Exact boundary controllability of semilinear wave equations

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Control of PDEs and related topics
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with Jérôme Lemoine and Arnaud Münch

Framework and objective

Framework. Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, a bounded domain sufficiently smooth and $T > 0$. We denote

$$Q_T := \Omega \times (0,T), \qquad \Sigma_T := \partial\Omega \times (0,T).$$

We consider

$$\begin{cases} \partial_{tt} y - \Delta y + f(y) = 0, & Q_T, \\ y = v_{|\Gamma_0}, & \Sigma_T, \\ (y(\cdot,0), \partial_t y(\cdot,0)) = (u_0, u_1), & \Omega, \end{cases} \qquad (\star)$$

where $v \in L^2(\Sigma_T)$, $\Gamma_0 \subset \partial\Omega$ non-empty and $f \in \mathcal{C}(\mathbb{R})$ is a non-linear function.

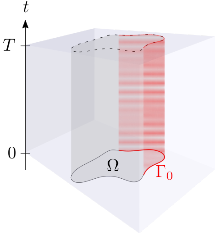

Figure – Control zone Γ_0, $d = 2$

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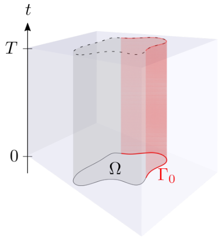

Figure – Control zone Γ_0, $d = 2$

Exact controllability problem.

Given $T > 0$, $\Gamma_0 \subset \partial\Omega$ and $(u_0, u_1), (z_0, z_1)$ in an appropriate space, find (if possible) a pair (y, v) solution of $(\star)$ such that

$$(y(\cdot, T), \partial_t y(\cdot, T)) = (z_0, z_1)? \tag{1}$$

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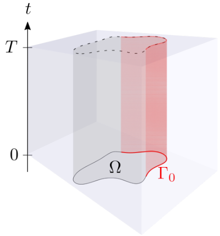

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Given $T > 0$, $\Gamma_0 \subset \partial\Omega$ and $(u_0, u_1), (z_0, z_1)$ in an appropriate space, find (if possible) a pair (y, v) solution of $(\star)$ such that

$$(y(\cdot, T), \partial_t y(\cdot, T)) = (z_0, z_1)? \qquad (1)$$

For (u_0, u_1) and (z_0, z_1) fixed, we call *state-control pair* of $(\star)$ any pair (y, v) solution of $(\star)$ satisfying (1).

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First controllability result

Controllability result [Zua91, Theorem 2.1] [1]

Assume that f is globally lipschitz and T large enough. The system $(\star)$ is exactly controllable in $L^2(\Omega) \times H^{-1}(\Omega)$.

1. E. Zuazua, *Exact boundary controllability for the semilinear wave equation*, in Nonlinear Partial Differential Equations and Their Applications, Collège de France Seminar, Vol. X (Paris, 1987-1988), Pitman Res. Notes Math. Ser. 220, Longman, Harlow, UK, 1991,pp. 357-391.

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Linearization

$$\begin{cases} \partial_{tt} y - \Delta y + \dfrac{f(z) - f(0)}{z} y = -f(0), & Q_T, \\ y = v_{|\Gamma_0}, & \Sigma_T \\ (y(\cdot,0), \partial_t y(\cdot,0)) = (u_0, u_1), & \Omega. \end{cases} \tag{2}$$

Fixed point operator

$$\begin{aligned} \Lambda := L^2(Q_T) &\to L^2(Q_T) \\ z &\mapsto y \end{aligned}$$

where (y, v) is a state-control pair of (2) such that $v := \arg\min_v \|v\|^2_{L^2(\Sigma_T)}$.

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Controllability result $d=1$, distributed control case [Zua93, Theorem 1][2]

Let T large enough. Assume that $f \in \mathcal{C}^1(\mathbb{R})$ such that

$$\exists \beta > 0, \qquad \limsup_{|r|\to\infty} \frac{|f(r)|}{|r| \ln^2 |r|} \leq \beta.$$

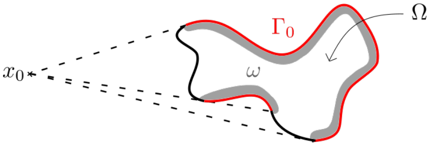
If β is small enough then, the system $(\star)$ is exactly controllable at time T in $H_0^1(\Omega) \times L^2(\Omega)$.

2. E. Zuazua, *Exact controllability for semilinear wave equations in one space dimension*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 10 (1993), pp. 109-129.

Most general result known - distributed control case - Carleman setting

Let $x_0 \in \mathbb{R}^d \setminus \overline{\Omega}$ and $\varepsilon > 0$. We denote

$$\Gamma_0 := \{x \in \partial\Omega;\ (x - x_0) \cdot \nu(x) > 0\},$$
$$\mathcal{O}_\varepsilon(\Gamma_0) := \{x \in \mathbb{R}^d;\ \mathrm{dist}(x, \Gamma_0) < \varepsilon\}.$$



We suppose that $\omega = \mathcal{O}_\varepsilon(\Gamma_0) \cap \Omega$ and

$$T > \max\left\{8 \max_{x \in \overline{\Omega}} |x - x_0|^2, 1 + 24\sqrt{d} \max_{x \in \overline{\Omega}}\{2(x - x_0) \cdot \nu(x)\}\,(2 + d)\right\}$$

If, moreover, the function $f \in \mathcal{C}^1(\mathbb{R})$ satisfies

$$\exists \beta > 0, \qquad \limsup_{|r| \to \infty} \frac{|f(r)|}{|r| \ln^p |r|} \leq \beta, \qquad 0 \leq p < 3/2.$$

Controllability result [FLZ19, Theorem 4.5] [3]

Under the above conditions, if β is small enough, the system $(\star)$ is exactly controllable in $H_0^1(\Omega) \times L^2(\Omega)$.

3. X. Fu, Q. Lü, X. Zhang, *Carleman Estimates for Second Order Partial Differential Operators and Applications : A Unified Approach*, Springer Briefs in Math., Springer, Cham, 2019.

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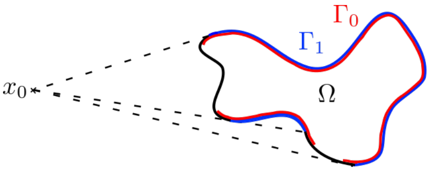
$$\begin{cases} \partial_{tt} y - \Delta y + f(y) = 0, & Q_T, \\ y = v_{|\Gamma_0}, & \Sigma_T, \\ (y(\cdot,0), \partial_t y(\cdot,0)) = (u_0, u_1), & \Omega. \end{cases} \qquad (\star)$$

First main result (existence of a control)

For any $x_0 \in \mathbb{R}^d \setminus \overline{\Omega}$, let

- $\Gamma_1 := \{x \in \partial\Omega : (x - x_0) \cdot \nu(x) > 0\}$
- $\Gamma_0 \subset \partial\Omega$ such that $\mathrm{dist}(\Gamma_1, \partial\Omega \setminus \Gamma_0) > 0$
- $T > 2 \max_{x \in \overline{\Omega}} |x - x_0|$.



Assume that $f \in C^0(\mathbb{R})$ satisfies

$$\exists \beta > 0, \qquad \limsup_{|r| \to \infty} \frac{|f(r)|}{|r| \ln^p |r|} \leq \beta, \qquad 0 \leq p < 3/2.$$

Theorem [CLM24] [4]

If β is small enough then, the system $(\star)$ is exactly controllable in $L^2(\Omega) \times H^{-1}(\Omega)$.

4. C., Lemoine, Münch, On the exact boundary controllability of semilinear wave equations. SIAM J. Control and Optimization 62(4), 1953-1976 (2024).

First main result (existence of a control)

- Extends and generalizes to any dimension [BLM23] [5] devoted to the case $d = 1$.

- Improve [FLZ19, Theorem 4.5] [6]

	[FLZ19]	[CLM24]
Regularity of f	$\mathcal{C}^1(\mathbb{R})$	$\mathcal{C}^0(\mathbb{R})$
Regularity of (u_0, u_1)	$H_0^1(\Omega) \times L^2(\Omega)$	$L^2(\Omega) \times H^{-1}(\Omega)$
Lower bound of T	$\max\left\{8 \max_{x\in\overline{\Omega}} \lvert x - x_0\rvert^2, 1 + 24\sqrt{d} \max_{x\in\overline{\Omega}}\{2(x - x0) \cdot \nu(x)\} (2 + d)\right\}$	$2 \max_{x\in\overline{\Omega}} \lvert x - x_0\rvert$

- The conditions on Γ_0, T and (u_0, u_1) are those of the linear case, which are optimal.

5. K. Bhandari, J. Lemoine, A. Münch, *Exact boundary controllability of 1d semilinear wave equations through a constructive approach*, Math. Control Signals Systems, 1 (2023).

6. X. Fu, Q. Lü, X. Zhang, *Carleman Estimates for Second Order Partial Differential Operators and Applications : A Unified Approach*, Springer Briefs in Math., Springer, Cham, 2019.

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Linearize the system by introducing an operator Λ_S

Linearization + Fixed point

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Zero order linearization.

$$\begin{cases} \partial_{tt} y - \Delta y = -f(z), & Q_T, \\ y = v_{|\Gamma_0}, & \Sigma_T, \\ (y(\cdot,0), \partial_t y(\cdot,0)) = (u_0, u_1), & \Omega. \end{cases} \qquad (\star\star)$$

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Fixed point operator.

$$\Lambda_s := \mathcal{C}(s) \subset L^\infty(0,T;L^2(\Omega)) \to \mathcal{C}(s)$$
$$z \mapsto y$$

where (y,v) is an optimal state-control pair of $(\star\star)$ with $(z_0,z_1) = (0,0)$ for the cost

$$\mathcal{J}_s(y,v) := s \int_{Q_T} \rho^2 |y|^2 \,dx\,dt + \int_{\Sigma_T} \eta^{-2} \Psi^{-1} \rho^2 |v|^2 \,dx\,dt,$$

involving Carleman weight $\rho(s;x,t)$ and parameter $s > 0$.

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Goal. Prove the existence of a fixed-point for Λ_s for at least one s.

Outline of the proof

- Verify assumptions for the Schauder theorem.

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- *A priori* estimate of the linear optimal state-control pair (with respect to the initial data (u_0, u_1), the second member $f(z)$ and the parameter s).

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- Verify assumptions for the Schauder theorem.
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- Carleman inequality (depending of s).

Weight function

For any $\mu \in (0,1)$, $\lambda > 0$ and for some $M_0 > 0$ large enough so that
$\psi(x,t) := |x - x_0|^2 - \mu\left(t - \frac{T}{2}\right) + M_0 > 1$ in $\overline{Q_T}$,

$$\phi(x,t) := e^{\lambda\psi(x,t)}, \qquad e^{-sc} \leq \rho(s;x,t) := e^{-s\phi(x,t)} \leq e^{-s}, \qquad \forall (x,t) \in Q_T, \quad c = \|\phi\|_{L^\infty(Q_T)}$$

Carleman estimate [BdBE2013] [7]

Under the geometric condition, there exists $s_0 > 0$, $\lambda > 0$ and $C > 0$ such that for any $s \geq s_0$,

$$s\int_{Q_T} \rho^{-2}(s)\left(|\partial_t w|^2 + |\nabla w|^2\right) \mathrm{d}x\,\mathrm{d}t + s^3\int_{Q_T} \rho^{-2}(s)\,|w|^2\,\mathrm{d}x\,\mathrm{d}t + \underbrace{s\int_\Omega \rho^{-2}(s;x,0)\left(|\partial_t w(x,0)|^2 + |\nabla w(x,0)|^2\right)\mathrm{d}x}_{\text{initial energy}}$$

$$+ s^3\int_\Omega \rho^{-2}(s;x,0)\,|w(x,0)|^2\,\mathrm{d}x \leq C\Bigg(\underbrace{\int_{Q_T} \rho^{-2}(s)\,|\partial_{tt}w - \Delta w|^2\,\mathrm{d}x\,\mathrm{d}t}_{\text{source}} + \underbrace{s\int_{\Sigma_T} \eta^2(t)\Psi(x)\rho^{-2}(s)\,|\partial_\nu w|^2\,\mathrm{d}x\,\mathrm{d}t}_{\text{observation}}\Bigg),$$

for any $w \in \mathcal{H} := \left\{w \in \mathcal{C}^0([0,T];H_0^1(\Omega)) \cap \mathcal{C}^1([0,T];L^2(\Omega));\ \partial_{tt}w - \Delta w \in L^2(Q_T)\right\}$.

7. L. Baudouin, M. de Buhan, S. Ervedoza, *Global Carleman Estimates for Waves and Applications*, Communications in Partial Differential Equation, 38(5), 823-859.

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A priori estimate on (y, v)

For any $r \in [0,1]\backslash\{1/2\}$. The following estimate holds

$$\begin{aligned} &\|\rho(s)y\|_{L^2(Q_T)} + s^{-2}\|\rho(s)y\|_{L^\infty(0,T;L^2(\Omega))} + s^{-1/2}\left\|\rho(s)\eta^{-1}\Psi^{-1/2}v\right\|_{L^2(\Sigma_T)} \\ &\leq C_r\left(s^{r-3/2}\|\rho(s)f(z)\|_{L^2(0,T;H^{-r}(\Omega))} + s^{-1/2}\|\rho(0)(u_0,u_1)\|_{L^2(\Omega)\times H^{-1}(\Omega)}\right). \end{aligned} \tag{E_1}$$

Goal : Prove the existence of s and of a fixed-point for

$$\begin{aligned}\Lambda_s : \mathcal{C}(s) &\to \mathcal{C}(s)\\ z &\mapsto y.\end{aligned}$$

$\rightarrow$ Employ the Schauder theorem

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- **Define** $\mathcal{C}(s)$ to have a **stability property** $\left(\text{i.e. } z \in \mathcal{C}(s) \Rightarrow \Lambda_s(z) = y \in \mathcal{C}(s)\right)$:

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Recall : *A priori* estimate on y

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$$\begin{aligned}&\|\rho(s)y\|_{L^2(Q_T)} + s^{-2}\|\rho(s)y\|_{L^\infty(0,T;L^2(\Omega))}\\ &\quad\leq C_r\left(s^{r-3/2}\|\rho(s)f(z)\|_{L^2(0,T;H^{-r}(\Omega))} + s^{-1/2}\|\rho(0)(u_0,u_1)\|_{L^2(\Omega)\times H^{-1}(\Omega)}\right).\end{aligned} \qquad (E_1)$$

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For any $p \in (1, 3/2)$. The following estimate holds

$$\begin{aligned}&\|\rho(s)y\|_{L^2(Q_T)} + s^{-2}\|\rho(s)y\|_{L^\infty(0,T;L^2(\Omega))}\\ &\qquad \leq sC\left(s^{-p}\alpha_2 + \beta c^p + e^{-s}\left(s^{-p-1}\alpha_1 T^{1/2}|\Omega|^{1/2} + s^{-1/2}\left(\|u_0\|_{L^2(\Omega)} + \|u_1\|_{H^{-1}(\Omega)}\right)\right)\right).\end{aligned}$$

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$$\leq Cs\Big(s^{-p}\alpha_2 + \underbrace{\beta c^p}_{\Rightarrow \beta < \frac{1}{Cc^p}} + e^{-s}\Big(s^{-p-1}\alpha_1 T^{1/2}|\Omega|^{1/2} + s^{-1/2}\big(\|u_0\|_{L^2(\Omega)} + \|u_1\|_{H^{-1}(\Omega)}\big)\Big)\Big).$$

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→ For β small enough, there exists s large enough such that $\mathcal{C}(s)$ is stable under the map Λ_s.

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- **Continuity.** The map $\Lambda_s : \mathcal{C}(s) \to \mathcal{C}(s)$ is continuous for the $L^\infty(0,T;L^2(\Omega))$-norm.
- **Relative compactness.** $\Lambda_s(\mathcal{C}(s))$ is a relatively compact subset of $\mathcal{C}(s)$ relative to the $L^\infty(0,T;L^2(\Omega))$-norm.

Relative compactness : Any sequence $(y^n)_n$ of $\Lambda_s(\mathcal{C}(s))$ admits a subsequence $(y^{n_k})_k$ that converges in $\mathcal{C}(s)$ for the $L^\infty(0,T;L^2(\Omega))$-norm.

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$$y^n \in \mathcal{C}^0([0,T];L^2(\Omega)) \cap \mathcal{C}^1([0,T];H^{-1}(\Omega))$$

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Key point - Additional regularity property on the optimal controlled pair.

Assume that $(u_0,u_1,f(z)) \in H_0^{1-r}(\Omega)\times H^{-r}(\Omega)\times L^2(0,T;H^{-r}(\Omega))$, $r\in(0,1)\neq\{\frac{1}{2}\}$. The controlled pair (y,v), which minimize $\mathcal{J}_s$, belongs to the space

$$\Big(\mathcal{C}^0([0,T];H^{1-r}(\Omega))\cap\mathcal{C}^1([0,T];H^{-r}(\Omega))\Big)\times H^{1-r}(0,T;L^2(\partial\Omega)),$$

and satisfies

$$\begin{aligned}&\|(\rho y)_t\|_{L^\infty(0,T;H^{-r}(\Omega))}+\|\rho y\|_{L^\infty(0,T;H^{1-r}(\Omega))}\\&\qquad\le C\left(\|\rho f(z)\|_{L^2(0,T;H^{-r}(\Omega))}+\|\rho(0)u_0\|_{H^{1-r}(\Omega)}+\|\rho(0)u_1\|_{H^{-r}(\Omega)}\right).\end{aligned}$$

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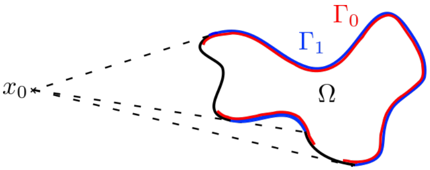
$\rightarrow$ work with $(y^n-y^0)_n$ which is solution of linear wave equation associated with the data $(0,0,f(z_n)-f(z_0))$.

Recall of the result

For any $x_0 \in \mathbb{R}^d \backslash \overline{\Omega}$, let

- $\Gamma_1 := \{x \in \partial\Omega : (x - x_0) \cdot \nu(x) > 0\}$
- $\Gamma_0 \subset \partial\Omega$ such that $\mathrm{dist}(\Gamma_1, \partial\Omega \setminus \Gamma_0) > 0$
- $T > 2 \max_{x \in \overline{\Omega}} |x - x_0|$.



Assume that $f \in C^0(\mathbb{R})$ satisfies

$$\exists \beta > 0, \qquad \limsup_{|r| \to \infty} \frac{|f(r)|}{|r| \ln^p |r|} \leq \beta, \qquad 0 \leq p < 3/2.$$

Theorem [CLM24] [8]

If β is small enough then, the system $(\star)$ is exactly controllable in $L^2(\Omega) \times H^{-1}(\Omega)$.

8. C., Lemoine, Münch, On the exact boundary controllability of semilinear wave equations. SIAM J. Control and Optimization 62(4), 1953-1976 (2024).

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$$\begin{cases} \partial_{tt} y - \Delta y + f(y) = 0, & Q_T, \\ y = v_{|\Gamma_0}, & \Sigma_T, \\ (y(\cdot,0), \partial_t y(\cdot,0)) = (u_0, u_1), & \Omega. \end{cases} \qquad (\star)$$

Assume that $f \in \mathcal{C}^1(\mathbb{R})$ and that there exists $0 \leq p < 3/2$ such that f satisfies

$$\exists \alpha, \beta > 0, \quad |f'(r)| \leq \alpha + \beta \ln^p(r), \quad \forall r \in \mathbb{R}.$$

Proposition [CLM24] [9]

For any s large enough and β small enough (s.t. $\beta < \frac{1}{Cc^p}$). Then,

$$\|\rho(s)\left(\Lambda_s(z_1) - \Lambda_s(z_2)\right)\|_{L^2(Q_T)} \leq C\left(s^{-p}\alpha + \beta c^p\right)\|\rho(s)(z_1 - z_2)\|_{L^2(Q_T)}.$$

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Theorem [CLM24] [9]

For any $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, there exists of a non trivial sequence $(y_k, v_k)_k$ that strongly converges to a state-control pair (y, v) for system $(\star)$. Moreover, the convergence is at least linear for the norm $\|\rho(s)\cdot\|_{L^2(Q_T)} + \|\rho(s)\cdot\|_{L^2(\Sigma_T)}$ where s is chosen sufficiently large.

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Thank you for your attention !