Global propagation of analyticity and unique continuation for semilinear waves

Joint work with C. Laurent (CNRS, LMR)

Control of PDEs and related topics

Cristóbal Loyola

Laboratoire Jacques-Louis Lions
Sorbonne Université
Paris, France

July 2, 2025

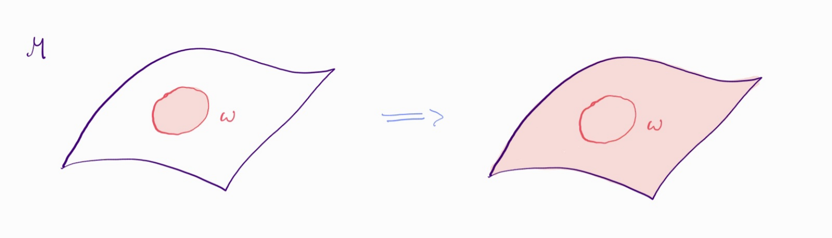
Outline

Overview

Propagation of information

Let $\omega \subset \mathcal{M}$ and consider some PDE $Pu = 0$. We are interested in the following:

- **Propagation of analyticity**: if the solution is analytic in time on $(0,T) \times \omega$, is the full solution analytic in time on $(0,T) \times \mathcal{M}$?
- **Unique continuation**: if the solution is zero on $[0,T] \times \omega$, is the solution identically zero in $[0,T] \times \mathcal{M}$?



The semilinear wave equation

Let $T > 0$. Consider the semilinear wave equation

$$\begin{cases} \partial_t^2 u - \Delta_g u + f(u) = 0 & (t,x) \in (0,T) \times \mathrm{Int}(\mathcal{M}) \\ u = 0 & (t,x) \in (0,T) \times \partial\mathcal{M} \\ (u, \partial_t u)(0) = (u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M}), \end{cases} \tag{1}$$

where $\mathcal{M}$ is a smooth compact Riemannian manifold with boundary of dimension $d = 3$ with metric g. The nonlinearity $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ is assumed to be energy subcritical, in the sense that there exists $C > 0$ such that

$$f(0) = 0, \ |f(s)| \leqslant C(1 + |s|)^p \text{ and } |f'(s)| \leqslant C(1 + |s|)^{p-1}$$

with $1 \leqslant p < 5$.

The well-posedness of (1) is handled by means of Strichartz estimates: for some admissible exponents (q, r), one has

$$\|u\|_{L^q([0,T], L^r(\mathcal{M}))} \leqslant C\|(u_0, u_1)\|_{H_0^1 \times L^2}.$$

Geometric assumption

Let ω be a nonempty open subset of $\mathcal{M}$. We assume that (ω, T) satisfy the Geometric Control Condition:

(GCC) every ray of geometric optics traveling at speed 1 meets ω in time $t \in (0, T)$.

Theorem (BLR '92)

If (ω, T) satisfies the (GCC), then

$$\begin{cases} \partial_t^2 u - \Delta_g u = 0 & (t,x) \in (0,T) \times \mathrm{Int}(\mathcal{M}) \\ u = 0 & (t,x) \in (0,T) \times \partial\mathcal{M} \\ (u, \partial_t u)(0) = (u_0, u_1), & \end{cases} \implies \|(u_0, u_1)\|_{H^1 \times L^2} \leqslant C \|\partial_t u\|_{L^2([0,T]\times\omega)}.$$

Bardos, Claude, Gilles Lebeau, and Jeffrey Rauch.
Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary.
SIAM journal on control and optimization 30, no. 5 (1992): 1024-1065.

Propagation of analyticity

Theorem (Laurent - L., 24')

Let $\sigma \in (1/2, 1]$. Let $(u, \partial_t u) \in \mathcal{C}([0,T], H^{1+\sigma} \cap H_0^1 \times H_0^\sigma(\mathcal{M}))$ be solution of (1). Assume that the above setting holds and assume moreover that:

- (ω, T) satisfies the (GCC).
- $t \in (0,T) \mapsto \chi u(t,\cdot) \in H^{1+\sigma}(\mathcal{M}) \cap H_0^1(\mathcal{M})$ is analytic for any cutoff function $\chi \in C_c^\infty(\mathcal{M})$ whose support is contained in ω.
- $s \mapsto f(s)$ is real analytic.

Then $t \mapsto u(t,\cdot)$ is analytic from $(0,T)$ to $H^2(\mathcal{M}) \cap H_0^1(\mathcal{M})$.

Laurent, Camille, and Cristóbal Loyola.
Global propagation of analyticity and unique continuation for semilinear waves.
arXiv preprint arXiv:2407.02866 (2024).

Unique continuation

Theorem (Laurent - L., 24') (Unique continuation)

Assume that (ω, T) satisfies the (GCC) and that f is energy subcritical and real analytic. If one solution $U = (u, \partial_t u) \in \mathcal{C}([0,T], H_0^1 \times L^2(\mathcal{M}))$ with finite Strichartz norms of (1) satisfies $\partial_t u = 0$ in $[0,T] \times \omega$, then $\partial_t u = 0$ in $[0,T] \times \mathcal{M}$ and u is an equilibrium point of (1), that is, solution of

$$\begin{cases} -\Delta u + f(u) = 0 & x \in \mathrm{Int}(\mathcal{M}), \\ u = 0 & x \in \partial\mathcal{M}. \end{cases} \tag{2}$$

If, moreover, the nonlinearity satisfies $sf(s) \geqslant 0$ for all $s \in \mathbb{R}$, then $u \equiv 0$.

Overview

How do we tackle the (nonlinear) unique continuation problem?

Set $z = \partial_t u$ and observe that

$$\begin{cases} \partial_t^2 z - \Delta z + f'(u)z = 0 & (t,x) \in [0,T] \times \mathrm{Int}(\mathcal{M}) \\ z_{|\partial\mathcal{M}} = 0 & (t,x) \in [0,T] \times \partial\mathcal{M} \\ z = 0 & (t,x) \in [0,T] \times \omega. \end{cases}$$

so the problem is reduced to show a UCP for linear waves with potential $V = f'(u)$.

- Classical strategy: Carleman estimates. Backdraw: not so natural geometric assumptions (more or less of Morawetz type).

  Many references: Fursikov-Imanuvilov ('96), Duyckaerts-Zhang-Zuazua ('08), Baudouin-de Buhan-Ervedoza ('13), Shao ('19), ...

- Partial analyticity framework allows less restrictive geometry assumption.

  References: Robbiano-Zuily ('98), Hörmander ('97), Tataru ('95-'98).

How do we tackle the (nonlinear) unique continuation problem?

Set $z = \partial_t u$ and observe that

$$\begin{cases} \partial_t^2 z - \Delta z + f'(u)z = 0 & (t,x) \in [0,T] \times \mathrm{Int}(\mathcal{M}) \\ z_{|\partial\mathcal{M}} = 0 & (t,x) \in [0,T] \times \partial\mathcal{M} \\ z = 0 & (t,x) \in [0,T] \times \omega. \end{cases}$$

so the problem is reduced to show a UCP for linear waves with potential $V = f'(u)$.

- Classical strategy: Carleman estimates. Backdraw: not so natural geometric assumptions (more or less of Morawetz type).

  Many references: Fursikov-Imanuvilov ('96), Duyckaerts-Zhang-Zuazua ('08), Baudouin-de Buhan-Ervedoza ('13), Shao ('19), ...

- Partial analyticity framework allows less restrictive geometry assumption.

  References: Robbiano-Zuily ('98), Hörmander ('97), Tataru ('95-'98).

How do we tackle the (nonlinear) unique continuation problem?

Set $z = \partial_t u$ and observe that

$$\begin{cases} \partial_t^2 z - \Delta z + f'(u)z = 0 & (t,x) \in [0,T] \times \mathrm{Int}(\mathcal{M}) \\ z_{|\partial\mathcal{M}} = 0 & (t,x) \in [0,T] \times \partial\mathcal{M} \\ z = 0 & (t,x) \in [0,T] \times \omega. \end{cases}$$

so the problem is reduced to show a UCP for linear waves with potential $V = f'(u)$.

- Classical strategy: Carleman estimates. Backdraw: not so natural geometric assumptions (more or less of Morawetz type).

  Many references: Fursikov-Imanuvilov ('96), Duyckaerts-Zhang-Zuazua ('08), Baudouin-de Buhan-Ervedoza ('13), Shao ('19), ...
- Partial analyticity framework allows less restrictive geometry assumption.

  References: Robbiano-Zuily ('98), Hörmander ('97), Tataru ('95-'98).

A general UCP result for linear waves

Theorem (Robbiano-Zuily, Hörmander, Tataru '99)

Let $\mathcal{M}$ be a compact Riemannian manifold with (or without) boundary and Δ_g the Laplace-Beltrami operator on $\mathcal{M}$. Let ω a nonempty open subset of $\mathcal{M}$ and $T > 2\sup_{x\in\mathcal{M}} \text{dist}(x,\omega)$. Let V be bounded and analytic on the variable $t \in (0,T)$ and let $(u_0, u_1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$ and associated solution u of

$$\begin{cases} \partial_t^2 u - \Delta_g u + Vu = 0 & \text{in } (0,T)\times \text{Int}(\mathcal{M}) \\ u|_{\partial\mathcal{M}} = 0 & \text{in } (0,T)\times\partial\mathcal{M} \\ (u,\partial_t u)(0) = (u_0,u_1). \end{cases}$$

If u satisfies $u = 0$ on $[0,T]\times\omega$, then $u = 0$ on $[0,T]\times\mathcal{M}$.

Overview

Abstract result

Rough assumptions:

- A skew-adjoint with compact resolvent, with *Sobolev spaces* X^σ.
- Observability for $\mathbf{C} \in \mathcal{L}(X^\sigma)$: $\|W_0\|^2_{X^\sigma} \leqslant \mathfrak{C}^2_{\mathrm{obs}} \int_0^T \|\mathbf{C}e^{tA}W_0\|^2_{X^\sigma}\,dt$
- F is a *compact* nonlinearity with gain of derivatives $X^\sigma \mapsto X^{\sigma+\varepsilon}$.
- F is analytic.

Theorem

Let $T^* > T$. Then, any $U \in \mathbb{B}_{R_0}(C^0([0,T], X^\sigma))$ solution of

$$\begin{cases} \partial_t U = AU + F(U), & \text{on } [0,T], \\ \mathbf{C}U(t) = 0, & \text{for } t \in [0,T], \end{cases}$$

is real analytic in $t \in (0, T^*)$ with value in X^σ.

Abstract result

Rough assumptions:

- A skew-adjoint with compact resolvent, with *Sobolev spaces* X^σ.
- Observability for $\mathbf{C} \in \mathcal{L}(X^\sigma)$: $\|W_0\|_{X^\sigma}^2 \leqslant \mathfrak{C}_{\mathsf{obs}}^2 \int_0^T \|\mathbf{C}e^{tA}W_0\|_{X^\sigma}^2 \, dt$
- F is a *compact* nonlinearity with gain of derivatives $X^\sigma \mapsto X^{\sigma+\varepsilon}$.
- F is analytic.

Theorem

Let $T^* > T$. Then, any $U \in \mathbb{B}_{R_0}(C^0([0,T], X^\sigma))$ solution of

$$\begin{cases} \partial_t U = AU + F(U), & \text{on } [0,T], \\ \mathbf{C}U(t) = 0, & \text{for } t \in [0,T], \end{cases}$$

is real analytic in $t \in (0, T^*)$ with value in X^σ.

A word on the proof

We consider the following splitting into low and high frequency

$$U = \mathcal{P}_n^L U + \mathcal{P}_n^H U := U_L + U_H$$

which translates into the following nonlinear system

$$\begin{cases} \partial_t U_L = AU_L + \mathcal{P}_n^L F(U_L + U_H) \\ \partial_t U_H = AU_H + \mathcal{P}_n^H F(U_L + U_H) \\ \mathbf{C}U_H = -\mathbf{C}U_L. \end{cases}$$

The idea now is to find a large n and an operator $\mathcal{R}$ such that

$$U(t) = U_L(t) + \mathcal{R}(U_L)(t).$$

Here the high-frequency component U_H will depend analytically on the low-frequency component. Moreover, U_L will be proved to be analytic in time (it solves an ODE in a Banach space).

A word on the proof

We consider the following splitting into low and high frequency

$$U = \mathcal{P}_n^L U + \mathcal{P}_n^H U := U_L + U_H$$

which translates into the following nonlinear system

$$\begin{cases} \partial_t U_L = AU_L + \mathcal{P}_n^L F(U_L + U_H) \\ \partial_t U_H = AU_H + \mathcal{P}_n^H F(U_L + U_H) \\ \mathbf{C}U_H = -\mathbf{C}U_L. \end{cases}$$

The idea now is to find a large n and an operator $\mathcal{R}$ such that

$$U(t) = U_L(t) + \mathcal{R}(U_L)(t).$$

Here the high-frequency component U_H will depend analytically on the low-frequency component. Moreover, U_L will be proved to be analytic in time (it solves an ODE in a Banach space).

Nonlinear reconstruction operator

Inspired by ideas of Hale and Raugel, we have the following.

Theorem (Nonlinear reconstruction)

Let $R_0 > 0$. There exist $n \in \mathbb{N}$ and a nonlinear Lipschitz (reconstruction) operator $\mathcal{R}$

$$\mathcal{R}: \quad \mathbb{B}_{R_0}(C^0([0,T], \mathcal{P}_n^L X^\sigma)) \longrightarrow \mathbb{B}_{R_0}(C^0([0,T], \mathcal{P}_n^H X^\sigma))$$

so that for any $U \in \mathbb{B}_{R_0}(C^0([0,T], X^\sigma))$ that satisfy

$$\begin{cases} \partial_t U = AU + F(U), & \text{on } [0,T], \\ \mathbf{C}U(t) = 0, & \text{for } t \in [0,T], \end{cases}$$

then, $\mathcal{P}_n^H U = \mathcal{R}(\mathcal{P}_n^L U)$. Moreover, $\mathcal{R}$ extends holomorphically in a small neighborhood.

Further results

We point out some further results obtained in the present work.

- The global propagation of analyticity can be reduced to the zero observation case.
- If we assume that g is an analytic metric, then we are able to prove that the solution u is analytic both in time t and space x.
- In the subcritical defocusing case, we obtain the observability estimate

$$\|(u_0, u_1)\|_{H^1 \times L^2} \leqslant C \|\partial_t u\|_{L^2([0,T]\times\omega)}$$

  with C depending on the size of the initial data.
- A result for the Plate equation is obtained as an application of this abstract result.

Further results

We point out some further results obtained in the present work.

- The global propagation of analyticity can be reduced to the zero observation case.
- If we assume that g is an analytic metric, then we are able to prove that the solution u is analytic both in time t and space x.
- In the subcritical defocusing case, we obtain the observability estimate

$$\|(u_0, u_1)\|_{H^1 \times L^2} \leqslant C \|\partial_t u\|_{L^2([0,T] \times \omega)}$$

with C depending on the size of the initial data.
- A result for the Plate equation is obtained as an application of this abstract result.

Further results

We point out some further results obtained in the present work.

- The global propagation of analyticity can be reduced to the zero observation case.
- If we assume that g is an analytic metric, then we are able to prove that the solution u is analytic both in time t and space x.
- In the subcritical defocusing case, we obtain the observability estimate

$$\|(u_0, u_1)\|_{H^1 \times L^2} \leqslant C \|\partial_t u\|_{L^2([0,T]\times\omega)}$$

  with C depending on the size of the initial data.
- A result for the Plate equation is obtained as an application of this abstract result.

Further results

We point out some further results obtained in the present work.

- The global propagation of analyticity can be reduced to the zero observation case.
- If we assume that g is an analytic metric, then we are able to prove that the solution u is analytic both in time t and space x.
- In the subcritical defocusing case, we obtain the observability estimate

$$\|(u_0, u_1)\|_{H^1 \times L^2} \leqslant C \|\partial_t u\|_{L^2([0,T]\times\omega)}$$

  with C depending on the size of the initial data.
- A result for the Plate equation is obtained as an application of this abstract result.

Overview

Nonlinear Schrödinger equation

Let P be a polynomial function with real coefficients, satisfying $P(0) = 0$. Let us consider the Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta_g u = P'(|u|^2)u & \text{in } (0,T) \times \mathcal{M}, \\ u(0) = u_0, \end{cases} \tag{3}$$

where $\mathcal{M}$ is a compact Riemannian surface.

If $\omega \subset \mathcal{M}$ satisfies the (GCC) and $\partial_t u = 0$ in $(0,T) \times \omega$, does u vanish everywhere, namely, $u \equiv 0$?

Dehman, B., P. Gérard, and G. Lebeau.
Stabilization and control for the nonlinear Schrödinger equation on a compact surface.
Mathematische Zeitschrift 254, no. 4 (2006): 729-749.

(L. '25) Unique continuation for nonlinear Schrödinger

Assume that ω satisfies the (GCC). If one solution $u \in C([0,T], H^1(\mathcal{M}))$ of (3) with finite Strichartz norm satisfies $\partial_t u = 0$ in $(0,T) \times \omega$, then $\partial_t u = 0$ in $(0,T) \times \mathcal{M}$ and u is an equilibrium point of (3), that is, solution of

$$-\Delta_g u + P'(|u|^2)u = 0, \quad x \in \mathcal{M}.$$

Moreover, if there exists $C > 0$ such that $P'(r) \geqslant C$ for $r \geqslant 0$, then $u = 0$ in $(0,T) \times \mathcal{M}$.

Loyola, C.
In the oven.
2025.

Thanks for your attention!

Funding

This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 945322.