Incompressible fluids driven by degenerate forces

Manuel Rissel

NYU Shanghai

Control of PDEs and related topics

Incompressible fluid in $\mathbb{T}^N$

Navier-Stokes (N–S) system:

$$\begin{aligned} \partial_t u - \nu \Delta u + (u \cdot \nabla)\, u + \nabla p &= f + \mathbb{I}_\omega \xi, \\ \nabla \cdot u &= 0, \\ u(\cdot, 0) &= u_0. \end{aligned}$$

$u(x,t)$:	velocity
$p(x,t)$:	pressure
$f(x,t)$:	known body force
$u_0(x)$:	initial state
$\nu > 0$:	viscosity
$\xi(x,t)$:	interior control supported in $\omega \subset \mathbb{T}^N$

(Global) approximate controllability

$$\forall u_0, u_1, T > 0, \varepsilon > 0 \, \exists \xi \colon \|u(\cdot, T) - u_1\| < \varepsilon$$

Finite-dimensional controls

$$\xi(x,t) = \alpha_1(t)\psi_1(x) + \cdots + \alpha_{42}(t)\psi_{42}(x)$$

Finite-dimensional controls for fluid equations

Case $\omega \neq \mathbb{T}^N$:

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Some open problems

Chapter

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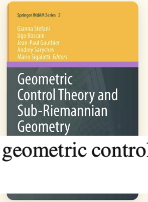


Andrei A. Agrachev

Abstract We discuss some challenging open problems in the geometric control theory and sub-Riemannian geometry.

Some literature for case $\omega = \mathbb{T}^N$:

- A. A. Agrachev and A. V. Sarychev, 2006 (CMP)
- A. Shirikyan, 2006 (CMP)
- A. Shirikyan, 2008 (Phys. D)
- V. Nersesyan, 2015 (Nonlinearity)
- V. Nersesyan, 2021 (SICON)
- ...

Finite-dimensional controls for transport equation?

Are there choices of $u^\star$ such that

$$\partial_t v + (u^\star \cdot \nabla) v = g,$$

is approximate controllable with control

$$g \in L^2((0,1); \mathcal{H}_0),$$

where

$$\mathcal{H}_0 \coloneqq \operatorname{span}\{\sin(x_1), \cos(x_1), \sin(x_2), \cos(x_2)\}?$$

Universally fixed divergence-free vector field

$$u^\star(x,t) := \gamma_1(t)\begin{bmatrix}\sin(x_2)\\0\end{bmatrix} + \gamma_2(t)\begin{bmatrix}\cos(x_2)\\0\end{bmatrix} + \gamma_3(t)\begin{bmatrix}0\\\sin(x_1)\end{bmatrix} + \gamma_4(t)\begin{bmatrix}0\\\cos(x_1)\end{bmatrix},$$

where

$$\gamma_1,\ldots,\gamma_4 \in W^{1,2}((0,1);\mathbb{R})$$

have a special structure, first introduced by Kuksin, Nersesyan, and Shirikyan, 2020 (GAFA).

Lemma (R., V. Nersesyan)

For all $v_1 \in H^m$ *and* $\varepsilon > 0$ **there is** $g \in \mathrm{L}^2((0,1); \mathcal{H}_0))$ **such that** *the solution*

$$v(x,t) = \int_0^t g(\Phi^{u^\star}(x,1,s), s)\, ds$$

to

$$\partial_t v + (u^\star \cdot \nabla) v = g, \quad v(\cdot, 0) = 0$$

satisfies

$$\|v(\cdot,1) - v_1\|_m < \varepsilon.$$

Here, Φ^V is flow of vector field V:

$$\frac{d}{dt}\Phi^V(x,s,t) = V(\Phi^V(x,s,t),t),\ \Phi^V(x,s,s) = x.$$

Open $\omega \subset \mathbb{T}^2$ with $\mathbb{T}^2 \setminus \omega$ simply-connected.

Theorem (V. Nersesyan, R., 2025 (CPAM))

The N–S in $\mathbb{T}^2$ is approximately controllable in any time $T > 0$ with control

$$\xi(x,t) = \alpha_1(t)\vartheta_1(x,\alpha(t)) + \cdots + \alpha_8(t)\vartheta_8(x,\alpha(t)),$$

where

$$\operatorname{supp}(\xi) \subset \omega \times (0,T).$$

J.-M. Coron's return method

Ansatz:

$$u^\delta(\cdot,t) = \delta^{-1}\overline{y}(\cdot,\delta^{-1}t) + U(\cdot,\delta^{-1}t) + \mathrm{error}^\delta(\cdot,t),$$
$$\xi^\delta(\cdot,t) = \delta^{-2}\overline{\xi}(\cdot,\delta^{-1}t) + \delta^{-1}\eta(\cdot,\delta^{-1}t)$$

- $\overline{y}$ flushes information through ω and $(\overline{y},\overline{\xi})$ satisfy Euler system

$$\partial_t\overline{y} + (\overline{y}\cdot\nabla)\overline{y} + \nabla\overline{p} = \mathbb{I}_\omega\overline{\xi}, \quad \nabla\cdot\overline{y} = 0, \quad \overline{y}(\cdot,0) = \overline{y}(\cdot,1) = 0$$

- $\partial_t U + (\overline{y}\cdot\nabla)U + (U\cdot\nabla)\overline{y} + \nabla P = \eta, \quad U(\cdot,0) = u_0, \quad U(\cdot,1) = u_1.$

(In 2D, $v := \nabla\wedge U$, satisfies

$$\partial_t v + (\overline{y}\cdot\nabla)v + (U\cdot\nabla)\nabla\wedge\overline{y} = \nabla\wedge\eta, \quad v(\cdot,0) = u_0, \quad v(\cdot,1) = u_1$$)

- Approximate controllability: $\mathrm{error}^\delta(\cdot,\delta) \longrightarrow 0$, as $\delta \longrightarrow 0$.

Approach

- Return method with constant-in-space reference trajectory $\overline{y}$.
- Approximate controllability of 2D transport problem

$$\partial_t v + (\overline{y} \cdot \nabla) v = \mathbb{I}_\omega \eta, \quad v(\cdot, 0) = 0.$$

Goal: find control η of desired structure.

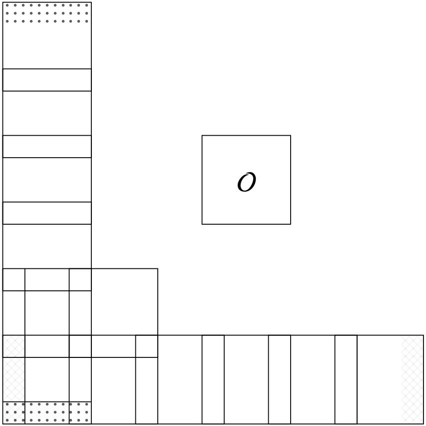
- Take generating $u^\star$ and determine control $g \in L^2((0,1); \mathcal{H}_0)$ for

$$\partial_t V + (u^\star \cdot \nabla) V = g, \quad V(\cdot, 0) = 0.$$

- Construct η as a rearrangement

$$v(1) = \int_0^1 \eta(\Phi^{\overline{y}}(x,1,s), s) \, \mathrm{d}s = \int_0^1 g(\Phi^{u^\star}(x,1,s), s) \, \mathrm{d}s = V(1).$$

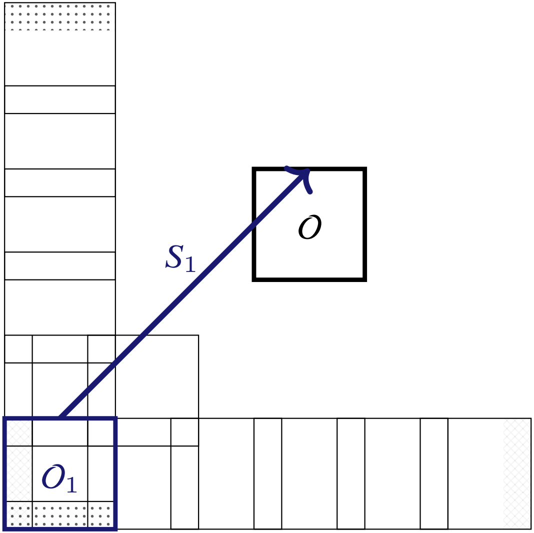
Cover $\mathbb{T}^2$ by overlapping squares $\mathcal{O}_1, \ldots, \mathcal{O}_M$, fix reference square $\mathcal{O} \subset \omega$.



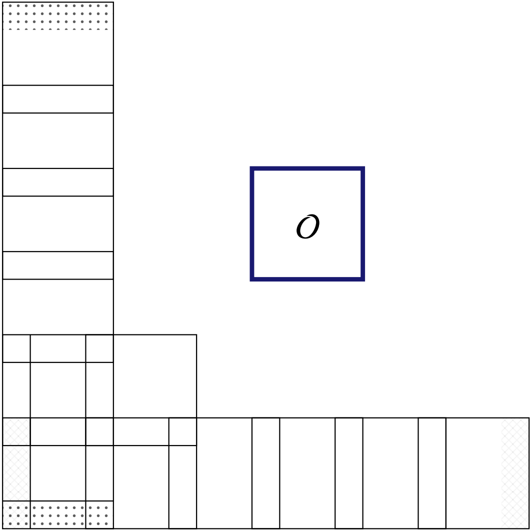
Equal-distant partition of time interval:

$$0 < t_c^0 < t_a^1 < t_b^1 < t_c^1 < t_a^2 < t_b^2 < t_c^2 < \cdots < t_a^M < t_b^M < t_c^M < 1.$$

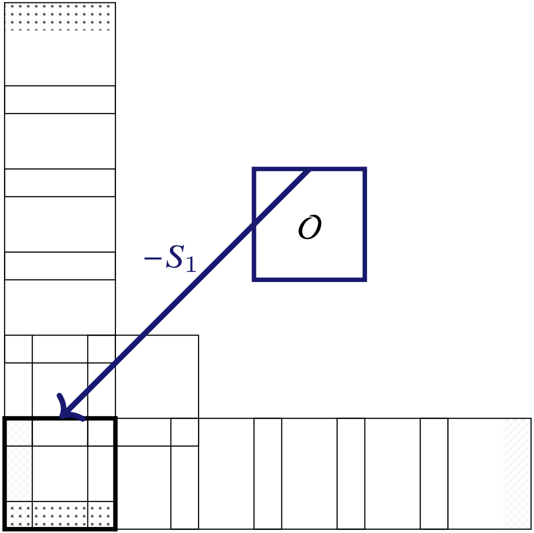
During $[t_0^c, t_1^a]$, information from $\mathcal{O}_1$ is transported along $\bar{y}$ to $\mathcal{O}$.



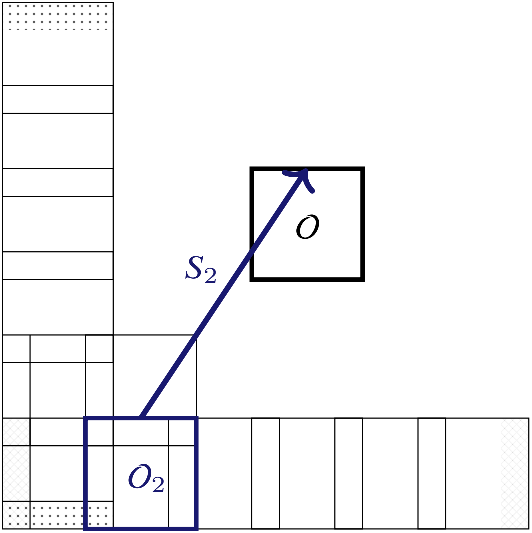
During $[t_1^a, t_1^b]$, the flow rests.



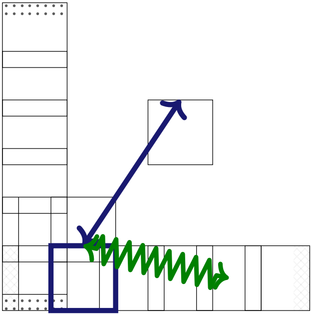
During $[t_1^b, t_1^c]$, information is transported along $\overline{y}$ from $\mathcal{O}$ to $\mathcal{O}_1$.



During $[t_1^c, t_2^a]$, information from $\mathcal{O}_2$ is transported to $\mathcal{O}$.



$$\eta(x,t) := \underbrace{\chi(x)}_{\substack{\text{supported} \\ \text{in } \omega}} \sum_{k=1}^{M} \frac{\mathbb{I}_{[t_a^k, t_b^k]}(t)}{t_b^k - t_a^k} g\left(\Phi^{u^\star}\left(\Phi^{\overline{y}}(x,t,0), 0, \frac{t - t_a^k}{t_b^k - t_a^k}\right), \frac{t - t_a^k}{t_b^k - t_a^k}\right)$$



$$\int_0^1 \eta(\Phi^{\overline{y}}(x,1,s), s)\,\mathrm{d}s = \int_0^1 g(\Phi^{u^\star}(x,1,s), s)\,\mathrm{d}s$$

Main issue from the start: using constant-in-space $\overline{y}$ not suitable to get finite-dimensional controls (no mixing effect)

On the other hand: according to the return method ansatz, if $\overline{y}$ is not constant-in-space one would need to obtain desired controls for

$$\partial_t V + (\overline{y} \cdot \nabla)V + (\nabla^{\perp}(-\Delta)^{-1}V \cdot \nabla)\nabla \wedge \overline{y} = \mathbb{I}_{\omega} g,$$

instead for

$$\partial_t V + (\overline{y} \cdot \nabla)V = \mathbb{I}_{\omega} g,$$

Boussinesq system controlled in velocity and temperature

$$\partial_t u - \nu \Delta u + (u \cdot \nabla)\, u + \nabla p = \theta e_2 + \mathbb{I}_\omega \xi,$$
$$\nabla \cdot u = 0,$$
$$\partial_t \theta - \tau \Delta \theta + (u \cdot \nabla)\theta = \mathbb{I}_\omega \eta.$$

$\omega \subset \mathbb{T}^2$ **arbitrary** nonempty open set. We construct universal finite-dimensional spaces

$$\mathscr{F}_v \subset C^\infty(\mathbb{T}^2;\mathbb{R}^2), \quad \mathscr{F}_t \subset C^\infty(\mathbb{T}^2;\mathbb{R})$$

such that the following statement holds.

Theorem (R. 2025, arXiv:2506.19764)

For any given

$$\varepsilon, \nu, \tau, T > 0, \quad k \in \mathbb{N}_0, \quad u_0 \in L^2_{\mathrm{div}}, \quad u_1 \in H^k_{\mathrm{div}}, \quad \theta_0 \in L^2, \quad \theta_1 \in H^k$$

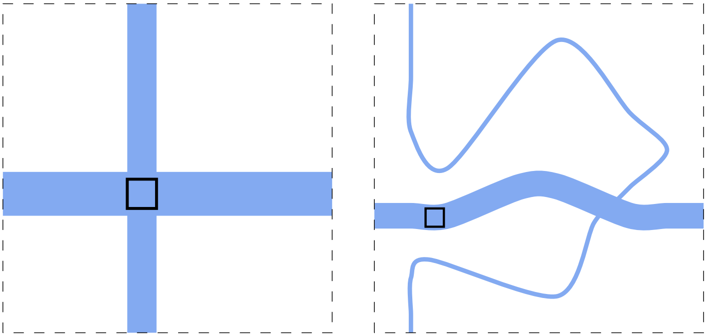
there exist controls

$$\xi \in C^\infty((0,T);\mathscr{F}_v), \quad \eta \in C^\infty((0,T);\mathscr{F}_t)$$

such that the solution to the Boussinesq system satisfies

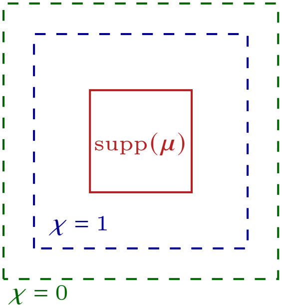
$$\|u(\cdot,T) - u_1\|_k + \|\theta(\cdot,T) - \theta_1\|_k < \varepsilon.$$

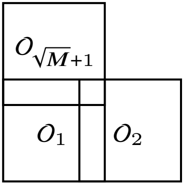
Special case: ω contains two cuts of the torus:



If a closed square of side-length L fits inside ω, our approach yields completely explicit representations of $\mathscr{F}_v$ and $\mathscr{F}_t$ with

$$\dim(\mathscr{F}_v) \le 2 + 18\lceil 2\pi/L\rceil^2 + 8\lceil 2\pi/L\rceil^4,$$
$$\dim(\mathscr{F}_t) \le 2 + 8\lceil 2\pi/L\rceil^2.$$


supp(μ)
$\chi = 1$
$\chi = 0$


$\mathcal{O}_{\sqrt{M}+1}$
$\mathcal{O}_1$
$\mathcal{O}_2$

We know already that

$$\partial_t v + (u^\star \cdot \nabla) v = g^\star$$

is approximately controllable with $\mathcal{H}_0$-valued (4-dimensional) control

$$g^\star(x,t) = \alpha_1(t)\sin(x_1) + \alpha_2(t)\sin(x_2) + \beta_1(t)\cos(x_1) + \beta_2(t)\cos(x_2).$$

Idea: construct a particular "finite-dimensional" return method trajectory $\overline{U}$ that allows controlling

$$\partial_t V + (\overline{U} \cdot \nabla) V = \mathbb{I}_\omega G$$

with control (roughly) of the form

$$G(x,t) = \mu(x) \sum_{i=1}^{M} \mathbb{I}_{[t_a^i, t_b^i]}(t) g^\star(x - S_i, t - t_a^i).$$

$\omega \subset \mathbb{T}^2$ arbitrarily fixed nonempty open set.

Lemma

There exists a D_ω-dimensional vector space $\mathscr{H}_\omega \subset C^\infty(\mathbb{T}^2;\mathbb{R}^2)$ and a vector field $\overline{y} = \overline{y}_\omega \in C_0^\infty((0,1);\mathscr{H}_\omega)$ with the properties

- $\forall h \in \mathscr{H}_\omega,\ \forall x \in \mathbb{T}^2:\ \mathrm{div}(h)(x) = 0.$
- $\forall h \in \mathscr{H}_\omega,\ \exists \varphi_h \in C^\infty(\mathbb{T}^2;\mathbb{R}),\ \forall x \in \mathbb{T}^2 \setminus \omega:\ h(x) = \nabla\varphi_h(x).$
- $\Phi^{\overline{y}}(\mathrm{Neighborhood}(\mathcal{O}_i), 0, [t_a^i, t_b^i]) = \mathrm{Neighborhood}(\mathcal{O}_i) + S_i.$

We take

$$\overline{U}(x,t) := \overline{y}(x,t) + \sum_{i=1}^{M} \mathbb{I}_{[t_a^i, t_b^i]}(t)\nabla^\perp[\chi(x)\phi^\star(x - S_i, t - t_a^i)],$$

where $\phi^\star$ is stream function of $u^\star$.

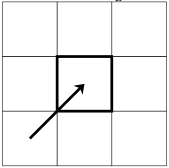
$u^\star$ on $[0, T^\star]$

G	**H**	**I**
D	**E**	**F**
A	**B**	**C**

$g^\star$ on $[0, T^\star]$

7	**8**	**9**
4	**5**	**6**
1	**2**	**3**

$\overline{U}$ on $[0, t_a^1]$



G on $[0, t_a^1]$

–	–	–
–	–	–
–	–	–

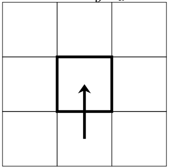
$\overline{U}$ on $[t_a^1, t_b^1]$

–	–	–
–	**A**	–
–	–	–

G on $[t_a^1, t_b^1]$

–	–	–
–	**1**	–
–	–	–

$\overline{U}$ on $[t_b^1, t_a^2]$



G on $[t_b^1, t_a^2]$

–	–	–
–	–	–
–	–	–

$\overline{U}$ on $[t_a^2, t_b^2]$

–	–	–
–	**B**	–
–	–	–

G on $[t_a^2, t_b^2]$

–	–	–
–	**2**	–
–	–	–

Theorem (R., 2025, arXiv:2506.19764)

There are finite-dimensional spaces

$$\mathscr{F}_v \subset C^\infty(\mathbb{T}^2;\mathbb{R}^2), \quad \mathscr{F}_t \subset C^\infty(\mathbb{T}^2;\mathbb{R})$$

such that for any given data

$$\nu, \tau, \varepsilon, T > 0, \quad m \in \mathbb{N}, \quad u_0 \in L^2_{\mathrm{div}}, \quad \theta_0 \in L^2, \quad \theta_1 \in H^m,$$

there exists $\delta_0 > 0$ *so that for each* $\delta \in (0, \delta_0)$ *there are controls*

$$\xi \in C^\infty((0,\delta);\mathscr{F}_v), \quad \eta \in C^\infty((0,\delta);\mathscr{F}_t)$$

for which the associated solution to the Boussinesq system satisfies

$$\|u(\cdot,\delta) - u_0\|_m + \|\theta(\cdot,\delta) - \theta_1\|_m < \varepsilon.$$

Vorticity-temperature formulation:

$$\partial_t w - \nu \Delta w + (u \cdot \nabla)\, w = \partial_1 \theta,$$
$$\partial_t \theta - \tau \Delta \theta + (u \cdot \nabla)\, \theta = 0.$$

One can prove the following scaling limits:

- If $(w, \theta)(\cdot, 0) = (w_0, \theta_0 - \delta^{-1}\xi)$, then

$$w(\cdot, \delta) \longrightarrow w_0 - \partial_1 \xi$$

as $\delta \longrightarrow 0$.

- If $(w, \theta)(\cdot, 0) = (w_0 + \delta^{-1/2}\xi, \theta_0)$, then

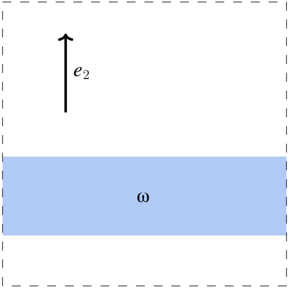
$$w(\cdot, \delta) - \delta^{-1/2}\xi \longrightarrow w_0 - (\nabla^{\perp}(-\Delta)^{-1}\xi \cdot \nabla)\xi$$

as $\delta \longrightarrow 0$.

Advertisements

V. Nersesyan, R., 2024 & R., 2024

Boussinesq system in $\mathbb{T}^2$ is approximately controllable using only a temperature control supported in a strip.



$$\partial_t u - \Delta u + (u \cdot \nabla)\, u + \nabla p = \theta e_2,$$
$$\nabla \cdot u = 0,$$
$$\partial_t \theta - \Delta \theta + (u \cdot \nabla)\theta = \mathbb{I}_\omega \xi.$$

Enhanced relaxation

Zero average solutions to advection-diffusion equation

$$\partial_t \phi - \Delta\phi + (v \cdot \nabla)\phi = 0,$$

with divergence-free v may decay faster in L^2 than solutions to diffusion equation

$$\partial_t \phi - \Delta\phi = 0.$$

Characterizations of good divergence-free v are available: *e.g.*,

- P. Constantin et al., 2008 (Ann. Math.)
- A. Kiselev et al., 2008 (Indiana Univ. Math. J.)

Question: Can one take v as a solution to a (controlled) fluid equation?

Incompressible Euler system with degenerate (four-dimensional) control:

$$\partial_t v + (v \cdot \nabla) v + \nabla p = \gamma_1(t) \begin{bmatrix} \sin(x_2) \\ 0 \end{bmatrix} + \gamma_2(t) \begin{bmatrix} \cos(x_2) \\ 0 \end{bmatrix} + \gamma_3(t) \begin{bmatrix} 0 \\ \sin(x_1) \end{bmatrix} + \gamma_4(t) \begin{bmatrix} 0 \\ \cos(x_1) \end{bmatrix}$$

Passive scalar advection-diffusion equation:

$$\partial_t \phi - \Delta \phi + (v \cdot \nabla) \phi = 0,$$
$$\phi(\cdot, 0) = \phi_0.$$

Theorem (K. Koike, V. Nersesyan, R., M. Tucsnak; arXiv:2506.22233)

For every $\tau, \delta > 0$, there exist $\gamma_1, \ldots, \gamma_4 \in L^2([0,\tau];\mathbb{R})$ such that

$$\|\phi(\tau, \cdot)\|_{L^2(\mathbb{T}^2)} < \delta$$

for every $\phi_0 \in L^2(\mathbb{T}^2)$ with $\|\phi\|_{L^2(\mathbb{T}^2)} \leqslant 1$ and $\int_{\mathbb{T}^2} \phi(x)\, dx = 0$.

Thank you!