Small-time bilinear control for a class of nonlinear parabolic evolution equations

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Università Mercatorum

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Outline

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1. **Introduction to bilinear control problems**
2. Local/semi-global controllability to eigensolutions for bilinear parabolic problems
3. Controllability in small time of nonlinear parabolic problems

Controllability of evolution equations

Dynamical system: $u' = f(u, \mathrm{p})$

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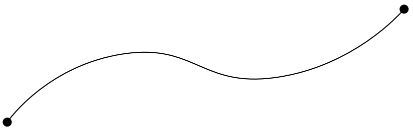

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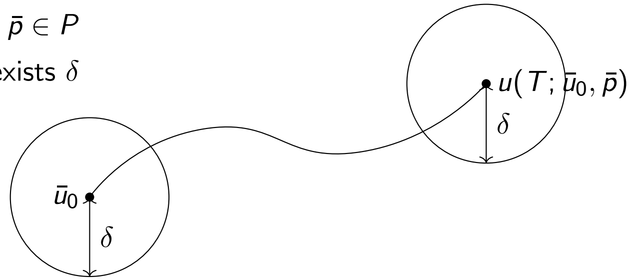

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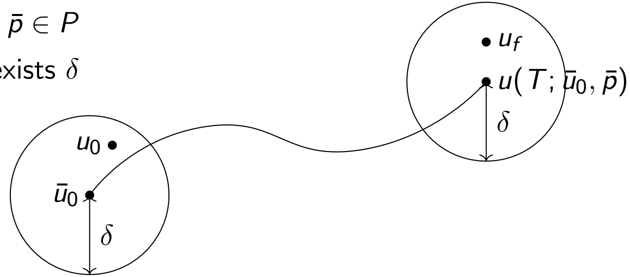

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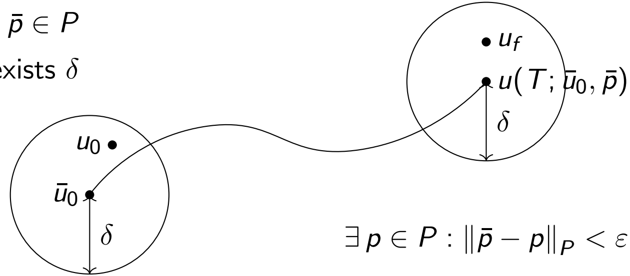

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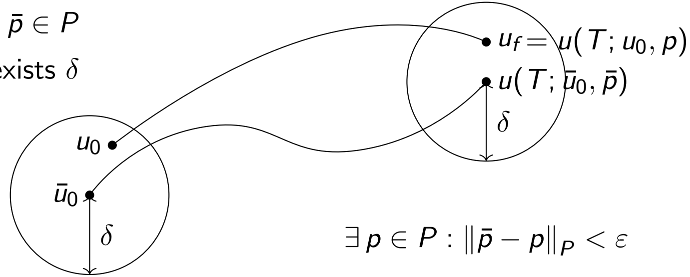

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Different kinds of control systems

Boundary control:

$$\begin{cases} u' = Au + Bu \\ u = \mathbf{p}|_{\partial\Omega} \\ u(0) = 0 \end{cases}$$

Locally distributed control:

$$\begin{cases} u' = Au + Bu + \mathbf{p}\mathbb{1}_\omega \\ u = g|_{\partial\Omega} \\ u(0) = 0 \end{cases}$$

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$$\begin{cases} u' = Au + \mathbf{p}Bu \\ u = g|_{\partial\Omega} \\ u(0) = u_0 \end{cases} \tag{BCS}$$

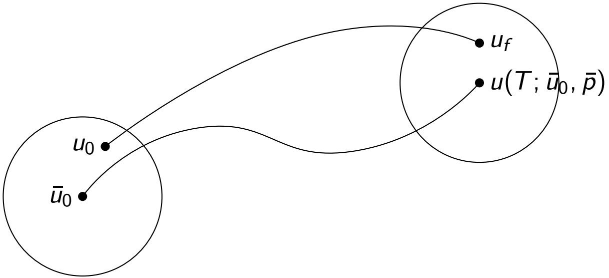
Theorem (Ball, Marsden, Slemrod 1982)

Let X be a Banach space with $\dim(X)=+\infty$. Let A generate a C^0-semigroup of bounded linear operators on X and $B : X \to X$ be a bounded linear operator. Let $u_0 \in X$ be fixed, and let $u(t; p, u_0)$ denote the unique solution of (BCS) for $p \in L^1_{loc}([0, +\infty), \mathbb{R})$. The set of states accessible from u_0 defined by

$$S(u_0) = \{u(t; p, u_0); t \geq 0, p \in L^r_{loc}([0, +\infty), \mathbb{R}), r > 1\}$$

is contained in a countable union of compact subsets of X and, in particular, has a dense complement.

Bilinear control systems



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Literature on exact bilinear controllability of hyperbolic pbms

Exact controllability of bilinear **hyperbolic** equations (nonexhaustive list):

- K. Beauchard, C. Laurent. "Local controllability of 1D linear and nonlinear Schrödinger equations with bilinear control." J. de Math. Pures et Appl. (2010)

  $\rightsquigarrow$ controllability in $H^3_{(0)}(0,1)$
- K. Beauchard "Local controllability and non-controllability for a 1D wave equation with bilinear control." J. of Diff. Eq. (2011)

  $\rightsquigarrow$ controllability in $H^3_{(0)}(0,1) \times H^2_{(0)}(0,1)$
- M. Morancey. "Simultaneous local exact controllability of 1D bilinear Schrödinger equations." Ann. de l'Inst. Henri Poincare (C) Non Linear Analysis. (2014)

  $\rightsquigarrow$ controllability in $(H^3_{(0)}(0,1))^N$
- A. Duca. "Global exact controllability of bilinear quantum systems on compact graphs and energetic controllability." SIAM J. on Contr. and Opt. (2020)

  $\rightsquigarrow$ controllability in $H^{2+d}_{\mathcal{G}}$
- P. Cannarsa, P. Martinez, C Urbani. "Bilinear control of a degenerate hyperbolic equation." SIAM J. of Math. An., vol. 55, n. 6, pp 6517–6553 (2023)

  $\rightsquigarrow$ controllability in $H^3_{(\alpha)}(0,1) \times H^2_{(\alpha)}(0,1)$

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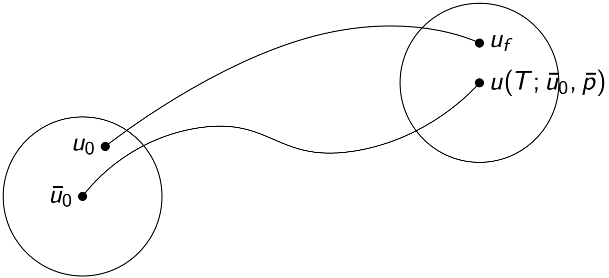

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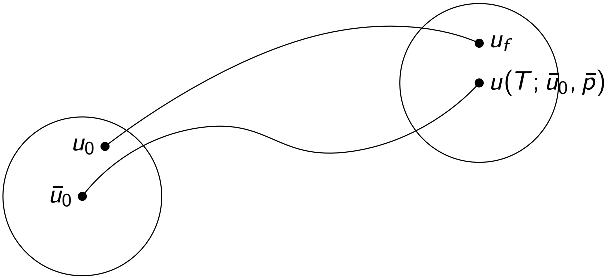

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Controllability to eigensolutions

$(X, \langle \cdot, \cdot \rangle)$ separable Hilbert space.

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$A : D(A) \subset X \to X$ densely defined linear operator:

$$\begin{aligned} &(a) \quad A \text{ is self-adjoint },\\ &(b) \quad \exists\, \sigma > 0 \;:\; \langle Ax, x\rangle \geq -\sigma \left\|x\right\|^2, \forall x \in D(A), \\ &(c) \quad \exists\, \lambda > -\sigma \text{ such that } (\lambda I + A)^{-1} : X \to X \text{ is compact,} \end{aligned} \tag{SAC}$$

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Trajectories: **eigensolutions** $\psi_j = e^{-\lambda_j t}\varphi_j$: solutions of (BCP) for $p = 0$ and $u_0 = \varphi_j$, for all $j \in \mathbb{N}^*$.

Controllability to eigensolutions

Definition

Let $T > 0$ and let A satisfy (SAC). The pair $\{A, B\}$ is called **j-null controllable in time T** if there exists a constant $N(T) > 0$ such that for every $y_0 \in X$ one can find a control $p \in L^2(0, T)$ satisfying $\|p\|_{L^2(0,T)} \leq N(T)\,\|y_0\|$, and for which $y(T) = 0$, where $y(\cdot)$ is the solution of

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Theorem (of Stabilization. Alabau-Boussouira, Cannarsa, Urbani 2021)

Let $\{A, B\}$ be a j-null controllable pair. Then, (BCP) is superexponentially stabilizable to ψ_j:

$$\|u(t) - \psi_j(t)\| \leq Me^{-\rho e^{\omega t}} \qquad \forall t \geq 0.$$

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Theorem (of Controllability. Alabau-Boussouira, Cannarsa, U. 2022)

Let $\{A, B\}$ be a j-null controllable pair and $N(\tau) \leq e^{C/\tau}$ for τ small. Then, for any $T > 0$, (BCP) is exactly controllable to ψ_j:

$$u(T) = \psi_j(T).$$

Sufficient conditions for j-null controllability

Theorem (Alabau-Boussouira, Cannarsa, Urbani 2021)

Let $A : D(A) \subset X \to X$ satisfy (SAC) and be such that $\exists\, \alpha > 0$ for which its eigenvalues fulfill the gap condition

$$\sqrt{\lambda_{k+1} - \lambda_1} - \sqrt{\lambda_k - \lambda_1} \geq \alpha, \quad \forall\, k \in \mathbb{N}^*. \tag{GAP}$$

Let $B : X \to X$ be a bounded linear operator such that

$$i) \quad \langle B\varphi_j, \varphi_k \rangle \neq 0, \quad \forall\, k \in \mathbb{N}^*,$$

$$ii) \quad \exists\, \tau > 0 \,:\, \sum_{k \in \mathbb{N}^*} \frac{e^{-2\lambda_k \tau}}{|\langle B\varphi_j, \varphi_k \rangle|^2} < +\infty.$$

Then, the pair $\{A, B\}$ is j-null controllable.

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Theorem (Alabau-Boussouira, Cannarsa, Urbani 2022)

Let $A : D(A) \subset X \to X$ satisfy (SAC) and (GAP). Let $B : X \to X$ be a bounded linear operator such that there exist $b, q > 0$ for which

$$\langle B\varphi_j, \varphi_j \rangle \neq 0 \quad \text{and} \quad |\lambda_k - \lambda_1|^q |\langle B\varphi_j, \varphi_k \rangle| \geq b, \quad \forall\, k \neq j.$$

Then, the pair $\{A, B\}$ is j-null controllable in any time $T > 0$ with control cost $N(\cdot)$ that satisfies $N(\tau) \leq e^{C/\tau}$, for τ small.

Semi-global controllability to the ground state

Theorem (Alabau-Boussouira, Cannarsa, Urbani 2022)

Let A and B satisfy the hypotheses of Exact Controllability Theorem. Then, there exists a constant $r_1 > 0$ such that for any $R > 0$ there exists $T_R > 0$ such that for all $u_0 \in X$ that satisfy

$$|\langle u_0, \varphi_1 \rangle - 1| < r_1,$$
$$||u_0 - \langle u_0, \varphi_1 \rangle \varphi_1|| \leq R,$$

problem (BCP) is exactly controllable to the ground state solution $\psi_1(t) = e^{-\lambda_1 t}\varphi_1$ in time T_R.

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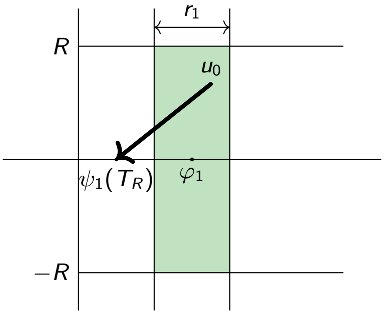

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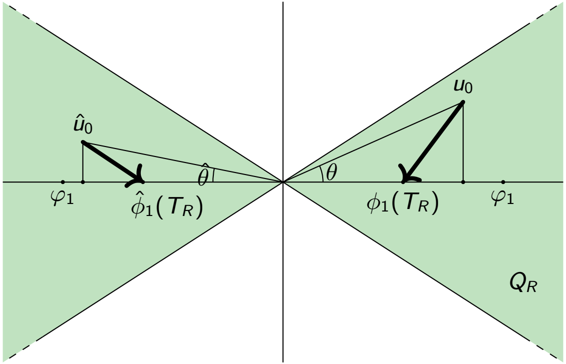

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Setting of the problem

Let $\mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$, $d \in \mathbb{N}^*$ and consider

$$\begin{cases} \partial_t \psi(t,x) = \Delta\psi(t,x) - \kappa\psi^{p+1}(t,x) + \langle u(t), Q(x)\rangle\psi(t,x), & x \in \mathbb{T}^d,\ t > 0, \\ \psi(0,x) = \psi_0(x), \end{cases} \tag{NHE}$$

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- $\boldsymbol{Q} = (Q_1, \ldots, Q_q, \mu_1, \mu_2) : \mathbb{T}^d \to \mathbb{R}^{q+2}$ potentials, $q \in \mathbb{N}$, $q \geq 2d+1$,

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Setting of the problem

Let $\mathbb{T}^d = \mathbb{R}^d / 2\pi\mathbb{Z}^d$, $d \in \mathbb{N}^*$ and consider

$$\begin{cases} \partial_t \psi(t,x) = \Delta\psi(t,x) - \kappa\psi^{p+1}(t,x) + \langle \boldsymbol{u}(t), Q(x)\rangle \psi(t,x), & x \in \mathbb{T}^d,\ t > 0, \\ \psi(0,x) = \psi_0(x), \end{cases} \tag{NHE}$$

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We prove:

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More precisely...

Main results

Proposition (Duca, Pozzoli, Urbani 2025)

(assumptions)

(i) Let $\psi_0, \psi_1 \in H^s(\mathbb{T}^d, \mathbb{R})$ be such that $\mathrm{sign}(\psi_0) = \mathrm{sign}(\psi_1)$. For any $\epsilon > 0$ and $T > 0$, there exist $\tau \in (0, T]$ and $u \in L^2((0,\tau), \mathbb{R}^{q+2})$ such that the solution $\psi(t; \psi_0, u)$ of (NHE) satisfies

$$\|\psi(\tau; \psi_0, u) - \psi_1\|_{L^2} < \epsilon$$

(ii) Let $\psi_0, \psi_1 \in H^s(\mathbb{T}^d, \mathbb{R})$ be such that $\psi_0, \psi_1 > 0$ (or $\psi_0, \psi_1 < 0$). For any $\epsilon > 0$ and $T > 0$, there exists $u \in L^2((0, T), \mathbb{R}^{q+2})$ such that the solution $\psi(t; \psi_0, u)$ of (NHE)

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Theorem (Duca, Pozzoli, Urbani 2025)

(assumptions), then (NHE-1D) is locally exactly controllable to the ground state solution c_0 in any positive time: for any $T > 0$ there exists $R_T > 0$ such that, for any

$$\psi_0 \in \{\psi \in H^3(\mathbb{T}, \mathbb{R}) : \|\psi - c_0\|_{H^1} < R_T\},$$

there exists $u \in H^1((0, T), \mathbb{R}^{q+2})$ such that $\psi(T; \psi_0, u) = c_0$.

Local well-posedness

Proposition (D-P-U 2025)

Let $s > d/2$ and $Q \in H^s(\mathbb{T}^d, \mathbb{R}^{q+2})$. For any $\psi_0 \in H^s(\mathbb{T}^d, \mathbb{R})$ and $u \in L^2_{loc}(\mathbb{R}^+, \mathbb{R}^{q+2})$ there exists a maximal time $\mathcal{T} = \mathcal{T}(\psi_0, u) > 0$ and a unique mild solution $\psi \in C^0([0,T], H^s(\mathbb{T}^d, \mathbb{R}))$, $\forall\, T < \mathcal{T}$, of (NHE) represented by

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i. if $\psi_0, \phi_0 \in B_{H^s(\mathbb{T}^d,\mathbb{R})}(0,R)$, with $R > 0$, and $u, v \in L^2_{loc}(\mathbb{R}^+, \mathbb{R}^{q+2})$, then for any $0 \le T \le \min\{\mathcal{T}(\psi_0,u), \mathcal{T}(\phi_0,v)\}$, there exists $C = C(u,v)$ such that

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ii. set $K = \|\psi\|_{C([0,T],H^s)} + \|\psi_0\|_{H^s} + \|u\|_{L^2}$. There exists $\delta = \delta(\mathcal{T}(\psi_0,u), K) > 0$ such that, for any $\hat{\psi}_0 \in H^s(\mathbb{T}^d, \mathbb{R})$ and $\hat{u} \in L^2((0,T), \mathbb{R}^{q+2})$ satisfying
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problem (NHE) admits a unique mild solution $\hat{\psi} \in C\big([0,T], H^s(\mathbb{T}^d,\mathbb{R})\big)$ with initial condition $\hat{\psi}_0$ and control $\hat{u}$.

Outline

Small-time limit of conjugated dynamics

Define the non-linear operator

$$\mathbb{B}(\varphi)(x) = \sum_{j=1}^{d} \left(\partial_{x_j}\varphi(x)\right)^2, \qquad \forall\, \varphi \in C^1(\mathbb{T}^d, \mathbb{R}).$$

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$$e^{\delta^{-1/2}\varphi}\psi(\delta; e^{-\delta^{-1/2}\varphi}\psi_0, \delta^{-1}u) \to e^{\mathbb{B}(\varphi)+\langle u, Q\rangle}\psi_0 \quad \text{in } H^s, \text{ as } \delta \to 0^+.$$

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Corollary (D-P-U 2025)

Let $s > d/2$ and $1 \in \mathrm{span}\{Q_1, ..., Q_q\}$. Let $\psi_0 \in H^s(\mathbb{T}^d, \mathbb{R})$. For any $\epsilon, T > 0$ there exists a constant control $u \in \mathbb{R}^{q+2}$ such that the solution $\psi(t; \psi_0, u)$ of (NHE) is defined in $[0, T]$ and

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Apply the limit of conjugate dynamics with $\varphi = 0$ and $-c = \sum_{j=1}^{q} u_j Q_j$ small enough.

An intermediate controllability result

Given $Q_1, \dots, Q_q \in C^\infty(\mathbb{T}^d, \mathbb{R})$, $q \in \mathbb{N}^*$, define the vector space

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To prove the Proposition:

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Small time global approximate controllability

Theorem (D-P-U 2025)

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- observe that $\|e^{\phi_\eta}\psi_0 - \psi_1\|_{L^2(\mathbb{T}^d)} \le \|\psi_0 - \psi_1\|_{L^2(Z_\eta)} < \frac{\varepsilon}{3}$ for η small enough

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- observe that $H^1(\mathbb{T}^d)$ is dense in $L^2(\mathbb{T}^d) \Longrightarrow \exists\, \tilde{\phi}_\eta \in H^1(\mathbb{T}^d)$ such that $\|e^{\tilde{\phi}_\eta}\psi_0 - e^{\phi_\eta}\psi_0\|_{L^2(\mathbb{T}^d)} < \frac{\varepsilon}{3}$

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- Jurdjevic, Kupka "*Polynomial control systems*", 1985 **(finite dimensional control systems)**
- Agrachev, Sarychev "*Navier-Stokes equations: controllability by means of low modes forcing*", 2005 **(infinite dimensional control systems)**
- Agrachev, Sarychev "*Controllability of 2D Euler and Navier-Stokes equations by degenerate forcing*", 2006 **(infinite dimensional control systems)**
- Shirikyan "*Approximate controllability of three-dimensional Navier-Stokes equations*", 2006
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- Glatt-Holtz, Herzog, Mattingly "*Scaling and saturation in infinite-dimensional control problems with applications to stochastic partial differential equations*", 2018
- Nersesyan, "*Approximate controllability of nonlinear parabolic PDEs in arbitrary space dimension*", 2021
- Duca, Nersesyan "*Bilinear control and growth of Sobolev norms for the nonlinear Schrödinger equation*", 2025 **(bilinear controls)**

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Theorem (Duca, Nersesyan 2025)

Assume that

$$\{Q_1, \ldots, Q_q\} = \{1, \sin\langle k, x\rangle), \cos\langle k, x\rangle\}_{k \in L},$$

for some $L \subset \mathbb{Z}^d$. Then, $\mathcal{H}_\infty$ is dense in $H^s(\mathbb{T}^d, \mathbb{R}), s \geq 0$, if and only if

- L is a generator,
- for any $l, m \in L$, there exists $\{n_j\}_{j=1}^r \subset L$ such that $l \not\perp n_1, n_j \not\perp n_{j+1}, j = 1, \ldots, r-1$, and $n_r \not\perp m$.

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In our result we assume that $Q_1, \dots, Q_q \in C^\infty(\mathbb{T}^d, \mathbb{R})$ and

$$\left\{1, \cos\langle k, x\rangle,\ \sin\langle k, x\rangle\right\}_{k \in L} \subset \mathcal{H}_0,$$

with

$$L = \{(1, 0, \dots, 0), (0, 1, \dots, 0), \dots, (0, \dots, 1, 0), (1, \dots, 1)\} \subset \mathbb{Z}^d.$$

Literature

- A. Y. Khapalov, *Global non-negative controllability of the semilinear parabolic equation governed by bilinear control*, ESAIM Control Optim. Calc. Var., 7, 269–283, 2002
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- in presence of a (potentially high power) polynomial nonlinearity

Outline

Properties of the operator and assumptions

➔ 1-dimensional case: $\mathbb{T}$

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We need the solution to be globally (in time) defined:

Proposition (D-P-U 2025)

Let $p \in 2\mathbb{N}$, $\psi_0 \in H^3(\mathbb{T}, \mathbb{R})$, $Q \in H^3(\mathbb{T}, \mathbb{R}^{q+2})$, $u \in H^1_{loc}((0, +\infty), \mathbb{R}^{q+2})$ and $\kappa \geq 0$. Then, for any $T > 0$ there exists a unique mild solution $\psi \in C^0([0, T], H^3(\mathbb{T}, \mathbb{R}))$ of (NHE-1D).

Local controllability to the ground state solution

Theorem (D-P-U 2025)

Let $\kappa \geq 0$ and $p \in 2\mathbb{N}$. Suppose that Assumptions on Q is satisfied. Then, (NHE-1D) is locally exactly controllable to the ground state solution c_0, in any positive time. In other words, for any $T > 0$ there exists $R_T > 0$ such that, for any

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- proof that the distance of the solution w.r.t. the target is zero a time T

Outline

Global small time controllability to the ground state solution

Theorem (D-P-U 2025)

Let $d = 1$, $\kappa \geq 0$ and $p \in 2\mathbb{N}$. Suppose that $Q_1, \dots, Q_q \in C^\infty(\mathbb{T}, \mathbb{R})$ and

$$\{1, \cos x, \ \sin x\} \subset \mathcal{H}_0.$$

Assume moreover that $Q_1 = 1$, $\mu_1, \mu_2 \in H^3(\mathbb{T}, \mathbb{R})$ and

$$\begin{aligned}
&\langle \mu_1, c_0 \rangle_{L^2(\mathbb{T})} \neq 0, && \langle \mu_2, c_0 \rangle_{L^2(\mathbb{T})} = 0, \\
\exists\, b_1, q_1 > 0 \ : \ \lambda_k^{q_1} \left| \langle \mu_1, c_k \rangle_{L^2(\mathbb{T})} \right| \geq b_1, \ \text{and} \ &\langle \mu_1, s_k \rangle_{L^2(\mathbb{T})} = 0, && \forall\, k \in \mathbb{N}^*, \\
\exists\, b_2, q_2 > 0 \ : \ \lambda_k^{q_2} \left| \langle \mu_2, s_k \rangle_{L^2(\mathbb{T})} \right| \geq b_2, \ \text{and} \ &\langle \mu_2, c_k \rangle_{L^2(\mathbb{T})} = 0, && \forall\, k \in \mathbb{N}^*
\end{aligned}$$

Then, (NHE-1D) is exactly controllable to the ground state solution c_0 in any positive time from any positive state. More precisely, for any $T > 0$ and $\psi_0 \in H^3(\mathbb{T}, \mathbb{R})$ such that $\psi_0 > 0$, there exists $u \in L^2((0, T), \mathbb{R}^{q+2})$, such that

$$\psi(T; \psi_0, u) = c_0.$$

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Example of a suitable potential:

$$Q(x) = (1, \cos x, \sin x, x^3(2\pi - x)^3, x^3(x - \pi)^3(x - 2\pi)^3).$$

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THANK YOU! MERCI!
GRACIAS! GRAZIE!