x \in [b, 2\pi - b_1]$
$u(x) \in \Gamma\ \forall x \in [b, 2\pi - b]$
$u(x) = p_0\ \forall x \in [b, 2\pi - b_1]$
$u(x) = p_1\ \forall x \in [b_1, 2\pi - b_1]$
$u(x) \in \Gamma\ \forall x \in [b, 2\pi - b]$
$u(x) = p_1\ \forall x \in [b_1, 2\pi - b_1]$

Conclusion:

- Global controllability equals homotopy
- Small-time global controllability between steady states
- Interplay between: Analysis, Dynamics, Geometry

Further perspectives:

- Schrödinger maps, Yang-Mills etc.
- Small-time global controllability
- Higher dimensional $\mathcal{M}$
- Control and singularity formation
- Random and stochastic equations

Thank you!