Carleman-based reconstruction algorithm on a wave network

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Control of PDES and related topics

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Coefficient inverse problem in the wave equation

In a smooth bounded domain $\Omega \subset \mathbb{R}^n$, it writes for instance,

$$\begin{cases} \partial_{tt} y(t,x) - \Delta_x y(t,x) + p^*(x) y(t,x) = f(t,x), & (t,x) \in (0,T) \times \Omega, \\ y(t,x) = g(t,x), & (t,x) \in (0,T) \times \partial\Omega \\ (y(0,x), \partial_t y(0,x)) = (y^0(x), y^1(x)), & x \in \Omega. \end{cases}$$

- Given data: Source terms f, g ; initial data: (y^0, y^1);
- Unknown: the potential $p^* = p^*(x)$;
- Additional measurement : the flux $\partial_\nu y(t,x)$ on $(0,T) \times \partial\Omega$.

Motivation

- The determination in Ω of p^* from an additional measurement are inverse problems for which *uniqueness* and *stability* are well-known and proved using Carleman estimates.
- Classical reconstruction : from the measurement $d^* = \partial_\nu y[p^*]$, calculate

  $$\min J(p) = \frac{1}{2}\|\partial_\nu y[p] - d^*\|^2.$$

  But J is not convex and may have several local minima, so that the solution will depend on the initialization p_0. Algorithms not guaranteed to converge to the global minimum.
- Klibanov, Beilina and co-authors have worked a lot on related questions...

The Carleman-based reconstruction algorithm

- First goal : compute the PDE unknown coefficient with convergence estimates and no a priori first guess.
- Core idea : build a reconstruction algorithm (C-bRec)
  - from the appropriate Carleman estimates to build the cost functional;
  - using the structure of the proof of stability to prove the global convergence.
- Until now, the idea was applied to three reconstruction cases:
  - potential / wave speed in the wave equation ([Baudouin, de Buhan, Ervedoza 2013, 2017], [Baudouin, de Buhan, Ervedoza, Osses 2021]);
  - source term in a non linear heat equation by [Boulakia, de Buhan, Schwindt, 2020].

Outline

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Determination of the potential in the wave equation

$$\begin{cases} \partial_{tt} y - \Delta y + p^* y = f, & (0,T)\times\Omega, \\ y = g, & (0,T)\times\partial\Omega \\ (y(0), \partial_t y(0)) = (y^0, y^1), & \Omega. \end{cases}$$

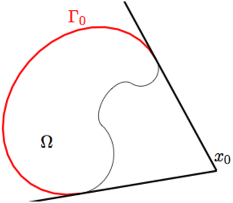
Is it possible to retrieve the potential $p^* = p^*(x)$, $x \in \Omega$ *from measurement of the flux* $d^* = \partial_\nu y[p^*](t,x)$ *on* $(0,T)\times\Gamma_0$ *?*

- Uniqueness: Given $p_1 \neq p_2$, can we guarantee $\partial_\nu y[p_1] \neq \partial_\nu y[p_2]$?
- Stability: If $\partial_\nu y[p_1] \simeq \partial_\nu y[p_2]$, can we guarantee that $p_1 \simeq p_2$?
- Reconstruction: Given $d^* = \partial_\nu y[p^*]$, can we compute p^* ?

• Known results: Uniqueness ([Klibanov 92], stability ([Yamamoto 99], [Imanuvilov, Yamamoto 01]), using Carleman estimates.

• Main question: Reconstruction : how to compute the potential from the boundary measurement ?

Stability Result ([Yamamoto 99], [Baudouin, Puel 01])



Let $x_0 \in \mathbb{R}^N \setminus \Omega$ and let Γ_0 and T satisfy

$$\{x \in \partial\Omega,\ (x - x_0)\cdot \nu(x) > 0\} \subset \Gamma_0 \quad ; \quad T > \sup_{x\in\Omega}\{|x - x_0|\}.$$

Let the potential p, the initial data y^0 and the solution $y[p]$ s.t.

$$\|p\|_{L^\infty(\Omega)} \leq m, \quad \inf_{x\in\Omega}\{|y^0(x)|\} \geq \gamma > 0, \quad y[p] \in H^1(0,T;L^\infty(\Omega)).$$

Then, one can prove uniqueness and local Lipschitz stability of the inverse problem for the wave equation: $\forall q \in L^\infty_{\leq m}(\Omega)$,

$$\|p - q\|_{L^2(\Omega)} \leq C\|\partial_\nu y[p] - \partial_\nu y[q]\|_{H^1((0,T);L^2(\Gamma_0))}.$$

Towards a (re)constructive approach

The idea is considering p^* as the fix point of a contracting application

$\rightsquigarrow$ construct a sequence $(q^k)_{k\in\mathbb{N}}$ converging towards p^*.

Based on the Bukhgeim-Klibanov method, it is easy to check that $Z = \partial_t \left(y[q^k] - y[p^*]\right)$ satisfies

$$\begin{cases} \partial_{tt} Z - \Delta_x Z + q^k(x) Z = (p^* - q^k)\partial_t y[p^*] =: h, & (t,x) \in (0,T)\times\Omega, \\ Z(t,x) = 0, & (t,x)\in(0,T)\times\partial\Omega \\ (Z(0,x), \partial_t Z(0,x)) = (0, (p^* - q^k) y^0), & x \in \Omega. \end{cases}$$

One should notice that Z was built to be the unique minimizer of the functional

$$J_h^k(z) = \int_0^T\int_\Omega e^{2s\varphi}|\partial_{tt} z - \Delta_x z + q^k(x) z - h|^2 + s\int_0^T\int_{\Gamma_0} e^{2s\varphi}|\partial_\nu z - \mu^k|^2,$$

where $\mu^k = \partial_t \left(\partial_\nu y[q^k] - \partial_\nu y[p^*]\right)$ on $\Gamma_0 \times (0,T)$.Then

$$p^* = q^k + \frac{\partial_t Z(0)}{y^0}$$

Be careful: h is unknown.

Idea: minimize another functional J_0^k associated to $h = 0$.

Carleman estimate [Baudouin, de Buhan, Ervedoza 13]

Assuming $q \in L^\infty_{\leq m}(\Omega)$, $L_q = \partial_{tt} - \Delta_x + q(x)$, $\varphi(t,x) = e^{\lambda(|x-x_0|^2 - \beta t^2)}$

$$\{x \in \partial\Omega,\ (x - x_0)\cdot\nu(x) > 0\} \subset \Gamma_0\ , \qquad \sup_{x\in\Omega} |x - x_0| < \beta T$$

$\exists s_0 > 0$, $\lambda > 0$ and $M = M(s_0, \lambda, T, \beta, x_0, m) > 0$ such that

$$s \int_0^T\!\!\int_\Omega e^{2s\varphi} \left(|\partial_t w|^2 + |\nabla w|^2 + s^2|w|^2\right) dxdt + s^{1/2} \int_\Omega e^{2s\varphi(0)} |\partial_t w(0)|^2\, dx$$
$$\leq\ M \int_0^T\!\!\int_\Omega e^{2s\varphi} |L_q w|^2\, dxdt + Ms \int_0^T\!\!\int_{\Gamma_0} e^{2s\varphi}\, |\partial_\nu w|^2\ d\sigma dt,$$

for all $s > s_0$ and $w \in L^2(-T,T; H^1_0(\Omega))$ satisfying

$$\begin{cases} L_q w \in L^2(\Omega \times (0,T)) \\ \partial_\nu w \in L^2((0,T)\times\Gamma_0), \\ w(0,x) = 0,\ \forall x \in \Omega. \end{cases}$$

$\rightsquigarrow$ but also Imanuvilov, Zhang, Klibanov,...

Carleman based Reconstruction Algorithm

Initialization: $q^0 = 0$ or any initial guess.
Iteration: Given q^k,
1 - Compute $y[q^k]$ the solution of

$$\begin{cases} \partial_t^2 y - \Delta y + q^k y = f, & \text{in } \Omega \times (0,T), \\ y = g, & \text{on } \partial\Omega \times (0,T), \\ y(0) = y_0, \quad \partial_t y(0) = y_1, & \text{in } \Omega, \end{cases}$$

and set $\mu^k = \partial_t \left(\partial_\nu y[q^k] - \partial_\nu y[p^*]\right)$ on $\Gamma_0 \times (0,T)$.

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2 - Introduce the functional

$$J_0^k(z) = \int_0^T \int_\Omega e^{2s\varphi} |L_{q^k} z|^2 + s \int_0^T \int_{\Gamma_0} e^{2s\varphi} |\partial_\nu z - \mu^k|^2,$$

on the space
$\mathcal{T}^k = \{z \in L^2(0,T; H_0^1(\Omega)), z(t=0) = 0,$
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Theorem

Assume some geometric and time conditions. Then, $\forall s > 0$ and $k \in \mathbb{N}$, the functional J_0^k is continuous, strictly convex and coercive on $\mathcal{T}^k$ endowed with a suitable weighted norm.

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3 - Let Z^k be the unique minimizer of the functional J_0^k, and then set

$$\tilde{q}^{k+1} = q^k + \frac{\partial_t Z^k(0)}{y_0}$$

where y_0 is the initial condition.

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where y_0 is the initial condition.
4 - Finally, set $q^{k+1} = T_m(\tilde{q}^{k+1})$
where

$$T_m(q) = \begin{cases} q, & \text{if } |q| \leq m, \\ \text{sign}(q) m, & \text{if } |q| \geq m. \end{cases}$$

Algorithm's convergence [Baudouin, de Buhan & Ervedoza 13]

Theorem

Assuming the geometric and time conditions (among others), there exists a constant $M > 0$ such that $\forall s \geq s_0(m)$ and $k \in \mathbb{N}$,

$$\int_\Omega e^{2s\varphi(0)} (q^{k+1} - p^*)^2 \, dx \leq \frac{M}{\sqrt{s}} \int_\Omega e^{2s\varphi(0)} (q^k - p^*)^2 \, dx.$$

In particular, when s is large enough, the algorithm converges.

Remark : Convergence to the global minimum from any initial guess.

Proof

As proposed earlier, let us set $v^k = \partial_t \left(y[q^k] - y[p^*] \right)$ that solves

$$\begin{cases} \partial_t^2 v - \Delta v + q^k v = f^k, & \text{in } \Omega \times (0,T), \\ v = 0, & \text{on } \partial\Omega \times (0,T), \\ v(0) = 0, \quad \partial_t v(0) = (p^* - q^k) y^0, & \text{in } \Omega, \end{cases}$$

where $f^k = (p^* - q^k)\partial_t y[p^*]$.

By definition, $\mu^k = \partial_\nu v^k$ on $\Gamma_0 \times (0,T)$, and we notice that v^k is the unique minimizer of the functional:

$$J_h^k(w) = \int_0^T \int_\Omega e^{2s\varphi} |L_{q^k} w - f^k|^2 + s \int_0^T \int_{\Gamma_0} e^{2s\varphi} |\partial_\nu w - \mu^k|^2,$$

on the space $\mathcal{T}^k = \{ w \in L^2(0,T; H_0^1(\Omega)), w(t=0) = 0,$
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Proof II

Let us write the Euler Lagrange equations satisfied by:
Z^k minimizer of J_0^k

$$\int_0^T\int_\Omega e^{2s\varphi} L_{q^k} Z^k \, L_{q^k} w + s\int_0^T\int_{\Gamma_0} e^{2s\varphi}(\partial_\nu Z^k - \mu^k)\partial_\nu w = 0,$$

and v^k minimizer of J_h^k

$$\int_0^T\int_\Omega e^{2s\varphi}(L_{q^k} v^k - f^k) L_{q^k} w + s\int_0^T\int_{\Gamma_0} e^{2s\varphi}(\partial_\nu v^k - \mu^k)\partial_\nu w = 0,$$

for all $w \in \mathcal{T}^k$. Applying these to $w = Z^k - v^k$ and subtracting the two identities, we obtain:

$$\int_0^T\int_\Omega e^{2s\varphi}|L_{q^k} w|^2 + s\int_0^T\int_{\Gamma_0} e^{2s\varphi}|\partial_\nu w|^2 = \int_0^T\int_\Omega e^{2s\varphi} f^k \, L_{q^k} w,$$

implying ($2ab \leq a^2 + b^2$)

$$\frac{1}{2}\int_0^T\int_\Omega e^{2s\varphi}|L_{q^k} w|^2 + s\int_0^T\int_{\Gamma_0} e^{2s\varphi}|\partial_\nu w|^2 \leq \frac{1}{2}\int_0^T\int_\Omega e^{2s\varphi}|f^k|^2.$$

Proof III

The *LHS* is precisely the *RHS* of the Carleman estimate. Hence:

$$s^{1/2}\int_\Omega e^{2s\varphi(0)}|\partial_t w(0)|^2\,dx \leq M\int_0^T\int_\Omega e^{2s\varphi}|f^k|^2\,dxdt,$$

where $\partial_t w(0) = \partial_t Z^k(0) - \partial_t v^k(0)$. Moreover,

$$\partial_t Z^k(0) = (\tilde{q}^{k+1}-q^k)y^0, \quad \partial_t v^k(0) = (p^*-q^k)y^0, \quad f^k = (p^*-q^k)\partial_t y[p^*].$$

Therefore, since $\varphi(t)\leq\varphi(0)$ for all $t\in(0,T)$ we have:

$$s^{1/2}\int_\Omega e^{2s\varphi(0)}|y^0|^2|\tilde{q}^{k+1}-p^*|^2\,dx \leq M\|\partial_t y[p^*]\|^2_{L^2(0,T;L^\infty(\Omega))}\int_\Omega e^{2s\varphi(0)}|q^k-p^*|^2\,dx.$$

Using the positivity condition on y^0 and the fact that

$$|q^{k+1}-p^*| = |T_m(\tilde{q}^{k+1})-T_m(p^*)| \leq |\tilde{q}^{k+1}-p^*|$$

because T_m is Lipschitz and $T_m(p^*)=p^*$, we can deduce

$$\int_\Omega e^{2s\varphi(0)}(q^{k+1}-p^*)^2\,dx \leq \left(\frac{M}{\sqrt{s}}\right)^{k+1}\int_\Omega e^{2s\varphi(0)}(q^0-p^*)^2\,dx. \qquad \square$$

In theory, it works. But in practice ?

Two remarks:

- Discretizing the wave equation brings numerical artefacts...
- Minimizing a strictly convex and coercive quadratic functional based on a Carleman estimate means dealing with $e^{2se^{\lambda\psi}}$ for large parameters s and λ...

New goal: propose a numerically efficient algorithm.

Ideas: We need an algorithm constructed with at least

- a regularization term in the cost functional,
- a single parameter Carleman estimate.

$\rightsquigarrow$ [Baudouin, de Buhan, Ervedoza 2017]

New C-bRec algorithm [Baudouin, de Buhan, Ervedoza 2017]

The algorithm is also modified according to the following items :

- Single parameter Carleman estimate ;

  $\rightsquigarrow$ presence of an additional term on the right
  $$s^3 \int\int_{\mathcal{O}} e^{2s\varphi}|z|^2$$

- Preconditioning of the cost functional ;

  $\rightsquigarrow$ introduce the conjugate variable $y = e^{s\varphi} z$

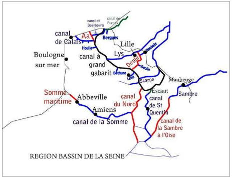
- Splitting of the observations by cut-off ;

  $$\rightsquigarrow v^k = \eta^\varphi \partial_t (y[q^k] - y[p^*])$$

... and the convergence result remains the same.

Outline

PDE on networks



Applications :

- control or stabilize the vibrations of elastic structures (as bridges, cranes,...),
- regulate the height of water in networks of irrigation canals,
- find the topography of the bottom in a network of irrigation canals,
- detect water losses by measurements in nodes,
- control gas flow in pipelines through compressors,
- determine the blood pressure leaving the heart with a finger pressure measurement,
- control road traffic on a network of roads or the flow of blood in a network of arteries,...

PDE on networks

On networks, the state is represented by several components

$$Z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_N(t) \end{bmatrix}$$

and the components are coupled together by boundary conditions.
If $p < N$ is the number of controls/observations, it is therefore necessary to pass the information on the remaining $N - p$ branches.
Goals:

- minimize the number of observations, feedbacks or controls,
- choice of placement of observations, feedback mechanisms or controls based on network topology and branch lengths.

An inverse problem on network

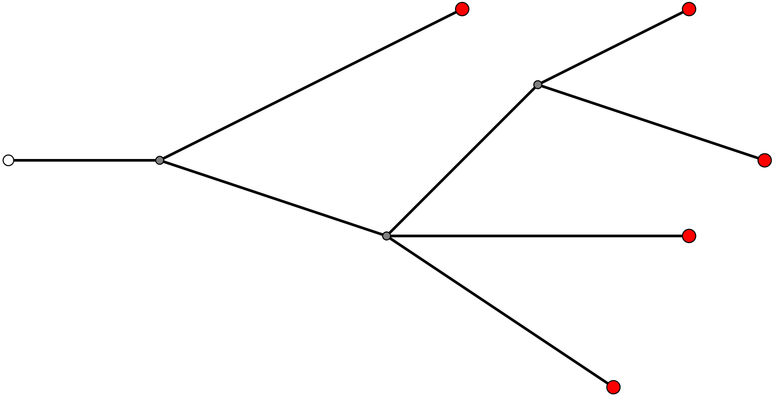


Figure: An 8 branches tree-shaped network $\mathcal{R}$, with an unobserved root node and 5 observed leaf nodes $\bullet$.

Notations

Let us thus consider a finite tree-shaped network $\mathcal{R}$.

- $\mathcal{J}$: the set of names of all branches of the network.
- We define the name of the branches by recurrence:
  - To the root branch, named 1, we associate its N_1 children branches denoted by $1_i \in \mathbb{N}$ for $i = 1..N_1$.
  - From a branch named $j \in \mathcal{J}$ we define the names of its N_j children branches by j_i for $i = 1..N_j$.
- ℓ_j: the length of the branch j.
- $\mathcal{J}_{ext} = \{j \in \mathcal{J}, N_j = 0\}$.
- $\mathcal{J}_{int} = \{j \in \mathcal{J}, N_j > 0\}$.
- f_j: the restriction of the function f on $\mathcal{R}$ to the branche j.
- $\displaystyle \int_{\mathcal{R}} f(x)dx := \sum_{j \in \mathcal{J}} \int_0^{\ell_j} f_j(x)dx,$
- $\displaystyle [f]_j := f_j(\ell_j) - \sum_{i=1}^{N_j} f_{j_i}(0), \quad \forall j \in \mathcal{J}_{int}.$

An inverse problem on network

On each branch $j \in \mathcal{J}$ of the network, we consider the one-dimensional wave equation system

$$\begin{cases} \partial_{tt} u_j(t,x) - \partial_{xx} u_j(t,x) + p_j(x) u_j(t,x) = g_j(t,x), & (t,x) \in (0,T) \times (0,\ell_j), \\ u_j(0,x) = u_j^0(x), \qquad \partial_t u_j(0,x) = u_j^1(x), & x \in (0,\ell_j), \end{cases}$$

with

$$\begin{cases} \text{for } j = 1, & u_1(t,0) = h_1(t), \\ \text{if } j \in \mathcal{J}_{ext}, & u_j(t,\ell_j) = h_j(t), \\ \text{if } j \in \mathcal{J}_{int}, & u_j(t,\ell_j) = u_{j_i}(t,0), \ \forall i \in \{1, \cdots, N_j\}, \\ & [\partial_x u]_j(t) = 0, \end{cases}$$

Inverse problem on a network

Inverse problem

Knowing, for each branch $j \in \mathcal{J}$, the source term g_j and the initial data (u_j^0, u_j^1), for the root and for each leaf $j \in \{1\} \cup \mathcal{J}_{ext}$ the boundary source term h_j, is it possible to identify the unknown potentials $p_j^*(x)$ for any $x \in (0, \ell_j)$, from the only extra knowledge of the flux of the solutions through the leaf nodes of the network, meaning:

$$d_i^*(t) = \partial_x u_i^*(t, \ell_i), \quad \text{for } i \in \mathcal{J}_{ext} \text{ and } t \in (0, T),$$

where u_i^* is the solution associated to potential p_i^*?

Lipschitz stability result [Baudouin, Crépeau, V. 2011]

Theorem

There exist a time $T_0 > 0$ *and a scalar* $\alpha_0 > 0$ *such that if*

1. *Time condition:* $T > T_0$,
2. *Regularity condition:* $u \in H^1(0, T; L^\infty(\mathcal{R}))$,
3. *Sign condition:* $|u^0| \geq \alpha^0 > 0$ *on the whole network* $\mathcal{R}$,

then for a fixed $m > 0$, *there exists a positive constant* $C = C(\mathcal{R}, T, m)$ *such that, if* p *and* p^* *belong to* $L^\infty_m(\mathcal{R}) = \{p \in L^\infty(\mathcal{R}), \|p\|_{L^\infty(\mathcal{R})} \leq m\}$, *we have*

$$\|p - p^*\|^2_{L^2(\mathcal{R})} \leq C \sum_{i \in \mathcal{J}_{ext}} \|\partial_x u_i(\cdot, \ell_i) - \partial_x u_i^*(\cdot, \ell_i)\|^2_{H^1(0,T)}.$$

Proof: based on the Bukhgeim and Klibanov method and a two parameters Carleman estimate.

Carleman weight function φ

$$\forall j \in \mathcal{J},\ \varphi_j(t,x) = (x - x_j)^2 - \beta t^2 + M_j, \quad (t,x) \in \mathbb{R} \times (0, \ell_j).$$

There exist $(x_j)_{j\in\mathcal{J}} \in \mathbb{R}^-$, $(M_j)_{j\in\mathcal{J}} \in \mathbb{R}^+$, $\beta \in (0,1)$ and $T > 0$ satisfying

$$\beta T > \sup_{j\in\mathcal{J}} (\ell_j - x_j)$$

such that it holds

(i) $\forall j \in \mathcal{J}_{int}$, $\varphi_{j_i}(t,0) = \varphi_j(t,\ell_j)$, $\quad \forall i \in \{1, \cdots N_j\}$.

(ii) The matrices $A_j^\varphi(t)$ satisfy for any $j \in \mathcal{J}_{int}$: $\exists \alpha_j^0 > 0, \beta_j > 0$, $\forall \xi \in \mathbb{R}^{N_j+1}$,

$$\begin{array}{lll} (A_j^\varphi(t)\xi,\xi) \geq \alpha_j^0 \|\xi\|^2, & \forall t, & |t| \leq T_j := \frac{\ell_j - x_j}{\beta}; \\ (A_j^\varphi(t)\xi,\xi) \geq \alpha_j^0 \|\xi\|^2 - \beta_j |\xi_{N_j+1}|^2, & \forall t, & T_j \leq |t| \leq T; \end{array}$$

where $A_j^\varphi(t)$ are $(N_j+1)\times(N_j+1)$ symmetric matrices defined by

$$A_j^\varphi(t) := \begin{pmatrix} \phi_{j_1}(0) - \phi_j(\ell_j) & -\phi_j(\ell_j) & \cdots & -\phi_j(\ell_j) & -\phi_j(\ell_j)[\phi]_j \\ & \ddots & \ddots & \vdots & \vdots \\ & & \ddots & -\phi_j(\ell_j) & \vdots \\ & & & \phi_{j_{N_j}}(0) - \phi_j(\ell_j) & -\phi_j(\ell_j)[\phi]_j \\ & & & & a_j(t) \end{pmatrix}$$

with $\phi(x) := \partial_x \varphi(t,x)$ and $a_j(t) = -\phi_j(\ell_j)[\phi]_j^2 + \left[(|\partial_t \varphi(t)|^2 - |\phi|^2)\phi\right]_j$.

First tool: one-parameter Carleman estimate [Baudouin, de Buhan, Crépeau, V. 2025]

Theorem

There exist $C > 0$, $s_0 > 0$ *such that for all* $s \geq s_0$, *for all* $p \in L^\infty_m(\mathcal{R})$,

$$s^{1/2} \int_{\mathcal{R}} e^{2s\varphi(0,x)} |\partial_t z(0,x)|^2 dx + s \int_{-T}^{T} \int_{\mathcal{R}} e^{2s\varphi} \left(|\partial_t z|^2 + |\partial_x z|^2 + s^2 |z|^2 \right) dxdt$$

$$\leq C \int_{-T}^{T} \int_{\mathcal{R}} e^{2s\varphi} |\partial_{tt} z - \partial_{xx} z + pz|^2 dxdt$$

$$+ Cs \sum_{i \in \mathcal{J}_{ext}} \int_{-T}^{T} e^{2s\varphi_i(t,\ell_i)} |\partial_x z_i(t,\ell_i)|^2 dt + Cs^3 \mathcal{I}(z,z),$$

satisfied by all $z \in H^1((-T,T); H^1_0(\mathcal{R}))$ *s.t.* $\partial_{tt} z - \partial_{xx} z \in L^2((0,T) \times \mathcal{R})$, *under Kirchhoff node condition and* $z(0,\cdot) = 0$ *in* $\mathcal{R}$, *and where*

$$\mathcal{I}(z,z) = \iint_{(|t|,x) \in \mathcal{O}} e^{2s\varphi} |z|^2 dxdt + \sum_{j \in \mathcal{J}_{int}} \int_{|t| \in \mathcal{O}_{T_j}} e^{2s\varphi_j(t,\ell_j)} |z_j(t,\ell_j)|^2 dt$$

with $\mathcal{O} = \cup_{j \in \mathcal{J}} \mathcal{O}_j$ *where* $\mathcal{O}_j = \{(t,x) \in (0,T) \times (0,\ell_j), |x - x_j| - \beta|t| < 0\}$ *and* $\mathcal{O}_{T_j} = \{t \in (0,T), |\ell_j - x_j| - \beta|t| < 0\}$ *defined only for* $x = \ell_j$, $j \in \mathcal{J}_{int}$.

The domains $\mathcal{O}_j$ and $\mathcal{O}_{T_j}$

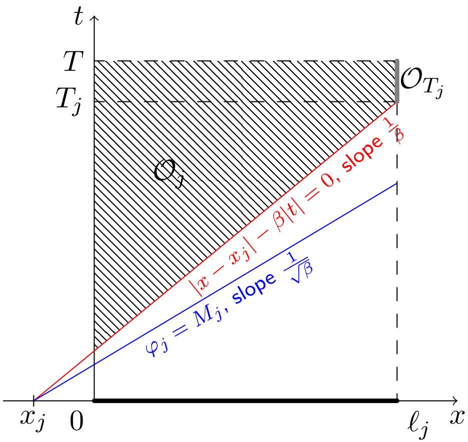


Figure: Illustration of domains $\mathcal{O}_j$ and $\mathcal{O}_{T_j}$ for the branch $(0, \ell_j)$, denoting $T_j = |l_j - x_j|/\beta$.

Second tool: properties of the cut-off function η^φ

$v^k = \eta^\varphi \partial_t \left(u^k - u^*\right)$ (with $\eta^\varphi \in C^2((0,T)\times\mathcal{R})$) is solution of

$$\begin{cases} \partial_{tt} v^k(t,x) - \partial_{xx} v^k(t,x) + p^k(x) v^k(t,x) = f^k(t,x), & \text{in } (0,T)\times\mathcal{R}, \\ v^k(0,x) = 0, \quad \partial_t v^k(0,x) = \eta^\varphi(0,x)(p^*(x) - p^k(x))u^0(x), & \text{in } \mathcal{R}, \end{cases}$$

where $f^k := \eta^\varphi(p^* - p^k)\partial_t u^* - [\eta^\varphi, \partial_{tt} - \partial_{xx}]\partial_t \left(u^k - u^*\right)$.
v^k satisfies also the continuity and the Kirchhoff law at the internal nodes, and the Dirichlet boundary condition at the external nodes.

v^k is built to be the unique minimizer of the functional

$$F_s[p^k, f^k, \mu^k](z) = \frac{1}{2}\int_0^T \int_{\mathcal{R}} e^{2s\varphi} |\partial_{tt} z - \partial_{xx} z + p^k z - f^k|^2 \, dx dt + \frac{s}{2} \sum_{i\in\mathcal{J}_{ext}} \int_0^T e^{2s\varphi_i(t,\ell_i)} |\partial_x z_i(t,\ell_i) - \mu_i^k(t)|^2 dt + \frac{s^3}{2}\mathcal{I}(z,z),$$

where we set, for all $i \in \mathcal{J}_{ext}$, $\mu_i^k(t) = \eta_i^\varphi(t,\ell_i)\partial_t \left(\partial_x u_i^k(t,\ell_i) - d_i^*(t)\right)$.

Properties expected from v^k

- Encoding $(p^k - p^*)$, which is the information we seek, through the initial speed data $\partial_t v^k(0,\cdot) = \eta^\varphi(0,\cdot)(p^* - p^k)u^0$

$$\rightsquigarrow \eta_j^\varphi(0,\cdot) = 1.$$

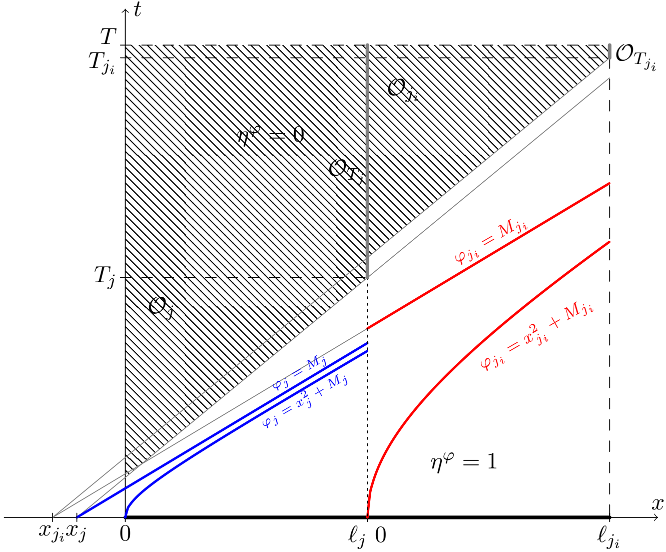
- Vanishing in the domains $\mathcal{O}$ and $\mathcal{O}_{T_j}$ so that $\mathcal{I}(v^k, v^k) = 0$

  $\rightsquigarrow \eta_j^\varphi = 0$ on some domain greater than $\mathcal{O} \cup \left(\cup_{j \in \mathcal{J}_{int}} \mathcal{O}_{T_j} \times \{\ell_j\}\right)$.

- Allowing the source term f^k solved by v^k to be manageable. We will ask for η^φ to vary (between 0 and 1) only in a small region of $(0,T) \times \mathcal{R}$. Actually, on each $(0,T) \times (0,\ell_j)$, it will be specifically possible (meaning *manageable*) where $M_j < \varphi_j < x_j^2 + M_j$.

- But it also has to be done properly across each internal node to ensure continuity and Kirchhoff law for v^k at those nodes.

Context of application of the cut-off functions η^φ over two consecutive branches j and j_i.



Third tool: properties of the cost functional F_s

Lemma

For all $s > 0$ large enough, $p \in L^\infty(\mathcal{R})$, $f \in L^2(0,T;L^2(\mathcal{R}))$ and $\mu \in L^2(0,T)$, the functional $F_s[p,f,\mu]$ recalled here

$$F_s[p,f,\mu](z) = \frac{1}{2}\int_0^T \int_{\mathcal{R}} e^{2s\varphi}|\partial_{tt}z - \partial_{xx}z + pz - f|^2\,dxdt$$
$$+ \frac{s}{2}\sum_{i\in\mathcal{J}_{ext}} \int_0^T e^{2s\varphi_i(t,\ell_i)}|\partial_x z_i(t,\ell_i) - \mu_i(t)|^2 dt + \frac{s^3}{2}\mathcal{I}(z,z),$$

is continuous, strictly convex and coercive on $\mathcal{T}$ defined by

$$\mathcal{T} = \Big\{ z \in C^0([0,T];H_0^1(\mathcal{R})) \cap C^1([0,T];L^2(\mathcal{R})), \partial_{tt}z - \partial_{xx}z \in L^2((0,T)\times\mathcal{R}),$$
$$z(0,\cdot) = 0 \text{ in } \mathcal{R}, \quad \text{and } [\partial_x z]_j(t) = 0,\ \forall j \in \mathcal{J}_{int}, t \in (0,T) \Big\}$$

and equipped with an appropriate weighed norm.
Thenceforth, the functional $F_s[p,f,\mu]$ admits a unique minimizer on the set $\mathcal{T}$.

The C-bRec algorithm on a network

Knowing, for each branch $j \in \mathcal{J}$, g_j, h_j and (u_j^0, u_j^1), we have the extra measured information at the leaves of the network $\mathcal{R}$:

$$d_i^*(t) = \partial_x u_i^*(t, \ell_i), \text{ for } i \in \mathcal{J}_{ext} \text{ and } t \in (0,T).$$

Initialisation: Choose any initial guess $p^0 \in L_m^\infty(\mathcal{R})$.
Iteration: Knowing $p^k \in L_m^\infty(\mathcal{R})$,

1. Calculate the solution u^k associated to p^k, and set

$$\forall i \in \mathcal{J}_{ext}, \forall t \in (0,T), \quad \mu_i^k(t) = \eta_i^\varphi(t, \ell_i)\partial_t \left(\partial_x u_i^k(t, \ell_i) - d_i^*(t)\right).$$

2. Minimize the functional $F_s[p^k, 0, \mu^k]$ defined by on the space $\mathcal{T}$ and denote w^k its unique minimizer.
3. Then set

$$\tilde{p}^{k+1} = p^k + \frac{\partial_t w^k(0,\cdot)}{u^0}, \quad \text{on } \mathcal{R}.$$

4. Finally, construct

$$p^{k+1} = T_m(\tilde{p}^{k+1}) := \begin{cases} \tilde{p}^{k+1}, & \text{if } |\tilde{p}^{k+1}| \leq m, \\ \text{sign}(\tilde{p}^{k+1})m, & \text{if } |\tilde{p}^{k+1}| > m. \end{cases}$$

Stopping criterion: Choose $\epsilon > 0$ and $K \in \mathbb{N}^*$ and stop the iterative loop as soon as

$$\sup_{j \in \mathcal{J}_{ext}} \frac{\left\|\partial_x u_i^k(t, \ell_i) - d_j^*\right\|_2}{\|d_j^*\|_2} \leq \epsilon, \quad \text{or} \quad \sup_{j \in \mathcal{J}} \frac{\|p_j^{k+1} - p_j^k\|_\infty}{m} \leq \epsilon,$$

or when the maximal number of iterations K is reached.

Convergence result [Baudouin, de Buhan, Crépeau, V. 25]

Theorem

Assume that $p^* \in L^\infty_m(\mathcal{R})$. Then there exists a constant $C > 0$ such that for all s large enough and for all $k \in \mathbb{N}$, it holds

$$\int_{\mathcal{R}} e^{2s\varphi(0)} |p^k - p^*|^2 \, dx \leq \left(\frac{C}{s^{1/2}} \right)^k \int_{\mathcal{R}} e^{2s\varphi(0)} |p^0 - p^*|^2 \, dx.$$

In particular, if s is large enough, the sequence $(p^k)_{k \in \mathbb{N}}$ given by the algorithm converges towards p^* when k tends to infinity.

Numerical example

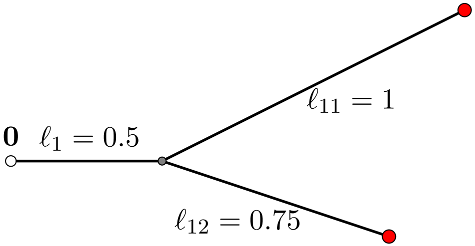


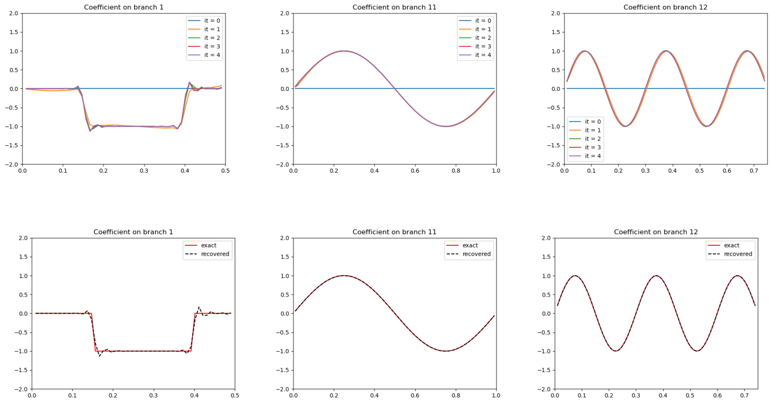
Figure: First setting - a 3 branches network, with observations at •.

Numerical values

u_0	u_1	g	h	m
(2,2,2)	(0,0,0)	(0,0,0)	(2,2,2)	2
ℓ_j	β	s	ϵ_1	ϵ_2
(0.5,1,0.75)	0.99	1	10^{-3}	10^{-2}
x_j	M_j	T	N_{xj}	N_t
(-0.3,-2.89,-2.89)	(7.71,0,0)	3.9	$100 * \ell_j$	$110 * T$

Table: Numerical values of the variables used for all the numerical examples.

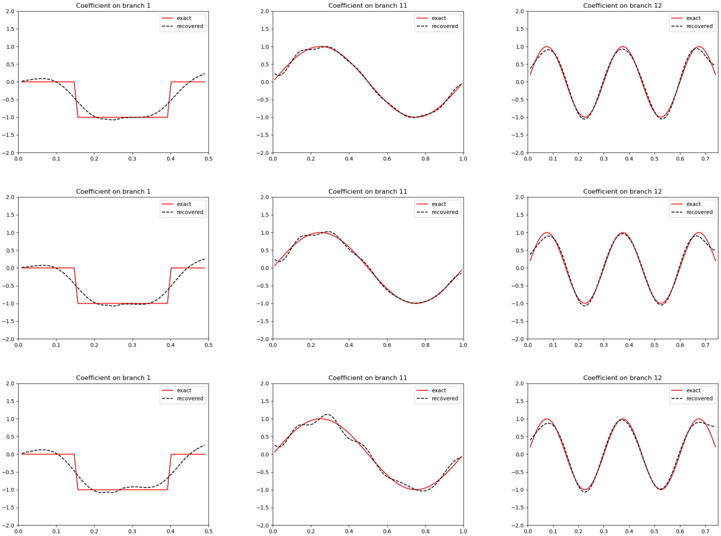
Simulations from data without noise



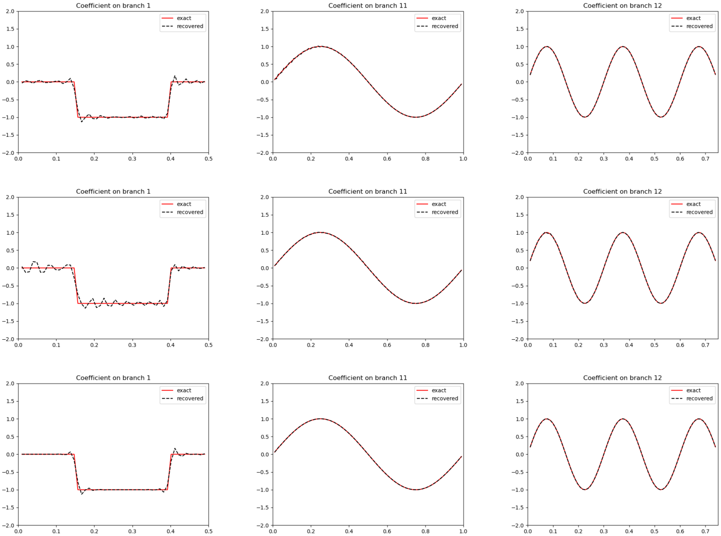
(a) $p_1^*(x) = -1_{[0.3,0.8]}(x/\ell_1)$ (b) $p_{11}^*(x) = \sin(2\pi x/\ell_{11})$ (c) $p_{12}^*(x) = \sin(5\pi x/\ell_{12})$

Figure: Top line: Convergence history of the reconstruction process. Bottom line: final reconstruction result (dotted black line) and exact coefficient (red line) for the three branches.

Simulations with several levels of noise: $\theta = 1\%$, $\theta = 2\%$, $\theta = 5\%$ noise in the data



Wrong choices of the parameters: $T = 1.5$, $T = 1.25$, without projection



A more complex network

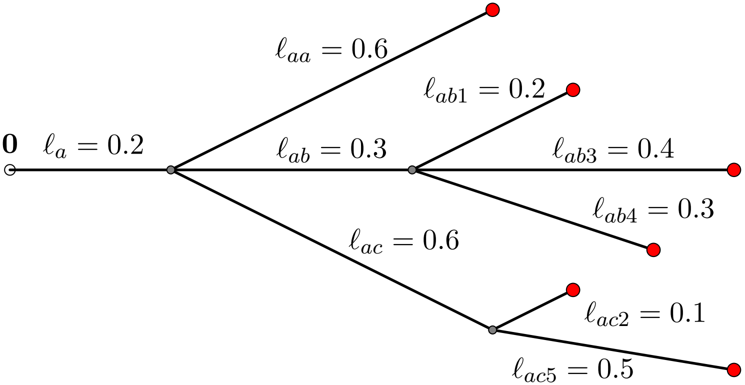


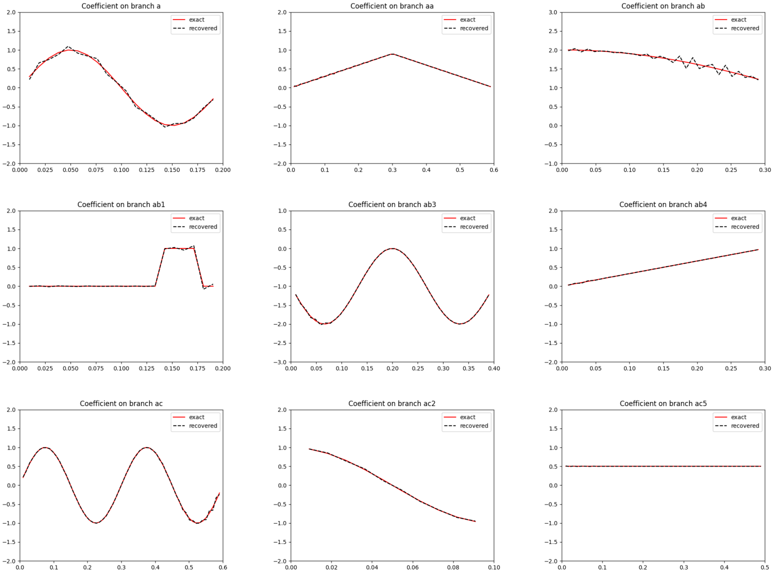
Figure: Second setting - an 9 branches network, with observation at •.

Numerical values for the nine-branches network

u_0	u_1	m
(2,2,2,2,2,2,2,2,2)	(0,0,0,0,0,0,0,0,0)	2
g	ℓ_j	β
(0,0,0,0,0,0,0,0,0)	(0.2,0.6,0.3,0.2,0.4,0.3,0.6,0.1,0.5)	0.99
h	x_j	ε_1
(2,2,2,2,2,2,2,2,2)	-(0.01,1.2,1.2,8.7,8.7,8.7,1.2,6.5,6.5)	10^{-3}
s	M_j	ε_2
1	(74.1,72.6,72.6,0,0,0,72.6,33.4,33.4)	10^{-2}
T	N_{xj}	N_t
9.15	$100 * \ell_j$	$110 * T$

Table: Numerical values of the variables used for the numerical examples of the nine-branches network.

Simulations of the nine-branches network



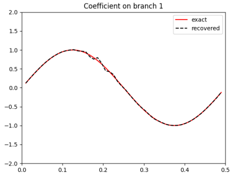
Conclusion

- Reconstruction of potentials on networks of wave equations.
- The C-bRec approach seems quite adaptable, even if it is to the price of appropriate one-parameter Carleman estimates.
- Other examples of network?
- Other equations? KdV equation?

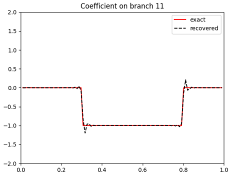
Discretization of the algorithm

- Discretization of the system: finite differences (explicit centered scheme) in space and time.
- Minimization of $F_s[p^k, 0, \mu^k]$: resolution of a variational formulation
  - approximation of the integrals using rectangle quadrature rules and standard centered finite differences,
  - attention must be paid to the discretization process of $\mathcal{T}$,
  - add viscosity terms to guarantee coercivity property uniformly with respect to discretization parameters (to handle high frequency spurious waves).
- Presence of large exponential factors in $F_s[p^k, 0, \mu^k]$:
  - to work on the conjugate variable $(y_j^k)_i^n = (w_j^k)_i^n e^{s\varphi_j(t^n, x_i)}$ that acts as a preconditioner of the linear system,
  - there are still exponential factors in the right hand side vector $\rightsquigarrow$ develop a progressive process to compute the solution as the aggregation of several problems localized in subdomains in which the exponential factors are all of the same order.

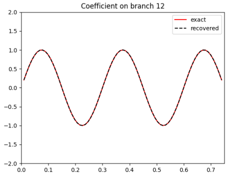
Simulations from data without noise: other potentials



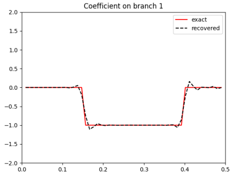
(a) $p_1^*(x) = \sin(2\pi x/\ell_1)$



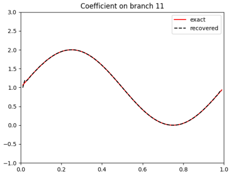
(b) $p_{11}^*(x) = -1_{[0.3,0.8]}(x/\ell_{11})$



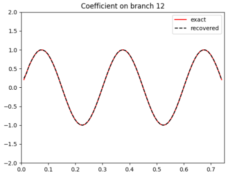
(c) $p_{12}^*(x) = \sin(5\pi x/\ell_{12})$



(d) $p_1^*(x) = -1_{[0.3,0.8]}(x/\ell_1)$



(e) $p_{11}^*(x) = 1 + \sin(2\pi x/\ell_{11})$



(f) $p_{12}^*(x) = \sin(5\pi x/\ell_{12})$