Convex-analytic techniques for constrained reachability of linear control problems

Ivan Hasenohr, *Camille Pouchol*, Yannick Privat,
Emmanuel Trélat, Christophe Zhang

MAP5, Université Paris Cité

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Reachability for a linear control problem

Linear control problem

$$\dot{y}(t) = Ay(t) + Bu(t), \quad y(0) = y_0$$

- $y(t) \in X$, $u(t) \in U$, X and U Hilbert spaces, $E := L^2(0, T; U)$
- $(A, D(A))$ operator generating a C_0-semigroup over X, denoted by $(S_t)_{t \geq 0}$
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Main notions of reachability

- ⋄ exact: does there exist $u \in E$ s.t. $y(T) = y_f$?
- ⋄ approximate: does there exist, for all $\varepsilon > 0$, $u_\varepsilon \in E$ s.t. $\|y(T) - y_f\|_X \leq \varepsilon$?

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If so, how to compute them? $\rightsquigarrow$ looking for constructive approaches.

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Given a final time $T > 0$, $y_0 \in X$ and a target $y_f \in X$, can one find $u \in E_{\mathcal{U}} := L^2(0, T; \mathcal{U})$ steering y_0 to y_f (or at least close to y_f), at time T?

Main notions of reachability under constraints $\mathcal{U} \subset U$

- exact: does there exist $u \in E_{\mathcal{U}}$ s.t. $y(T) = y_f$?
- approximate: does there exist, for all $\varepsilon > 0$, $u_\varepsilon \in E_{\mathcal{U}}$ s.t. $\|y(T) - y_f\|_X \leq \varepsilon$?

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Reachability for a linear control problem (bis)

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Duhamel's formula:

$$y(T) = L_T u + S_T y_0, \qquad L_T u := \int_0^T S_{T-t} B u(t)\, dt,$$

with $L_T \in L(E, X)$.

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Upon translating, assume $y_0 = 0$ so that $\mathcal{U}$-reachability rewrites as follows

- exact: does there exist $u \in E_{\mathcal{U}}$ s.t. $L_T u = y_f$?
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Upon translating, assume $y_0 = 0$ so that $\mathcal{U}$-reachability rewrites as follows

- exact: does one have $y_f \in L_T E_{\mathcal{U}}$?
- approximate: does one have $y_f \in \overline{L_T E_{\mathcal{U}}}$?

Constraints discussed in this talk

$\mathcal{U}$ is (closed, convex), bounded (joint work with Ivan Hasenohr, Yannick Privat, Christophe Zhang)

Motivation:

- ◇ bilateral constraints $m \leq u \leq M$,
- ◇ possibly with additional energy constraints like $\|u\|_2 \leq C$.

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$\mathcal{U} = P$ is unbounded, assumed to be a cone (joint work with Emmanuel Trélat, Christophe Zhang)

Motivation:

- ◇ sign constraints, $P = \{u \in U,\ u \geq 0\}$... convex
- ◇ sparsity constraints, $P = \{u \in U,\ |\mathrm{supp}(u)| \leq k\}$... not convex, at all

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- ◇ Necessary and sufficient reachability conditions
- ◇ Constructive methods

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Remark: exact and approximate reachability coincide in the closed, convex and bounded setting... but not in the closed, convex and unbounded conic setting! Even in dimension 2.

$$\ddot{x} = u, \qquad P = \mathbb{R}_+.$$

For all $a > 0, T > 0$,

$(0, a)$ is approximately but not exactly P-reachable in time T.

Separation argument: geometric intuition

Condition for non $\mathcal{U}$-reachability: find $p_f \in X$ such that $J(p_f) < 0$.

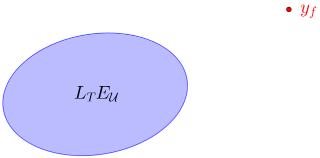

Figure: Separating y_f fom $L_T E_{\mathcal{U}}$. Courtesy of Ivan Hasenohr.

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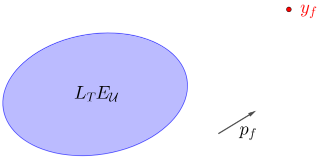

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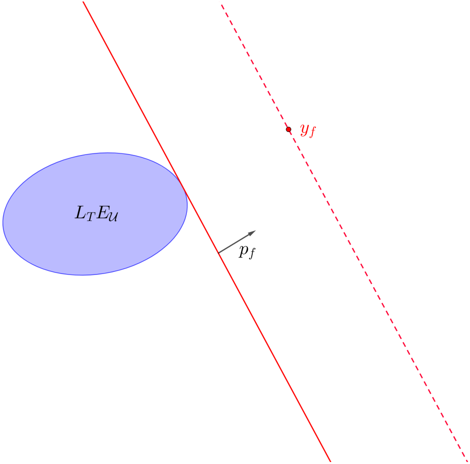

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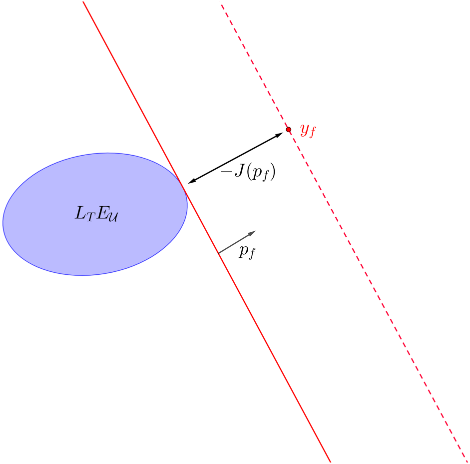

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Separation argument (1)

If there exists a strictly separating hyperplane between y_f and $L_T E_{\mathcal{U}}$, i.e., $p_f \in X$ s.t.

$$\sup_{u \in E_{\mathcal{U}}} \langle L_T u, p_f \rangle_X < \langle y_f, p_f \rangle_X,$$

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Above condition rewrites

$$\int_0^T \underbrace{\sup_{u \in \mathcal{U}} \langle u, L_T^* p_f(t) \rangle_U}_{\sigma_{\mathcal{U}}(L_T^* p_f(t))} \, dt < \langle y_f, p_f \rangle_X.$$

where $\sigma_{\mathcal{U}}(u) := \sup_{v \in \mathcal{U}} \langle u, v \rangle_U$ is the *support* function of $\mathcal{U}$.

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Define the dual functional

$$J(p_f) := \int_0^T \sigma_{\mathcal{U}}(L_T^* p_f(t)) \, dt - \langle y_f, p_f \rangle_X.$$

Separation argument (2)

Proposition

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In the case where $\mathcal{U}$ is convex, closed and bounded, the converse holds.

- ◇ Converse true by a separation argument (Hahn-Banach) in the weak topology.

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- Converse true by a separation argument (Hahn-Banach) in the weak topology.
- Tractable: $\sigma_{\mathcal{U}}$ explicitly computable for generic constraint sets.
- Can be generalised to convex closed sets $\mathcal{Y}_f$ in place of y_f.

1. Computer assisted-proofs of (non-) $\mathcal{U}$-reachability

2. Reachability conditions for conically constrained problems, with constructivity

Constraints:

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Bounded constraints

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In the (nonlinear or) linear ODE setting,

- ◇ huge body of literature to approximate the reachable set from inside or outside (more ambitious than fixed y_f, but much more costly)
- ◇ most often does not include numerical certification
- ◇ up to our knowledge, open problem in infinite dimension

Towards proofs of non $\mathcal{U}$-reachability

If there exists $p_f \in X$ s.t. $J(p_f) < 0$, then y_f is not $\mathcal{U}$-reachable in time T ('sharp', because the converse holds)

$$J(p_f) = \int_0^T \sigma_{\mathcal{U}}(L_T^* p_f(t))\, dt - \langle y_f, p_f \rangle_X .$$

Key remark: J takes finite values on the whole of X in the bounded case.

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Dual equation, through $L_T^* p_f(t) = B^* p(t)$

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- ◇ Need to discretise in time: the time-integral + the dual ODE/PDE,
- ◇ in infinite-dimension, need to discretise in space.

Standing assumption: some form of dissipativity for the semigroup $(S_t)_{t\geq 0}$

Towards computer-assisted proofs of non $\mathcal{U}$-reachability: finite dimension

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Using Euler's implicit scheme for the dual equation + Riemann sums for the time-integral

- Define a discretised proxy $J_\mathrm{d} : X \to \mathbb{R}$

$$J_\mathrm{d}(p_f) = \Delta t \sum_{n=1}^{N_t} \sigma_{\mathcal{U}}((\mathrm{Id} + \Delta t\, A^*)^{-1} p_f) - \langle y_f, p_f \rangle_X.$$

- Establish error bounds: for all $p_f \in X$

$$|J(p_f) - J_\mathrm{d}(p_f)| \leq C_1 \Delta t \|A^* p_f\|_X = e(p_f),$$

with explicit constant C_1.

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- Find $p_f \in X$ such that $J_{\mathrm{d}}(p_f) < 0$.
- Check, by means of interval-arithmetic, that $J_{\mathrm{d}}(p_f) + e(p_f) < 0$, hence $J(p_f) < 0$.

p_f is a dual certificate of non $\mathcal{U}$-reachability (of y_f in time T), cf *Computer-assisted proofs of non-reachability for linear finite-dimensional control systems* (HPPZ '25, to appear in SICON)

Towards computer-assisted proofs of non $\mathcal{U}$-reachability: infinite dimension

Standing assumption: some form of dissipativity for the semigroup $(S_t)_{t\geq 0}$

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- ⋄ Define a discretised proxy $J_{\mathrm{d}} : V_h \to \mathbb{R}$ where $V_h \subset X$ are approximation spaces

$$J_{\mathrm{d}}(p_{fh}) = \Delta t \sum_{n=1}^{N_t} \sigma_{\mathcal{U}}((\mathrm{Id} + \Delta t\, A_h^*)^{-1} p_{fh}) - \langle y_f, p_{fh} \rangle_X .$$

- ⋄ Establish error bounds, for all $p_f \in X, p_{fh} \in V_h$

$$|J(p_f) - J_{\mathrm{d}}(p_{fh})| \leq (C_1 \Delta t + C_2 h^2)\|A^* p_f\|_X + C_3\|p_f - p_{fh}\|_X =: e(p_f, p_{fh}),$$

with explicit constants C_1, C_2, C_3.

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  with explicit constants C_1, C_2, C_3.

- Find $p_{fh} \in V_h$ such that $J_{\mathrm{d}}(p_{fh}) < 0$,
- Interpolate $p_{fh} \in V_h$ into $p_f \in \mathcal{D}(A^*)$,
- Check, by means of interval-arithmetic, that $J_{\mathrm{d}}(p_{fh}) + e(p_f, p_{fh}) < 0$, hence $J(p_f) < 0$.

p_f is a dual certificate of non $\mathcal{U}$-reachability (of y_f in time T), cf upcoming preprint *Computer-assisted proofs of non-reachability for parabolic linear control problems* (HPPZ)

Example in finite dimension

$$A := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix}, \qquad B := \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Estimate:

$$|J(p_f) - J_{\mathrm{d}}(p_f)| \leq \frac{1}{2}\Delta t \; MT\|B\|\|A^* p_f\|\kappa(P)Q_2(\|N\|T).$$

$$\mathcal{U} = \{u \in \mathbb{R}^2, \; \|u\|_2 \leq M_2, \; \|u\|_\infty \leq M_\infty\}.$$

$$\mathcal{Y}_f = \left\{(y_1, y_2, y_3, y_4) \in \mathbb{R}^4, \; \|(y_1 - z_1, y_2 - z_2)\|_{\mathbb{R}^2} \leq \eta\right\},$$

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Theorem

Take $z_1 = z_2 = 0.5$, $\eta = 0.1$, $M_2 = 1.15$, $M_\infty = 1$ and $T = 1$. Then $\mathcal{Y}_f$ is not $\mathcal{U}$-reachable in time T thanks to the dual certificate

$$p_f = (0.62, 0.78, 0, 0)^T \quad \text{for which} \quad J(p_f) \in [-0.1146, -0.0717].$$

Example in infinite dimension

Consider $\partial_t y - \partial_{xx} y = \chi_\omega u$ & Dirichlet boundary conditions

$$\mathcal{U} = \{u \in L^2(0,1),\ 0 \leq u \leq M \text{ a.e.}\}, \quad \omega = (\tfrac{1}{5}, \tfrac{2}{5}) \cup (\tfrac{4}{5}, 1)$$

$\mathbb{P}_1$ finite elements + interpolating with cubic splines + using the estimate (with $M_0 = M|\omega|^{1/2},\ C = 2 + \frac{2}{\sqrt{3}}$)

$$|J(p_f) - J_{\mathrm{d}}(p_{fh})| \leq M_0 T \left((\tfrac{1}{2} + C)\Delta t + \tfrac{1}{2}(7 + 4\ln(2) + C)h^2\right) \|A^* p_f\| + \left(M_0 TC + \|y_f\|\right)\|p_f - p_{fh}\|,$$

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Theorem

For $M = 1,\ T = 1$, the target $y_f = \lambda \sin(\pi \cdot)$ is not $\mathcal{U}$-reachable in time T if $\lambda \ge 0.035$.

Consider $\partial_t y - \partial_{xx} y = \chi_\omega u$ & Dirichlet boundary conditions

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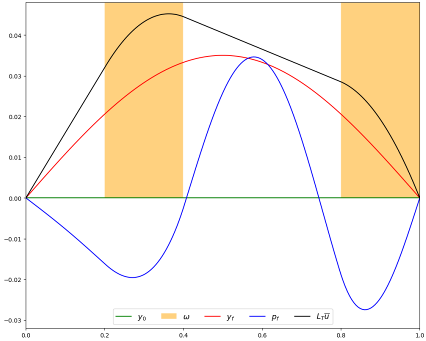

Figure: Target and optimal dual certificate. Courtesy of Ivan Hasenohr

Additional remarks

Crucial part: be able to find p_f (or p_{fh}) at which J_{d} is negative

- Done by 'minimising' J_{d}
- Found p_f can have huge $\|A^* p_f\|$: regularisation can be necessary
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Next in store? $\mathcal{U}$-reachability

- ◇ Possible but intractable in large dimensions
- ◇ Cannot deal with infinite-dimensional problems

1. Computer assisted-proofs of (non-) $\mathcal{U}$-reachability

2. Reachability conditions for conically constrained problems, with constructivity

Constraints:

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Unbounded (conic) constraints,

- ◇ HUM method (Lions '88), unconstrained
- ◇ sparsity in finite dimension, $k = 1$ (Zuazua '10)
- ◇ isotropic constraints (Berrahmoune '14 et '19)
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Constructive methods, based on an appropriate dual functional, also yield a sufficient reachability condition.

Goal: come up with a general recipe, cf preprint *Constructive reachability for linear control problems under conic constraints* (PTZ '25)

Unconstrained reachability: the HUM method (1)

Unconstrained case, i.e., $P = U$: find $u \in E$ s.t. $\|y(T) - y_f\|_X = \|L_T u - y_f\|_X \leq \varepsilon$.

For $\varepsilon \geq 0$, so-called dual functional, defined for $p_f \in X$

$$
\begin{aligned}
J_\varepsilon(p_f) &= \frac{1}{2}\int_0^T \|L_T^* p_f(t)\|_U^2 \, dt - \langle y_f, p_f\rangle_X + \varepsilon \|p_f\|_X \\
&= \frac{1}{2}\|L_T^* p_f\|_E^2 - \langle y_f, p_f\rangle_X + \varepsilon \|p_f\|_X
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Provides a sufficient condition (actually also necessary) for reachability

- ◇ approximate: $\forall p_f \in X, \quad L_T^* p_f = 0 \implies \langle y_f, p_f \rangle_X = 0,$
- ◇ exact: $\exists c > 0, \ \forall p_f \in X, \quad |\langle y_f, p_f \rangle_X| \leq c \|L_T^* p_f\|_E.$

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Constructive: if $p_f^\star$ minimises J_ε, then $u^\star := L_T^* p_f^\star$ steers 0 to $\overline{B}(y_f, \varepsilon)$ at time T.

Unconstrained reachability: the HUM method (2)

Dual functional, of what? Reachability, i.e., existence of $u \in E$ s.t. $\|y(T) - y_f\|_X \leq \varepsilon$, is equivalent to

$$\pi_\varepsilon = \inf_{u \in E,\ \|y(T)-y_f\|_X \leq \varepsilon} \underbrace{\frac{1}{2}\|u\|_E^2}_{=:F(u)} < +\infty.$$

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Put constraints into objective function

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where

$$G(L_T u) = \begin{cases} 0 & \text{if } \|y(T) - y_f\|_X = \|L_T u - y_f\|_X \leq \varepsilon \\ +\infty & \text{else} \end{cases}$$

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Fenchel-Rockafellar Theorem:

$$\begin{aligned} \pi_\varepsilon &= -\inf_{p_f \in X} F^*(L_T^* p_f) + G^*(-p_f) \\ &= -\inf_{p_f \in X} \underbrace{\tfrac{1}{2}\|L_T^* p_f\|_E^2 - \langle y_f, p_f \rangle + \varepsilon \|p_f\|_X}_{J_\varepsilon(p_f)} \end{aligned}$$

and π_ε is attained if finite.

Fenchel conjugate, gauge and support function

H Hilbert, $x \in H$

For $f : H \to]-\infty, +\infty]$,

- subdifferential

$$\partial f(x) := \{p \in H,\ \forall y \in H,\ f(y) \geq f(x) + \langle p, y - x \rangle\},$$

- Fenchel conjugate

$$f^\star(x) := \sup_{y \in H} \big(\langle x, y \rangle - f(y)\big).$$

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For C closed and convex,

- indicator function δ_C defined by $\delta_C(x) = 0$ if $x \in C$, $+\infty$ else,
- support function $\sigma_C := \delta_C^*$, i.e. par définition

$$\sigma_C(x) = \sup_{y \in C} \langle x, y\rangle,$$

- gauge of C,

$$j_C(x) := \inf\{\alpha > 0,\ x \in \alpha C\}.$$

Closed convex case

Constraints: defined by a closed convex cone P_r

Choose $\mathcal{U}_r$ convex, closed, bounded that generates P_r (i.e., s.t. $\mathrm{cone}(\mathcal{U}_r) = P_r$). Typically $\mathcal{U}_r = P_r \cap \overline{B}(0,1)$.

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$$\text{Set} \quad F(u) := \frac{1}{2} \int_0^T j_{\mathcal{U}_r}^2(u(t))\, dt.$$

- Cost F enforces the constraints: $F(u) < +\infty \iff u \in L^2(0, T; P_r)$.
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$$\pi_\varepsilon = \inf_{u\in E,\ \|y(T)-y_f\|_X \le \varepsilon} F(u) = \inf_{u\in E} F(u) + G(L_T u), \qquad G = \delta_{\overline{B}(y_f,\varepsilon)}.$$

$$\pi_\varepsilon = -\inf_{p_f\in X} F^*(L_T^* p_f) + G^*(-p_f)$$

$$= -\inf_{p_f\in X} \underbrace{\frac{1}{2}\int_0^T \sigma^2_{\mathcal{U}_r}(L_T^* p_f(t))\, dt - \langle y_f, p_f\rangle + \varepsilon\|p_f\|_X}_{J_\varepsilon(p_f)}$$

and π_ε is attained if finite.

How constructive is this?

If $p_f^\star$ is dual optimal, then any optimal control $u^\star$ must satisfy $u^\star \in \partial F^*(L_T^* p_f^\star)$. Necessary optimality condition... not sufficient in general. Becomes sufficient if subdifferential reduced to a singleton.

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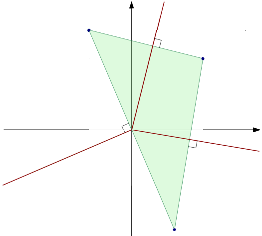
Uniqueness holds as soon as

$$L_T^* p_f^\star(t) \notin \mathrm{sing}(\mathcal{U}_r) \text{ for a.e. } t \in (0,T),$$

"hence" as soon as

$$\forall p_f \neq 0, \quad B^* S_t^* p_f \notin \mathrm{sing}(\mathcal{U}_r) \text{ for a.e. } t > 0, \tag{H}$$

where $\mathrm{sing}(\mathcal{U}_r) := \{q \in U,\ \arg\max_{v \in \mathcal{U}_r} \langle q, v\rangle_U \text{ is not a singleton}\}$



Closed convex case - reachability

Functionals of interest, for $\varepsilon > 0$:

$$J_\varepsilon(p_f) = \frac{1}{2}\int_0^T \sigma^2_{\mathcal{U}_r}(L_T^* p_f(t))\,dt - \langle y_f, p_f\rangle + \varepsilon\|p_f\|_X.$$

Relevant condition

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Theorem

y_f is approximately P_r-reachable in time T iff

$$\forall p_f \in X, \quad F^*(L_T^* p_f) = 0 \implies \langle y_f, p_f\rangle_X \leq 0 \tag{C$_a$}$$

Under (C$_a$), for all $\varepsilon > 0$, J_ε admits a unique minimiser $p_f^\star$, and if (H) holds, then the unique $u_\varepsilon^\star \in \partial F^*(L_T^* p_f^\star)$ satisfies the constraints P_r and steers 0 to $\overline{B}(y_f, \varepsilon)$ at time T.

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- Sufficiency of (C$_a$): coercivity of J_ε,
- Necessity of (C$_a$): independent argument,
- Uniqueness of minimiser: study of optimality conditions + strict convexity.

Closed convex case - reachability

Functional of interest:

$$J_0(p_f) = \frac{1}{2}\int_0^T \sigma^2_{\mathcal{U}_r}(L_T^* p_f(t))\,dt - \langle y_f, p_f\rangle.$$

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$$\forall p_f \neq 0, \quad B^* S_t^* p_f(t) \notin \mathrm{sing}(\mathcal{U}_r) \text{ for a.e. } t > 0. \tag{H}$$

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y_f is exactly P_r-reachable in time T iff

$$\exists c > 0,\ \forall p_f \in X, \quad \langle y_f, p_f\rangle_X \leq c\, F^*(L_T^* p_f)^{1/2} \tag{C$_e$}$$

*Under (C_e), **and if** J_0 admits a minimiser $p_f^\star$ then any $u^\star \in \partial F^*(L_T^* p_f^\star)$ satisfies the constraints P_r and steers 0 to y_f in time T for each such minimiser $p_f^\star$.*

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Context: $B = \mathrm{Id}$, $X = U = L^2(\Omega)$

Typical example: $\partial_t y - \Delta y = u$ (+ Dirichlet boundary conditions).

Shape control: problem and relaxation

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Constraints P: for $0 < m < |\Omega|$

for a.e. $t \in (0, T)$, $u(t) = M(t)\, \chi_{\omega(t)}$ where $M(t) > 0$ and $|\omega(t)| \leq m$.

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Lemma 1: For $y_f \in X$, $T > 0$ fixed s.t. $y_f \geq S_T y_0 = 0$, y_f is P_r-approximately reachable.

Lemma 2: 'Relaxation is bound to work', i.e., $\mathrm{ext}(\mathcal{U}_r) = \mathcal{U}$.

What about Condition (H), i.e.,

$$\forall p_f \neq 0, \quad S_t^* p_f(t) \notin \mathrm{sing}(\mathcal{U}_r) \text{ for a.e. } t > 0? \tag{H}$$

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Bathtub lemma: study of the optimisation problem

$$\sigma_{\mathcal{U}_r}(q) = \sup_{v \in \mathcal{U}_r} \langle q, v \rangle_U = \sup_{v \in \mathcal{U}_r} \int_\Omega q(x)v(x)\, dx.$$

If all level sets of q have measure 0, then there exists a unique maximiser, i.e., $q \notin \operatorname{sing}(\mathcal{U}_r)$

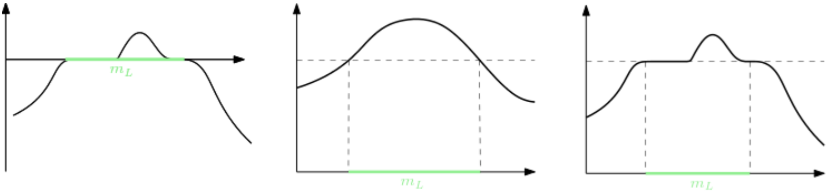

Figure: Relaxed in the bathtub. Courtesy of Christophe Zhang.

Shape (approximate) control: back to the original cone

Theorem

Let y_f, T *s.t.* $y_f \geq S_T y_0 = 0$. *If the adjoint semigroup satisfies*

$$\forall p_f \neq 0, \text{ all the level sets of } S_t^* p_f \text{ have measure } 0 \text{ for a.e. } t > 0,$$

then y_f *is approximately* P*-reachable in time* T*, whatever* $T > 0$ *and* $m > 0$ *are.*

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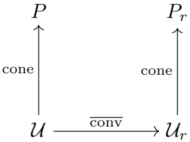
then y_f is approximately P-reachable in time T, whatever $T > 0$ and $m > 0$ are.

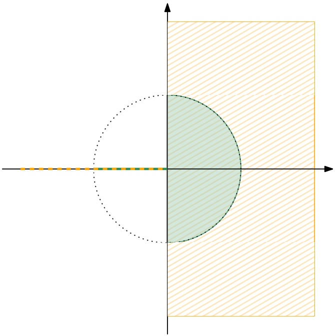
- Constructive: formula for the unique optimal control from the unique dual optimal variable
- Covers the heat (Dirichlet) equation, and more generally analytic-hypoelliptic operators (+ few non-restrictive properties)
- Nonnegative controllability result, 'optimal' (for the heat equation, say) because
  - $u \geq 0 \implies y(T) \geq 0$, by the parabolic comparison principle
  - if restriction on where the control acts, small-time obstructions
- Exact reachability: open problem.

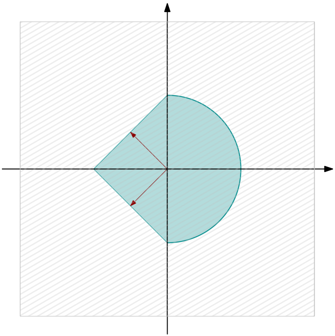
Constraints: defined by a cone P (containing 0).

(i) Choose a bounded set $\mathcal{U}$ generating P, i.e, $P = \mathrm{cone}(\mathcal{U})$.

(ii) Apply the previous recipe to $\mathcal{U}_r := \overline{\mathrm{conv}}(\mathcal{U})$, of associated cone $P_r := \mathrm{cone}(\mathcal{U}_r)$.







Relaxation usually works

Functional J_ε, associated to $\mathcal{U}_r$: yields $u^\star$ taking values in P_r... maybe even in P?

If $p_f^\star$ minimises J_ε, then any optimal control satisfies $u^\star \in \partial F^*(L_T^* p_f^\star)$, i.e.,

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$$u^\star \in \partial F^*(L_T^* p_f^\star) \iff \text{for a.e. } t \in (0,T),\ u^\star(t) \in \sigma_{\mathcal{U}_r}(L_T^* p_f^\star(t)) \operatorname*{arg\,max}_{v \in \mathcal{U}_r} \langle L_T^* p_f^\star(t), v\rangle_U.$$

Uniqueness ensured by

$$\forall p_f \neq 0, \quad B^* S_t^* p_f \notin \mathrm{sing}(\mathcal{U}_r) \ \text{ for a.e. } t > 0 \tag{H}$$

If (H) is satisfied, unique such control $u^\star$, which must be extremal:

$$u^\star(t) \in \sigma_{\mathcal{U}_r}(L_T^* p_f^\star(t))\, \mathrm{ext}(\mathcal{U}_r).$$

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Hence $u^\star(t)$ is in $\sigma_{\mathcal{U}_r}(L_T^* p_f^\star(t))\,\mathcal{U}$ generically, as soon as

$$\mathrm{ext}(\mathcal{U}_r) = \mathrm{ext}(\overline{\mathrm{conv}}(\mathcal{U})) \subset \mathcal{U} \tag{E}$$

Milman's theorem: (E) holds under any of the two hypothèses

- ◇ $\mathcal{U}$ is weakly closed,
- ◇ $\mathcal{U}_r$ is (strongly) compact and $\mathcal{U}$ is (strongly) closed.

Functionals of interest: for $\varepsilon > 0$

$$J_\varepsilon(p_f) = \frac{1}{2}\int_0^T \sigma^2_{\mathcal{U}_r}(L_T^* p_f(t))\,dt - \langle y_f, p_f\rangle + \varepsilon\|p_f\|_X.$$

Relevant conditions:

$$\forall p_f \neq 0, \quad B^* S_t^* p_f \notin \mathrm{sing}(\mathcal{U}_r) \text{ for a.e. } t > 0, \tag{H}$$

$$\mathrm{ext}(\mathcal{U}_r) = \mathrm{ext}(\overline{\mathrm{conv}}(\mathcal{U})) \subset \mathcal{U}. \tag{E}$$

Theorem

Assume that y_f is approximately P_r-reachable in time T, i.e. (C_a).

Then J_ε admits a unique minimiser $p_f^\star$ for all $\varepsilon > 0$, and if (H) and (E) are satisfied, then the unique $u_\varepsilon^\star \in \partial F^*(L_T^* p_f^\star)$ satisfies the constraints P and steers 0 to $\overline{B}(y_f, \varepsilon)$ in time T.

In particular, if (H) et (E) are satisfied, then y_f is approximately P-reachable in time T.

Functional of interest:

$$J_0(p_f) = \frac{1}{2}\int_0^T \sigma^2_{\mathcal{U}_r}(L_T^* p_f(t))\,dt - \langle y_f, p_f\rangle.$$

Relevant conditions:

$$\forall p_f \neq 0, \quad B^* S_t^* p_f \notin \mathrm{sing}(\mathcal{U}_r) \text{ for a.e. } t > 0, \tag{H}$$

$$\mathrm{ext}(\mathcal{U}_r) = \mathrm{ext}(\overline{\mathrm{conv}}(\mathcal{U})) \subset \mathcal{U}. \tag{E}$$

Theorem

Assume that y_f is exactly P_r-reachable in time T, i.e., (C_e).

Then, if J_0 admits a minimiser $p_f^\star$, and if (H) et (E) are satisfied, then for any such minimiser, the unique $u^\star \in \partial F^*(L_T^* p_f^\star)$ satisfies the constraints P and steers 0 to y_f in time T.

In particular, if (H) et (E) are satisfied, then y_f is exactly P-reachable in time T.

Context : $X = \mathbb{R}^n$, $U = \mathbb{R}^m$, A et B matrices

k-sparse controls

$$\text{for a.e. } t \in (0, T), \quad \|u(t)\|_0 \leq k$$

$$P^{(k)} := \{u \in \mathbb{R}^m,\ \|u\|_0 \leq k\}, \quad \text{closed, not convex (for } k \leq m-1\text{).}$$

Generator:

$$\mathcal{U}^{(k)} := P^{(k)} \cap \overline{B}_\infty(0,1) = \{u \in \mathbb{R}^m,\ \|u\|_0 \leq k,\ \|u\|_\infty \leq 1\}.$$

Sparsity in finite dimension: problem and relaxation

Context : $X = \mathbb{R}^n$, $U = \mathbb{R}^m$, A et B matrices

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Relaxation:

$$\mathcal{U}_r^{(k)} = \overline{\mathrm{conv}}(\mathcal{U}^{(k)}) = \{u \in \mathbb{R}^m,\ \|u\|_1 \leq k,\ \|u\|_\infty \leq 1\},$$

Remarks:

- $P_r =$ the whole $\mathbb{R}^m$ i.e., the relaxed problem is unconstrained,
- $\mathcal{U}^{(k)}$ is closed, hence Milman's theorem applies (hypothesis (E))

$$\mathcal{U}_r^{(k)} = \overline{\mathrm{conv}}(\mathcal{U}^{(k)}) = \{u \in \mathbb{R}^m,\ \|u\|_1 \leq k,\ \|u\|_\infty \leq k\},$$

Gauge and support functions

$$\forall u \in \mathbb{R}^m, \quad j_{\mathcal{U}_r^{(k)}}(u) = \max\left(\frac{\|u\|_1}{k}, \|u\|_\infty\right), \qquad \sigma_{\mathcal{U}_r^{(k)}}(u) = \sum_{i=1}^{k} |u_{(i)}|,$$

where, for $u \in \mathbb{R}^m$, $|u_{(1)}| \geq |u_{(2)}| \geq \ldots \geq |u_{(m)}|$.

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Associated cost

$$\forall u \in E, \quad F(u) = \tfrac{1}{2} \int_0^T j^2_{\mathcal{U}_r^{(k)}}(u(t))\, dt.$$

For $p_f \in \mathbb{R}^n$, letting $p(t) = S^*_{T-t} p_f$, the dual fonctional equals

$$\begin{aligned} J_0(p_f) &= \tfrac{1}{2} \int_0^T \sigma^2_{\mathcal{U}_r^{(k)}}(L_T^* p_f(t))\, dt - \langle y_f, p_f \rangle_{\mathbb{R}^n} \\ &= \tfrac{1}{2} \int_0^T \left(\sum_{i=1}^{k} |(B^* p(t))_{(i)}| \right)^2 dt - \langle y_f, p_f \rangle_{\mathbb{R}^n}. \end{aligned}$$

Sparsity in finite dimension: results

One can show that

$$\mathrm{sing}(\mathcal{U}_r^{(k)}) = \{u \in \mathbb{R}^m,\ |u_{(k)}| = |u_{(k+1)}|\},$$

hence (H) rewrites

$$\forall p_f \neq 0, \quad \{t > 0,\ |(B^* S_t^* p_f)_{(k)}| = |(B^* S_t^* p_f)_{(k+1)}|\} \text{ has zero measure,} \tag{H}$$

Proposition

Assume that the pair (A, B) is controllable.

If (H) holds, then for all y_f, T, y_f est is exactly reachable in time T by k-sparse controls.

- *Proof*: Relaxed problem is unconstrained, by Kalman's criterion 'relaxed' reachability holds and it can be checked that J_0 has a minimiser.
- Constructive: formulae for optimal controls as a function of minimisers of J_0.
- Open question: make (H) more explicit? (sufficient conditions available, but quite strong)

Consider, in the unconstrained case

$$\partial_t y(t,x) = \Delta y(t,x) + \chi_\omega(x) u(t,x), \qquad + \text{ Dirichlet boundary conditions,} \tag{1}$$

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rewritten in the form

$$\dot{y}(t) = Ay(t) + Bu(t)$$

avec $X = U = L^2(\Omega)$, $A = \Delta$, $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$, $B = u \mapsto \chi_\omega u$.

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For all $\omega \subset \Omega$ (of positive measure),

- ⋄ (1) is approximately controllable

$$\forall y_f \in X, \forall T > 0,\ \forall \varepsilon > 0, \quad \exists u_\varepsilon \in E \text{ s.t. } \|y(T) - y_f\|_X \leq \varepsilon.$$

- ⋄ (1) is **not** exactly controllable

Dual equation, $p_f \in X$

$$\begin{cases} \partial_t p + \Delta p = 0 \\ p(T, \cdot) = p_f, \\ p_{|\partial\Omega} = 0, \end{cases}$$

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Dual functional

$$\begin{aligned} J(p_f) &= \frac{1}{2}\int_0^T \int_\omega p^2(t,x)\,dx\,dt - \langle y_f, p_f \rangle_X + \varepsilon \|p_f\|_X \\ &= \frac{1}{2}\|\chi_\omega p\|_E^2 - \langle y_f, p_f \rangle_X + \varepsilon \|p_f\|_X \end{aligned}$$

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Key result: coercivity thanks to Holmgren's uniqueness theorem:

$$\Big(\forall (t,x) \in (0,T) \times \omega, \quad p(t,x) = 0\Big) \quad \Longrightarrow \quad p_f = 0.$$

J admits a unique minimiser $p_f^\star$, and the control

$$u^\star := \chi_\omega p^\star$$

steers 0 to $\overline{B}(y_f, \varepsilon)$ in time T.

$$\partial_t y(t,x) = \Delta y(t,x) + \chi_\omega(x) u(t,x) \qquad + \text{Dirichlet boundary conditions},$$

with

$$P = \{u \in L^2(\Omega),\ u \geq 0\}.$$

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First obstruction: monotonicity

$$\forall u \geq 0, \quad \forall t > 0, \quad y(t) \geq S_t y_0 = 0.$$

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First obstruction: monotonicity

$$\forall u \geq 0, \quad \forall t > 0, \quad y(t) \geq S_t y_0 = 0.$$

What if $y_f \geq 0$?

Theorem

If there exists $B(x,r) \subset \Omega \setminus \omega$, then one can build $y_f \geq 0$ s.t., for small enough T, y_f is not P-(approximately) reachable in time T.

Idea of proof (inspired from Pighin-Zuazua '18), build a separating hyperplane!

Dual equation, $p_f \in X$

$$\begin{cases} \partial_t p + \Delta p = 0 \\ p(T, \cdot) = p_f, \\ p_{|\partial\Omega} = 0, \end{cases}$$

Nonnegative reachability and obstructions (2)

Idea of proof (inspired from Pighin-Zuazua '18), build a separating hyperplane!

Dual equation, $p_f \in X$

$$\begin{cases} \partial_t p + \Delta p = 0 \\ p(T, \cdot) = p_f, \\ p_{|\partial\Omega} = 0, \end{cases}$$

Find $y_f \geq 0$, p_f such that equality below cannot hold with $y(T) = y_f$

$$\langle y(T), p_f \rangle_X = \int_0^T \langle p(t), \chi_\omega u(t) \rangle_X \, dt.$$

Nonnegative reachability and obstructions (2)

Idea of proof (inspired from Pighin-Zuazua '18), build a separating hyperplane!

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$$\begin{cases} \partial_t p + \Delta p = 0 \\ p(T, \cdot) = p_f, \\ p_{|\partial\Omega} = 0, \end{cases} \qquad (\mathcal{D})$$

Find $y_f \geq 0$, p_f such that equality below cannot hold with $y(T) = y_f$

$$\langle y(T), p_f \rangle_X = \int_0^T \langle p(t), \chi_\omega u(t) \rangle_X \, dt.$$

Pick any $y_f \geq 0$, $y_f \neq 0$, and then p_f s.t.

(i) $p_f < 0$ sur $\mathrm{supp}(y_f)$,

(ii) pour T small enough, $p \geq 0$ on $(0, T) \times (\Omega \setminus B(x, r))$, where p solves $(\mathcal{D})$.

$$\Longrightarrow \qquad \langle y_f, p_f \rangle_X < 0 \quad \text{and} \quad \int_0^T \langle p(t), \chi_\omega u(t) \rangle_X \, dt \geq 0.$$