Inverse source problems approximation with mixed finite elements

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The inverse source problem for the wave equation

Consider the wave equation with source term $\lambda(t) f^{source}(x)$,

$$\begin{cases} w'' - \Delta w + a(x) w = \lambda(t) f^{source}(x) & \text{for } x \in \Omega,\ t > 0 \\ w = 0 & \text{on } \partial\Omega,\ t > 0, \\ w(0,x) = w^0(x) & \text{for } x \in \Omega \\ w'(0,x) = w^1(x) & \text{for } x \in \Omega \end{cases} \quad (1)$$

Inverse source problem: Given $T > 0$, find the source term $f^{source}(x)$ from the following observation y^{obs} of a single solution:

$$y^{obs}(t) = \partial_n w'(t)\big|_{\Gamma}, \quad t \in (0, T).$$

Here the initial data (w^0, w^1), the potential $a(x)$, the intensity of the source $\lambda(t)$ and the observation zone $\Gamma \subset \partial\Omega$ are known.

Natural questions

1. **Uniqueness:** Does the observation y^{obs} allow to determine f^{source}?
2. **Stability:** Find a constant $c > 0$ such that

$$\|f^{source} - \widehat{f}^{source}\|_{L^2(\Omega)} \leq c\|y^{obs} - \widehat{y}^{obs}\|_{L^2(\Gamma\times(0,T))},$$

   where y^{obs} and $\widehat{y}^{obs}$ are the observations associated to f^{source} and $\widehat{f}^{source}$, respectively.
3. **Reconstruction:** Find a convergent numerical algorithm to obtain f^{source} from y^{obs}.

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$$\begin{cases} w'' - \Delta w + aw = \lambda(t) f(x) \\ w(0,x) = 0 \\ w'(0,x) = 0 \end{cases} \Rightarrow \begin{cases} u'' - \Delta u + au = 0 \\ u(0,x) = 0 \\ u'(0,x) = f(x) \end{cases}$$

$$w(t) = \int_0^t \lambda(t-s) u(s)\,\mathrm{d}s.$$

$$\underbrace{w'(t) = \lambda(0) u(t) + \int_0^t \lambda'(t-s) u(s)\,\mathrm{d}s}$$

Linear Volterra equation of the second kind in u
(convolution equation)

Under the assumptions $\lambda \in H^1(0,T)$ and $\lambda(0) \neq 0$, the map $w' \to u$ is a linear and continuous in L^2 :

$$\int_0^T \int_\Gamma |\partial_n u(t,x)|^2\,\mathrm{d}x \leq C \int_0^T \int_\Gamma \left|\partial_n w'(t,x)\right|^2 \mathrm{d}x.$$

$$
\begin{aligned}
C_T \|f\|^2_{L^2(\Omega)} &\leq \int_0^T \int_\Gamma |\partial_n u(t,x)|^2 \, \mathrm{d}x && \text{observability inequality} \\
&\leq C \int_0^T \int_\Gamma \left|\partial_n w'(t,x)\right|^2 \, \mathrm{d}x && \text{continuity of Volterra op.}
\end{aligned}
$$

It follows that the stability property holds:

$$
\|f - \widehat{f}\|_{L^2(\Omega)} \leq \sqrt{\frac{C_T}{C}} \|y - \widehat{y}\|_{L^2(\Gamma \times (0,T))},
$$

y and $\widehat{y}$ being the observations associated to f and $\widehat{f}$, respectively.

The stability constant $\sqrt{\frac{C_T}{C}}$ depends on the observability constant.

Observability

Consider the homogeneous wave equation,

$$\begin{cases} u'' - \Delta u + a(x)u = 0 & \text{for } x \in \Omega, \quad t > 0 \\ u = 0 & \text{on } \partial\Omega, \quad t > 0, \\ u(0,x) = u^0(x), \quad u'(0,x) = u^1(x) & \text{for } x \in \Omega \end{cases} \tag{2}$$

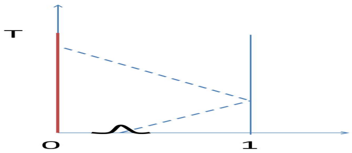
Observability: Given $\Gamma \subset \partial\Omega$ and $T > 0$, find constant $c > 0$ s. t.

$$\|u^0\|^2_{H^1_0(\Omega)} + \|u^1\|^2_{L^2(\Omega)} \leq c\|\partial_n u\|^2_{L^2(\Gamma\times(0,T))}.$$

for all solutions of the homogeneous wave equation.

- Observability implies stability of the inverse source problem (Puel-Yamamoto 95');
- Observability is known when $a \in L^\infty(\Omega)$ as long as (T, Γ) satisfies the GCC.

Observability is related with the speed of propagation. To observe at $x = 0$ we have to be aware of all disturbances induced by the initial data.



Let $T > 2$ and $E(t) =$
$$\frac{1}{2}\left(\int_0^1 |u'(t,x)|^2\,\mathrm{d}x + \int_0^1 |u_x(t,x)|^2\,\mathrm{d}x + \int_0^1 |a(x)||u(t,x)|^2\,\mathrm{d}x\right):$$

$$E(0) \le C_1 e^{C_2\sqrt{\|a(x)\|_{L^\infty}}} \int_0^T |u_x(t,0)|^2\,\mathrm{d}t.$$

Understanding the observability: proof (I)

The following proof for the continuous wave equation uses the lateral energy argument (Zuazua, 1993). Consider for $T > 2$ and $1 < \beta < T/2$

$$F(x) = \int_{\beta x}^{T-\beta x} \mathcal{E}(s,x)\,\mathrm{d}s,$$

where $\mathcal{E}(t,x) = \frac{1}{2}\left(|u'(t,x)|^2 + |u_x(t,x)|^2 + \|a\|_{L^\infty}|u(t,x)|^2\right)$.

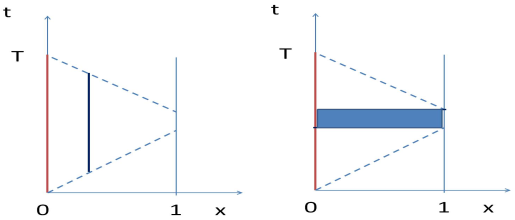
1. Prove that, for some constant $C > 0$,

$$F'(x) \leq CF(x).$$

2. By Gronwall's inequality:

$$F(x) \leq C_1 F(0) = C_1 \int_0^T |u_x(t,0)|^2\,\mathrm{d}t.$$

Understanding the observability: proof (II)



3. Using the conservation of the energy prove that

$$(T-2\beta)E(0) \leq \int_{\beta}^{T-\beta} \int_0^1 \mathcal{E}(t,x)\,\mathrm{d}x\mathrm{d}t \leq \int_0^1 F(x)\mathrm{d}x.$$

4. From 2. and 3. conclude that:

$$(T-2\beta)E(0) \leq C_1 F(0) = C_1 \int_0^T |u_x(t,0)|^2\,\mathrm{d}t.$$

Fourier approach $a = 0$ (no potential)

- $E(0) = \frac{1}{2}\left(\|u^0\|^2_{H^1_0(\Omega)} + \|u^1\|^2_{L(\Omega)}\right) = \frac{1}{2}\sum_{n\in\mathbb{Z}^*} |a_n^0|^2,$
  where a_n^0 are the Fourier coefficients of the initial data.
- $\int_0^T |u_x(t,0)|^2 \,\mathrm{d}t = \int_0^T \left|\sum_{n\in\mathbb{Z}^*} a_n^0 d_n e^{i\lambda_n t}\right|^2 \mathrm{d}t,$
  where d_n are the normal derivatives of the eigenfunctions in 0.
- Since $|d_n| > d$ and $\lambda_{n+1} - \lambda_n > \gamma$ (uniform positive gap), Ingham's inequality can be used to obtain that:

$$E(0) \leq C_1 \int_0^T |u_x(t,0)|^2 \,\mathrm{d}t.$$

Reconstruction Algorithm

1. Replace the wave equation by a convergent discretization depending on a parameter $h \to 0$. Define $F_h \sim f(x)$ and $Y_h \sim y$.
2. Prove stability for the discrete inverse source problem:

$$\|F_h - F_h^*\| \leq \kappa_h \|Y_h - Y_h^*\|$$

3. Implement an inversion algorithm to recover F_h from Y_h:

$$\min_{F_h} \mathcal{J}_h(F_h) := \min_{F_h} \frac{1}{2} \left\| Y_h - y^{obs} \right\|^2$$

(least squares approximation)

4. This minimization problem has a unique solution $\widehat{F}_h$ which is an approximation of f:

$$\lim_{h \to 0} \|\widehat{F}_h - F_h^{source}\| = 0.$$

Since $F_h^{source} \approx f^{source}$, then $\widehat{F}_h \approx f^{source}$.

Convergence proof (I)

Let Y_h^{obs} be the discretization of the observation y^{obs} and F_h^{source} be the discretization of the source term f^{source}. From the convergence of the numerical scheme we have that:

$$\lim_{h\to 0} \mathcal{J}_h(F_h^{source}) = \frac{1}{2} \lim_{h\to 0} \left\| Y_h^{obs} - y^{obs} \right\|^2 = 0. \qquad (3)$$

Since $\widehat{F}_h$ is a minimizer of $\mathcal{J}_h$, from (3) we deduce that

$$\lim_{h\to 0} \mathcal{J}_h(\widehat{F}_h) = 0. \qquad (4)$$

Convergence proof (II)

On the other hand, from the stability property, we obtain that

$$\|\widehat{F}_h - F_h^{source}\|^2 \leq \kappa_h^2 \|\widehat{Y}_h - Y_h^{obs}\|^2$$
$$\leq 4\kappa_h^2 \left(\mathcal{J}_h(\widehat{F}_h) + \mathcal{J}_h(F_h^{source})\right),$$

which, together with (3) and (4), implies that

$$\lim_{h\to 0} \|\widehat{F}_h - F_h^{source}\| = 0. \qquad (5)$$

Convergence difficulty

Convergence of the algorithm relies on two properties

- A **convergent numerical approximation** of the observation $Y_h \sim y$ (non-standard approximation result).
- A **uniform stability result** (with respect to the discretization parameter) that can be deduced from a **uniform observability** result for the homogenous wave equation.

It turns out that a convergent discretization for the wave equation does not always guarantee the convergence of the algorithm!

In the usual numerical schemes (finite differences, finite elements) the observability constant blows up as $h \to 0$!

This has been the object of active research in this and other related problems as control and stabilization of PDE's.

References

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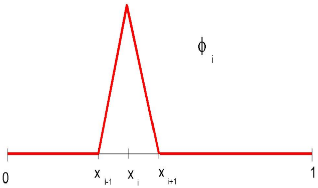
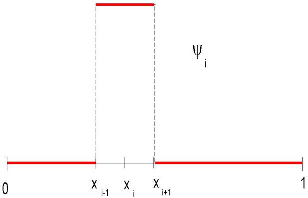
- Baudouin, Ervedoza and Osses 2015: Extension to rectangular domains in higher dimensions.
- Baudouin, Buhan and Ervedoza 2017: Improved algorithm based on a suitable penalization from the Carleman estimate.
- Cîndea and Münch 2015: Mixed formulation with a finite elements space-time discretization of the wave equation.

Main idea (F. Brezzi and M. Fortin - 91): $\Omega = (0,1)$, $h = \frac{1}{N+1}$, $x_j = hj, \quad 1 \leq j \leq N,$

$$w \sim w_h = \sum_{j=1}^{N} w_h^j \phi_j, \quad w' \sim w_h' = \sum_{j=1}^{N} v_h^j \psi_j,$$

$$f \sim f_h = \sum_{j=1}^{N} f_h^j \psi_k, \quad y = w_x(0,t) \sim y_h = \frac{w_h^1}{h}$$





MFE matrix formulation

$$\begin{cases} M_h W_h''(t) + K_h W_h(t) + L_h W_h(t) = \lambda(t) M_h F_h, & t > 0, \\ W_h(0) = W_h^0, \quad W_h'(0) = W_h^1. \end{cases}$$

$$K_h = \frac{1}{h}\begin{pmatrix} 2 & -1 & 0 & \ldots & 0 \\ -1 & 2 & -1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 2 \end{pmatrix}, \quad M_h = \frac{h}{4}\begin{pmatrix} 2 & 1 & 0 & \ldots & 0 \\ 1 & 2 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 2 \end{pmatrix},$$

$$L_h = h\begin{pmatrix} a_1 & 0 & 0 & \ldots\ldots & 0 \\ 0 & a_2 & 0 & \ldots\ldots & 0 \\ \ldots & \ldots & \ldots & \ldots\ldots & \ldots \\ 0 & 0 & 0 & \ldots\ldots & a_N \end{pmatrix}, \quad W_h = \begin{pmatrix} w_h^1 \\ w_h^2 \\ \ldots \\ w_h^N \end{pmatrix}, \quad F_h = \begin{pmatrix} f_h^1 \\ f_h^2 \\ \ldots \\ f_h^N \end{pmatrix}.$$

$$a_j = a(x_j), \quad 1 \leq j \leq N.$$

Understanding the case $a = 0$ (no potential)

In the 1-D case, the spectrum can be computed for the continuous and discrete approximations (finite differences (FD) or mixed finite elements (MFE)):

Continuous	$k\pi$	π
FD	$\frac{2}{h}\sin\left(\frac{k\pi h}{2}\right)$	h
MFE	$\frac{2}{h}\tan\left(\frac{k\pi h}{2}\right)$	π

Spectrums in the case $a = 0$

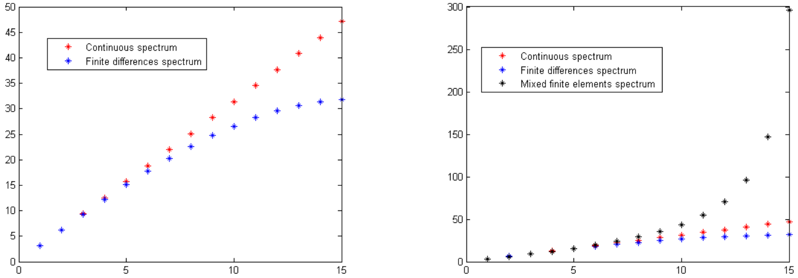


Figure: Spectrums compared

Uniform observability in the case $a = 0$

- For $a = 0$ uniform observability holds if mixed finite elements are used (C. Castro-SM 06', C. Castro-SM-A. Munch 08')!

  Explicit eigenvalues $\Rightarrow$ uniform gap $\Rightarrow$ Ingham inequality $\Rightarrow$ uniform observability.

- **Question:** Is this mixed finite elements approach robust enough to deal with an L^∞-potential?

  We consider the following homogeneous discrete equation and look for a uniform observability result:

$$\begin{cases} M_h U_h''(t) + K_h U_h(t) + L_h U_h(t) = 0, & t > 0, \\ U_h(0) = U_h^0, \quad U_h'(0) = U_h^1. \end{cases}$$

Main result: Uniform observability in the case $a \neq 0$

Theorem (C. Castro-SM 24')

Assume L_h is positive defined. There exist constants $C, T_0 > 0$, independent of h, such that for any $T > T_0$ we have

$$E_h(0) = \frac{1}{2}\left(\langle M_h U^1, U^1\rangle + \langle K_h U^0, U^0\rangle + \langle L_h U^0, U^0\rangle\right)$$

$$\leq C \int_0^T \left(\left| \frac{u_h^1(t)}{h} \right|^2 + \left| \frac{(u_h^1)'(t)}{2} \right|^2 \right) dt$$

- The observability constant $C = C(a, T)$ is uniform with respect to h!
- The time T should be sufficiently large!
- There are two terms in the observation!

Uniform observability in the case $a \neq 0$: the proof

At the discrete level, define

$$\mathcal{E}_j(s) = \left| \frac{u^{j+1}(s) - u^j(s)}{h} \right|^2 + \left| \frac{(u^{j+1})'(s) + (u^j)'(s)}{2} \right|^2 + a_\infty \left| \frac{u^{j+1}(s) + u^j(s)}{2} \right|^2 .$$

Consider also $T > 2$, $1 < \beta < T/2$ and discrete version of $F(x)$:

$$F_j = \frac{1}{2} \int_{\beta x_j}^{T - \beta x_j} \mathcal{E}_j(s) ds.$$

Lemma

The following discrete version of $F'(x) \leq cF(x)$ holds

$$\frac{F_j - F_{j-1}}{h} \leq c(a_M) \left(\frac{F_j + F_{j-1}}{2} + R_j(\beta x_j) + R_j(T - \beta x_j) \right),$$

$$R_j(s) = \frac{1}{h} \int_{s-\beta h}^{s} \mathcal{E}_j(r) dr - \frac{\mathcal{E}_j(s - \beta h) + \mathcal{E}_j(s)}{2}.$$

Uniform observability in the case $a \neq 0$: the proof

The proof from the continuous case does not work directly! :(

Lemma

Let $g > 0$, $s \geq 0$ and ν_1, ν_2 be two real numbers such that,

$$0 < \nu_2 - \nu_1 \leq \frac{\pi}{2(s+g)}. \tag{6}$$

Then, the following estimate holds

$$\frac{f(s) + f(s+g)}{2} \leq \frac{1}{g} \int_s^{s+g} f(r)\,\mathrm{d}r, \tag{7}$$

for any function $f(r)$ of the form

$$f(r) = \left| b_1 e^{i\nu_1 r} + b_2 e^{i\nu_2 r} \right|^2, \quad b_1, b_2 \in \mathbb{C}.$$

The lemma is an immediate consequence of Hermite-Hadamard inequality:

- Note that (7) holds (with equality) if b_1 or b_2 is zero.
- Therefore, it is sufficient to show (7) for functions of the form $f(s) = \left|be^{i\zeta s} + 1\right|^2 = b^2 + 1 + 2b\cos(\zeta s)$.
- Under the hypothesis (6), f is a concave function in $[s, s+g]$ and, consequently, (7) holds.

Uniform observability in the case $a \neq 0$: the proof

For particular solutions having only two frequencies λ_n, λ_m with

$$|\lambda_n - \lambda_m| < \frac{\pi}{2T},$$

we have

$$\frac{F_j - F_{j-1}}{h} \leq c(a_M)\frac{F_j + F_{j-1}}{2}$$

and the uniform observability inequality is proved!

This is not enough to prove the uniform observability inequality for arbitrary initial data, but implies that there exists a uniform constant $\gamma > 0$ such that:

$$\lambda_{n+1} - \lambda_n > \gamma \quad (0 < |n| \leq N). \tag{8}$$

Now we can apply Ingham's inequality to show the uniform observability.

The inverse source problem

The approximate observation for the inverse source problem is:

$$Y_h = \begin{pmatrix} \frac{(w_h^1)'(t)}{h} \\ \\ \frac{(w_h^1)''(t)}{2} \end{pmatrix} \sim y = \begin{pmatrix} w_x'(t,0) \\ \\ 0 \end{pmatrix}, \quad t \in (0, T).$$

Given an observation $y^{obs} = \begin{pmatrix} w_x'(t,0) \\ \\ 0 \end{pmatrix}$ associated to an unknown source term f, define the least-squares functional:

$$\mathcal{J}_h(F_h) = \frac{1}{2} \left\| Y_h - y^{obs} \right\|^2_{[L^2(0,T)]^2}.$$

Convergence

Theorem

There exists T_0 *such that for any* $T > T_0$, *the functional* $\mathcal{J}_h$ *has a unique minimizer* $\widehat{F}_h \in \mathbb{C}^N$. *Moreover, if we further assume that* $a \in C[0,1]$, *we have*

$$\widehat{f}_h = \sum_{j=1}^{N} \widehat{F}_h^j \psi_j \text{ tends to } f \in L^2(0,1) \text{ as } h \to 0.$$

Numerical experiments: MFE

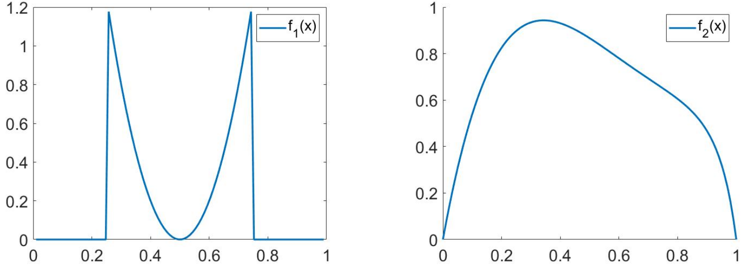


Figure: The two different source terms considered: a discontinuous one $f_1(x)$ (left) and a smooth one $f_2(x)$ (right).

Error estimates

MFE with $a(x) = 2 + cos(2\pi x)$. Error estimate: $e = \mathcal{O}(h)$

h	$\left\| \hat{f}_1 - \hat{f}_{1,h} \right\|_{L^2}$	$\left\| \hat{f}_2 - \hat{f}_{2,h} \right\|_{L^2}$
10^{-1}	3.4×10^{-2}	6.2×10^{-3}
10^{-2}	6.1×10^{-3}	1.1×10^{-4}
10^{-3}	7.4×10^{-4}	3.4×10^{-5}

Table: Error in the numerical reconstruction of the source term for different values of h and for two different source terms.

Time experiments

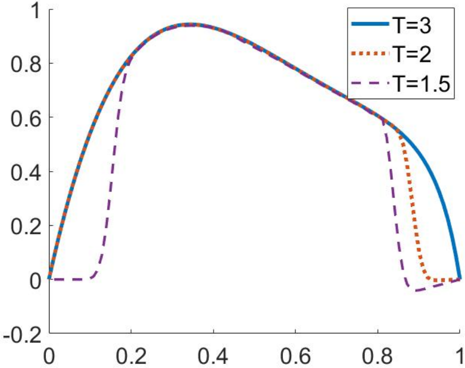


Figure: Reconstruction of the source $f_2(x)$ for different time observations T.

Thank you very much for your attention!