Observability of Schrödinger equations on abelian covers of compact hyperbolic surfaces

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$$\|u_0\|^2_{L^2(\Omega)} \leq C \int_0^T \|e^{it\Delta_g} u_0\|^2_{L^2(\omega)} \, \mathrm{d}t \qquad \textbf{(Obs)}$$

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Lebeau 1992 obtained the observability and control for the Schrödinger equation from an open subset ω at time T under the following geometric control condition (GCC):

Any (generalized) geodesic, meets ω in a time $t \leq T$.

Results in tori

- $\Omega = \mathbb{T}^d, d \geq 1$, ω open: Jaffard 1990, Haraux 1989 (in dimension 2), Komornik 1992 (in higher dimension).
- $\Omega = \mathbb{T}^d$, potential $V \in C^\infty$, ω open: Burq and Zworski 2012.
- $\Omega = \mathbb{T}^2$, potential $V \in L^2$, ω open: Bourgain, Burq, and Zworski 2013.
- $\Omega = \mathbb{T}^1, \mathbb{T}^2$, ω measurable: Burq and Zworski 2019.

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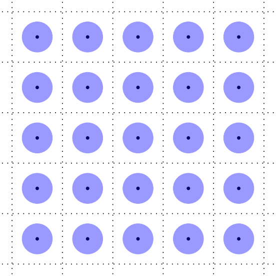

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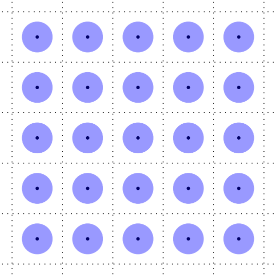

Täufer 2022 used Floquet–Bloch theorem (Floquet theorem, Bloch theorem) $\implies$ (**Obs**) from S.
$d = 1$, S thick: Su, Sun, and Yuan 2023 using a different method.
$d = 2$, ω measurable and $|\omega| > 0$: Le Balc'H and Martin 2023.

Bloch theory

The Bloch transform $\mathcal{F}$ is given by

$$\begin{aligned} \mathcal{F} : L^2(\mathbb{R}^d) &\longrightarrow \quad L^2([0, 2\pi]^d \times [0, 1]^d) \\ u &\longmapsto \quad \sum_{k \in \mathbb{Z}^d} e^{2\pi i \theta \cdot k} u(y + 2\pi k) \end{aligned}$$

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$\forall\theta \in [0,1]^d$, apply the Bloch transform to the Schrödinger equation, we obtain

$$\longrightarrow i\partial_t \mathcal{F}(u)(y,\theta) = -\Delta_\theta \mathcal{F}(u)(y,\theta)$$

where $\Delta_\theta = \Delta$ with θ-quasiperiodic boundary condition

Idea from Täufer

Rewrite the above as

$$\Longrightarrow \begin{cases} i\partial_t v = -\Delta_\theta v, \\ v(y_1, \cdots, \underbrace{0}_{k\text{-th}}, \cdots, y_d) = e^{2\pi i\theta_k} v(y_1, \cdots, \underbrace{2\pi}_{k\text{-th}}, \cdots, y_d), k = 1, 2, \cdots, d. \end{cases}$$

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Its observability inequality is given by

$$\|v_0\|^2_{L^2(\mathbb{T}^d)} \leq C_\theta \int_0^T \|e^{it\Delta_\theta} v_0\|^2_{L^2(\omega)} \, \mathrm{d}t, \quad \forall v_0 \in L^2(\mathbb{T}^2) \qquad (\mathbf{Obs}_\theta)$$

$(\mathbf{Obs})$ in $\mathbb{R}^d$ $\xRightarrow{\text{Bloch Theorem}}$ a family of $(\mathbf{Obs}_\theta)$ with constants C_θ uniformly bounded.

Several view points

quasiperiodic boundary condition determined by $\theta \in [0,1]^d$
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$\Updownarrow$

connection ∇^ρ on a unitary flat bundle $E^\rho \to \mathbb{T}^d$

More general setting

X: smooth connected Riemannian manifold with measure μ_X induced by the Riemannian metric g.
Γ: at most countable discrete group acting on X as a symmetry group, i.e., the Riemannian metric g is Γ-invariant.
The group action $\Gamma \times X \to X$: smooth, free and proper (or equivalently properly discontinuous).
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General scheme abstracted from the Euclidean case

Obs on X from the γ-invariant set $S = \bigcup_{\gamma \in \Gamma} \gamma\omega$ $\xrightarrow{\text{Bloch transform}}$ $\mathbf{Obs}_\rho$ over bundle $E^\rho \to M$ with uniform constants C_θ

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Moduli spaces of irreducible representations

Define

$$\mathscr{M}_\Gamma^n := \mathrm{Hom}_{\mathrm{irr}}(\Gamma, U(n))/U(n).$$

For torus, it only has $n = 1$ "parametrized space" (moduli space) of irreducible representations

$$\mathscr{M}_\Gamma^1 = \mathrm{Hom}_{\mathrm{irr}}(\Gamma, U(1)) = (\chi \mapsto (\chi(\gamma_1), \cdots, \chi(\gamma_d))) \simeq \mathbb{R}^d/\mathbb{Z}^d.$$

In general, we have (e.g., surface Σ_g with genus g)

$$\mathscr{M}_\Gamma := \bigcup_{n=1}^{\infty} \mathrm{Hom}_{\mathrm{irr}}(\Gamma, U(n))/U(n).$$

Abstract Bloch transform

See Kocábová and Šťovíček 2008(or Nagy and Rayan 2024)

- $\mathrm{End}(\mathbb{C}^n)$: space of endomorphisms of $\mathbb{C}^n$.
- $U(\rho, A) := (\mathrm{Ad}(U) \circ \rho, UAU^*)$: free, linear action of $PU(n)$ on $\mathrm{Hom}_{\mathrm{irr}}(\Gamma, U(n)) \times \mathrm{End}(\mathbb{C}^n)$.
- $\mathscr{V}^n := (\mathrm{Hom}_{\mathrm{irr}}(\Gamma, U(n)) \times \mathrm{End}(\mathbb{C}^n))/PU(n) \to \mathscr{M}_\Gamma^n$: Hermitian vector bundle over $\mathscr{M}_\Gamma^n$.
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Abstract Bloch transform

$\forall \psi \in C_{\mathrm{cpt}}(\Gamma) \otimes \mathscr{H}_0, \forall [\rho] \in \mathscr{M}_\Gamma$,

$$\mathscr{B}(\psi)([\rho]) := \left[\rho, \sum_{\gamma \in \Gamma} \psi(\gamma)\rho(\gamma)\right] \in \mathscr{V}_{[\rho]}^{\dim([\rho])} \otimes \mathscr{H}_0.$$

Observability on compact hyperbolic surface

Dyatlov and Jin 2017: $\forall a \in C_0^\infty(T^*M)$ and $a|_{S^*M} \neq 0 \Longrightarrow$
$\exists \mathbf{C}(a) > 0, \mathbf{h}_0(a) > 0$ s.t., $\forall 0 < h < \mathbf{h}_0(a)$ and $\forall u \in H^2(M)$

$$\|u\|_{L^2(M)} \leq \mathbf{C}(a, M) \left(\|\mathrm{Op}_h(a)u\|_{L^2(M)} + \frac{|\log h|}{h} \left\| \left(-h^2\Delta - 1\right) u \right\|_{L^2(M)} \right)$$

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We extend the above semiclassical control to flat bundles:

$$\begin{aligned} \|u\|_{L^2(M,F^\rho)} \leq \mathbf{C}(a, M) \Big(& \|\mathrm{Op}_h(a\mathrm{Id}_{F^\rho})u\|_{L^2(M,F^\rho)} \\ & + |\log h|/h \left\| \left(-h^2\Delta^\rho - \mathrm{Id}\right) u \right\|_{L^2(M,F^\rho)} \Big). \end{aligned}$$

The result

Theorem (Kocábová and Šťovíček 2008)

If Γ *is a countable discrete group of type* I (*i.e., has an abelian normal subgroup of finite index*), *then* $\mathscr{B}$ *is an isometric isomorphism between* $\ell^2(\Gamma) \otimes \mathscr{H}_0$ *and* $L^2(\mathscr{V}^n) \otimes \mathscr{H}_0$.

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The above Bloch Theorem + uniform semiclassical control on flat bundles $\Longrightarrow$ Observability inequality on X:

Theorem (Y. Gong–X. Fu–W.)

Γ *group of type* I, $M = X/\Gamma$ *compact hyperbolic surface,* S γ*-invariant nonempty open set.* $\Longrightarrow$ *observability inequality from* S.

Difficulties to hyperbolic plane

We failed to obtain observability of the Schrödinger equation from a Γ_g-invariant open nonempty set on the whole hyperbolic plane $\mathbb{H}$ (for heat equation on $\mathbb{H}$, it is equivalent to a thickness condition, proven by Rouveyrol 2024).

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Two main reasons:

- We do not have a strong version of hyperbolic Bloch Theorem for $\mathbb{H}$ over Σ_g.
- The moduli space of irreducible unitary representations is noncompact.

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