Observability on measurable sets of Schrödinger equations on tori

Control of PDEs and related topics Toulouse july 2nd 2025

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Observation and control for Schrödinger equations

- Let (M,g) be a compact Riemannian manifold, with boundary ∂M (possibly empty). Denote by $e^{it(\Delta+V)}u_0$ the solution to the linear Schrödinger equation

$$(i\partial_t + \Delta + V)u = 0, \qquad u\mid_{\partial M}= 0, \qquad u\mid_{t=0}= u_0 \in L^2(M).$$

- Observation of Schrödinger equation holds on $E_{t,x}$ iif

(OBS) $\quad \exists C > 0; \forall u_0 \in L^2(M), \|u_0\|_{L^2} \leq C\|1_E(t,x)e^{it(\Delta+V)}u_0\|_{L^2_{t,x}}.$

- Control of Schrödinger equation holds from $E_{t,x} \subset (0,T \times M$ iif (Cont)

$$\forall u_0, u_1 \in L^2(M), \exists f \in L^2((0,T) \times M);$$

$$(i\partial_t + \Delta + V)u = f1_E(t,x), \qquad u\mid_{\partial M}= 0, \qquad u\mid_{t=0}= 0, \quad u\mid_{t=T}= u_1$$

Theorem (HUM method)

(OBS) *and* (Cont) *are equivalent*

Observation from measurable sets: case of heat equation

Denote by $e^{t\Delta}u_0$ the solution to the linear heat equation

$$(\partial_t - \Delta)u = 0, \qquad u\,|_{\partial M} = 0, \qquad u\,|_{t=0} = u_0 \in L^2(M).$$

• Observation of heat equation holds from $E_{t,x} \subset (0,T) \times M$ iif

(OBS) $\quad \exists C > 0; \forall u_0 \in L^2(M), \|e^{T\Delta}u_0\|_{L^2} \leq C\|1_E(t,x)e^{it\Delta}u_0\|_{L^2_{t,x}}.$

Heat equation: observation

For heat equations, the picture is well advanced

Authors & References	Coefficients	E
Lebeau-Robbiano [95]	Smooth	open $_{t,x}$
Fursikov-Immanuvilov [95]	C^1	open $_{t,x}$
Miller[10]	Smooth	open $_{t,x}$
Phung-Wang [13]	Smooth	measurable$_t$ $\times$ open$_x$
Apraiz-Escauriaza		
-Wang-Zhang [14]	Constant	measurable$_{t,x}$
Burq-Moyano [22]	C^1	measurable$_{t,x}$
Burq-Moyano [22]	C^1	numerable$_t$ $\times$ hausdorf-dim$_x$ $> d - \delta$
Le Balch' Martin [24]	...	...
...	...	...

Table: Observability heat equations.

Geometric control for Schrodinger equations

The geometric control condition (T_0, ω) : All geodesics enter the control region ω in time $t < T_0$

Theorem (Lebeau 89)

Assume that the geometric control condition holds for ω an open set in time $T_0 > 0$. Then for any $T > 0$, any $V \in L^\infty$, observation (OBS) *holds on $(0, T) \times \omega$ (and hence also control* (Cont)*)*

Theorem (Burq, Hui Zhu 25)

Assume that the geometric control condition holds for ω an open set in time $T_0 > 0$. Then for any set of positive measure $F \subset (0, T)$, any $V \in L^\infty$, observation (OBS) *holds on $F_t \times \omega$ (and hence also control* (Cont)*)*

Schrödinger on the torus: eigenfunctions observation

For eigenfunctions of Δ on tori early results :Zygmund [72] and Connes [76]; Bourgain and Rudnick [12] for observability from hypersurfaces; (also works by Nazarov, Fontes-Merz, Egidi-Veselic, Germain-Moyano-Zhu)

Authors & References	d	ω
Zygmund [72]	$= 2$	measurable
Connes [76]	≥ 2	open

Table: Observability for toral eigenfunctions.

$$-\Delta e_n = \lambda_n^2 e_n \Rightarrow \|e_n\|_{L^2} \leq C\|e_n 1_\omega\|_{L^2}.$$

• Motivation was a Cantor–Lebesgue type result : if a sequence $(f_n)_{n\geq 0}$ of eigenfunctions converges pointwisely to zero on set of positive measure, then it converges to zero strongly in L^2.

• Parseval equality (and spectrum $\subset \mathbb{N}$): observation of eigenfunctions from $F \subset M$ is equivalent to observation of Schrödinger equations from $E = (0, 2\pi) \times F$ (using Ingham can be relaxed to any time interval of length $> \pi$). This was completely overlooked at the time

Schrödinger on the torus: observation

Results are mostly within the context of control theory. Haraux, Jaffard and Komornik used Kahane's theory of lacunary series , while Burq-Zworski, Macià Anantharaman used microlocal methods.

Authors & References	d	V	Ω
Haraux [89], Jaffard [90]	$= 2$	0	open
Komornik [92], Macià [10]	≥ 1	0	open
Burq–Zworski [12]	$= 1, 2$	smooth	open
Anantharaman–Macià [14]	≥ 1	Riemann integrable	open
Bourgain–Burq–Zworski [13]	$= 1, 2$	square integrable	open
Bourgain [14]	$= 3$	bounded	open
Burq–Zworski [19]	$= 1, 2$	0	$[0, T] \times \omega$
...			

Table: Observability for toral Schrödinger propagators, where $T > 0$ and $\omega \subset \mathbb{T}^d$ has a positive Lebesgue measure.

Conjecture

Observation (and control) hold on $\mathbb{T}^d$ for any $V \in L^\infty(\mathbb{T}^d))$ and any measurable (non trivial) observation domain $E \subset (0, T) \times \mathbb{T}^d$.

Our result

We consider on $C^\infty(\mathbb{T}^{1+d})$ the norm

$$\|f\|_Y = \|fe^{it\Delta}\|_{\mathcal{L}(L^2_x;L^2_{t,x})} = \sup_{\|u_0\|_{L^2_x}=1} \|fe^{it\Delta}u_0\|_{L^2_{t,x}}.$$

Notice that (taking $u_0 = 1$), $\|f\|_Y \geq \|f\|_{L^2_{t,x}}$,
and (using $\|e^{it\Delta}\|_{\mathcal{L}(L^2_x;L^\infty_t;L^2_x)} \leq 1$), $\|f\|_Y \leq \|f\|_{L^2_t;L^\infty_x}$.
Denote by Y the closure (in $L^2_{t,x}$) of $C^\infty_{t,x}$ for the Y-norm.

Theorem (Burq Zhu 25)

Assume that $V(x) \in Y \cap L^p_x$, *(*$p \geq \frac{d}{2}$, $d \geq 3$*), (*$p > 1$ $d = 1;2$*) and* $1_E(t,x) \in Y$. *Then* (OBS) *holds*

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Theorem (Burq Zhu 25)

Assume that $V(x) \in Y \cap L^p_x$, $(p \geq \frac{d}{2}, d \geq 3)$, $(p > 1\ d = 1; 2)$ *and* $1_E(t,x) \in Y$. *Then* (OBS) *holds*

Theorem

$L^\infty_{t,x} \subset Y \Leftrightarrow$, $\mathcal{F} = \{e^{it\Delta}u_0; \|u_0\|_{L^2} \leq 1\}$ *is uniformly* $L^2_{t,x}$ *integrable:*

$$\exists \Psi; \frac{\Psi(z)}{z} \to_{z\to+\infty} +\infty; \forall u_0, \|u_0\|_{L^2} \leq 1, \iint \Psi\left(|e^{it\Delta}u_0|^2\right) dtdx < +\infty.$$

Our result (continued)

Corollary (Burq Zhu 25)

Conjecture is true for $d = 1$*:* (OBS) *holds for all* $V \in L^p_x, p > 1$, $E_{t,x}$ *measurable*

Corollary (Burq Zhu 25)

For $d = 2$, (OBS) *holds for all* $V \in L^2_x$*, and all* $E = F_t \times G_{x_1,x_2}$*, F and G measurable (non trivial)*

Corollary (Burq Zhu 25)

For $d \geq 2$ (OBS) *holds for all* $V \in C^0$ *and* $E = E^1_{t,x_1} \times \cdots \times E^k_{x_{d-1},d_x}$*,* E^j *two dimensional measurable (non trivial), (one dimensional also OK)*

From Theorem to corollaries

The following conjecture implies equiintegrability (with $\Psi(t) = |t|^{\frac{p}{2}}$)

Conjecture (Bourgain et al.)

For any $p < \frac{2d}{d-2}$, there exists $C > 0$;

$$\|e^{it\Delta}u_0\|_{L^p(\mathbb{T}^{1+d})} \leq C\|u_0\|_{L^2}. \tag{0.1}$$

• For $d = 1$, (0.1) true for $p \leq 4$ which implies Corollary for $d = 1$ To get corollary for $d \geq 2$ we use weak forms of (0.1)

Lemma (Zygmund)

For any eigenfunctions of the Laplace operator on $\mathbb{T}^2$, $-(\partial_{x_1}^2 + \partial_{x_2}^2)e_\lambda = \lambda^2 e_\lambda$, we have $\|e_\lambda\|_{L^4(\mathbb{T}^2)} \leq C\|e_\lambda\|_{L^2(\mathbb{T}^2)}$

Corollary

On $\mathbb{T}^d$ we have

$$\|e^{it\Delta}u_0\|_{L^4_{y_1,y_2};L^2_{y_3,\dots,y_{d+1}}} \leq C\|u_0\|_{L^2(\mathbb{T}^d)}$$

Geometric control with measurable sets in time

$$\|u_0\|_{L^2}^2 \leq C \int_0^T \int_M 1_{t\in E} 1_{x\in\omega} |e^{it\Delta} u_0|^2(t,x)dxdt$$

- Regularize by convolution the characteristic function $1_E(t) \to \chi_\epsilon(t) = \rho_\epsilon \star 1_E$, $\|\rho_\epsilon - 1_E\|_{L^1(\mathbb{R})} \to 0$ when $\epsilon \to 0$.
- Prove HF observation for $\chi_\epsilon(t) \times 1_\omega(x)$ with HF constant uniform with respect to ϵ
- Pass to the limit $\epsilon \to 0$ to get HF observation by $1_E(t) \times 1_\omega(x)$
- Uniqueness result to pass from HF observation to general observation estimate

Key (simple) point 1

$$\iint |\chi_\epsilon - 1_E|(t) |e^{it\Delta} u_0|^2(t,x) dtdx \leq \|\chi_\epsilon - 1_E\|_{L^1(0,T)} \|e^{it\Delta} u_0\|^2_{L^\infty(0,T);L^2(M)}$$
$$\leq \|\chi_\epsilon - 1_E\|_{L^1(0,T)} \|u_0\|^2_{L^2(M)}.$$

Key (not so simple) point 2: Find a new proof of Lebeau's result with "explicit" HF constants (already somewhat present in Lebeau's approach to microlocal defect mesures)

Semi-classical HF estimates

Consider an eigenbasis of $L^2(M)$ $(e_n)_{n\in\mathbb{N}}$, where

$$-\Delta e_n = \lambda_n^2 e_n, \qquad \lambda_{n-1} \leq \lambda_n \to +\infty.$$

Let $\rho < 1$ and for $h << 1$,

$$E_{h,\rho} = \{u_0 \in L^2; u_0 = \sum_{k;\lambda_k \in (\rho h^{-1}, \rho^{-1} h^{-1})} u_{0,k} e_k\}.$$

Semi-classical Observation holds if

$$\exists C, h_0 > 0, \rho < 1; \forall h < h_0, \forall u_0 \in E_{h,\rho},$$

(SCO) $$\|u_0\|_{L^2}^2 \leq C \iint 1_E(t) 1_\omega(x) |e^{it\Delta} u_0|^2(t,x)\,dxdt.$$

Assume (SCO) does not hold. Then $\exists h_n \to 0, \rho_n \to 1, u_{0,n} \in E_{h_n,\rho_n}$;

$$\|u_{0,n}\|_{L^2}^2 = 1, \qquad \iint 1_E(t) 1_\omega(x) |e^{it\Delta} u_{0,n}|^2(t,x)\,dxdt \to 0, \quad n \to +\infty$$

Semi-classical measures: definition

Let $T^*M = M \times \mathbb{R}^d$. For $a \in C_0^\infty(\mathbb{R}_t \times T^*M))$ define

$$a(t,x,hD_x)u = \frac{1}{2\pi^d} \iint e^{\frac{i}{h}(x-y)\cdot\xi} a(t,x,h\xi)u(y)dyd\xi,$$

Lemma

There exists a non negative measure μ on $\mathbb{R}_t \times M \times \mathbb{R}^d)$ which satisfies

1. *After extracting a subsequence, with $u_n = e^{it\Delta}u_{0,n}$,*

$$\lim_{n\to+\infty} \Big(a(t,x,h_nD_x)u_n, u_n\Big)_{L^2_{t,x}} = \langle \mu, a(t,x,\xi)\rangle = \iint a(t,x,\xi)d\mu$$

2. *The measure μ is non zero: $\mu((0,T)\times T^*M) = T$.*
3. *The measure μ is supported by $\{\|\xi\| = 1\}$*
4. *The measure μ is invariant by the co-geodesic flow (reflected at the boundary according to the laws of geometric optics if $\partial M \neq \emptyset$)*
5. *The measure μ enjoys additional regularity:*

$$\mu \in L^\infty(\mathbb{R}_t; \mathcal{M}(T^*M))$$

Semi-classical measures: vanishing property

From

$$\iint 1_E(t)1_\omega(x)|e^{it\Delta}u_{0,n}|^2(t,x)dxdt \to 0, \quad n \to +\infty$$

Proposition

The measure μ vanishes on $E_t \times T^*\omega$.

$$\forall \chi \in C_0^\infty(\omega), \quad \lim_{n\to+\infty} \iint \rho_\epsilon(t)\chi(x)|e^{it\Delta}u_{0,n}|^2(t,x)dxdt$$

$$= \left(\rho_\epsilon(t)\chi(x)e^{it\Delta}u_{0,n}, e^{it\Delta}u_{0,n}\right)_{L^2_{t,x}} = \langle \mu, \rho_\epsilon(t)\chi(x)\rangle$$

$$\left|\lim_{n\to+\infty} \iint \rho_\epsilon(t)\chi(x)|e^{it\Delta}u_{0,n}|^2(t,x)dxdt\right|$$

$$= \lim_{n\to+\infty} \iint |\rho_\epsilon(t)-1_E(t)|\chi(x)|e^{it\Delta}u_{0,n}|^2(t,x)dxdt \le \|\rho_\epsilon(t)-1_E(t)\|_{L^1(0,T)}$$

And the result follows by dominated convergence: along a subsequence,

$$\rho_\epsilon(t) \to \text{ a.s. } 1_E(t), \text{ and } |\rho_\epsilon(t)| \le 1$$

$$\Rightarrow \lim_{\epsilon\to 0} \int \rho_\epsilon(t)\chi(x)d\mu = \int 1_E(t)\chi(x)d\mu = 0$$

The geometric control property and the contradiction

- For almost every $t \in E$, $\mu(t)$ vanishes above ω.
- From the invariance of μ by the (co)-geodesic flow, we deduce that for almost every $t \in E$, The measure μ vanishes on all geodesics starting from ω.
- From the Geometric control condition (all geodesics enter ω in finite time), we deduce (here $t \in E$ is fixed)

$$\text{For almost every } t \in E, \mu(t) \equiv 0$$

- Finally

$$\begin{aligned} 0 = \langle \mu, 1_E(t) \rangle &= \lim_{\epsilon \to 0} \langle \mu, \rho_\epsilon(t) \rangle \\ &= \lim_{\epsilon \to 0} \lim_{n \to +\infty} \iint \rho_\epsilon(t) |e^{it\Delta} u_{0,n}|^2 (t, x) dt dx \\ &= \lim_{\epsilon \to 0} \lim_{n \to +\infty} \int \rho_\epsilon(t) dt = \int 1_E(t) dt = |E| > 0 \end{aligned}$$

From semi-classical observation to observation : compactness-uniqueness

Theorem

Assume that $u = e^{it\Delta} u_0$ vanishes on $E_t \times \omega$, $|E| > 0$. Then it is identically 0.

Indeed, for $\varphi \in C_0^\infty(\omega)$, consider

$$t \mapsto \langle u(t,x)\varphi\rangle_{\mathcal{D}', C_0^\infty} = h_\varphi(t).$$

This function is holomorphic in the half plane $\mathbb{C}^- = \{z; \Im m z < 0\}$, has a continuous trace on the boundary of the half plane $\mathbb{R}$ which vanishes on a set of positive measure. As a consequence the function h_φ is identically zero in $\mathbb{C}^-$ (and its trace on $\mathbb{R}$ also). This implies that $u\mid_\omega = 0$ in $\mathcal{D}'(\omega)$, for all $t \in \mathbb{R}$ and hence $u \equiv 0$.

Back on tori: some ideas of proof: $(V = 0)$

1. Let $\chi(t,x) \in Y$ and $\chi_n \to \chi$ in Y, χ_n can actually be taken as a trigonometric polynomial. Modulo small error can assume χ is a trigonometric polynomial
2. Induction on the dimension of the torus
3. Reduction to (finitely many) lower dimensional tori via Granville-Spencer cluster structure of integer points on the characteristic manifold

$$\Sigma = \{(\tau, n) \in \mathbb{Z}^{1+d}; \tau = \sum_{j=1}^{d} n_j^2 = |n|^2\}$$

4. Treatment of low frequencies via a uniqueness theorem for solutions to Schrodinger equations

Granville-Spencer cluster structure

We work here with sublattices Λ of $\mathbb{Z}^d$. and affine sublattices $\Gamma = q + \Lambda$ for some $q \in \mathbb{Z}^d$, Let

$$\Sigma = \{(\tau, n) \in \mathbb{Z}^{1+d}; \tau = \sum_{j=1}^{d} n_j^2 = |n|^2\},$$

Lemma

Let $d \geq 1$ and let $S \subset \Sigma$
for all $R \geq R_0 >> 1$, there exists a partition

$$S = \cup_\alpha S_\alpha,$$

where

$$dist(S_\alpha, S_\beta) \geq 100R$$

and $diam(S_\alpha) \leq R^{K_d}$ and finally if S is included in an affine sublatice of dimension k, each S_α is included in another affine sublatice of dimension $\leq k - 1$.

Uniqueness for Schrodinger equation

We consider here solutions to the (elliptic) Schrödinger equation on $\mathbb{R}^k$

(elliptic) $$(-\Delta + V)u = 0.$$

An adaptation from De Figueiredo-Gosse (building on works by Carleman, Aronszajn, Jerison-Kenig) gives

Theorem

Assume that $V \in L^p_{loc}(\mathbb{R}^k)$, with $p \geq \frac{d}{2}$ (if $d \geq 3$) and $p > 1$ ((if $d = 1; 2$). If a weak solution $u \in H^1_{loc}$ to the equation (elliptic) *vanishes on a set of positive Lebesgue measure, then $u = 0$.*

Thank you for your attention !