Geometric conditions for the observability of the electromagnetic Schrödinger equation on $\mathbb{T}^2$

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Part I : Problem & Results

Electromagnetic Schrödinger equation

- The model :

$$\begin{cases} i\partial_t u = H_{\mathbf{A},V} u & \text{in } \mathbb{R}_t \times \mathbb{T}^2, \\ u(0,\cdot) = u_0 & \text{in } \mathbb{T}^2, \end{cases} \tag{1}$$

- $H_{\mathbf{A},V}$: The electromagnetic Schrödinger operator given by

$$H_{\mathbf{A},V}(z) := \left(\frac{1}{i}\nabla - \mathbf{A}(z)\right)^2 + V(z), \qquad z \in \mathbb{T}^2, \tag{2}$$

$$V \in C^\infty(\mathbb{T}^2, \mathbb{R}), \ \mathbf{A} = (A_1, A_2) \in C^\infty(\mathbb{T}^2, \mathbb{R}^2). \tag{3}$$

- Zero-flux magnetic field : $B =: \nabla \wedge \mathbf{A} = \partial_1 A_2 - \partial_2 A_1$, so that $⨍_{\mathbb{T}^2} B = 0$.
- Gauge-invariance : $\mathbf{A} \mapsto \mathbf{A}_\chi := \mathbf{A} + \nabla\chi$ for any $\chi \in C^\infty(\mathbb{T}^2)$,

$$e^{-itH_{\mathbf{A}_\chi,V}} = e^{i\chi} e^{-itH_{\mathbf{A},\mathbf{V}}} e^{-i\chi}.$$

- Main question : Observability for the Schrödinger propagator $e^{-itH_{\mathbf{A},V}}$ on $L^2(\mathbb{T}^2)$.

Brief review of literature for the Schrödinger observability on $\mathbb{T}^d$

$(\mathbf{Obs})_{T,\omega}$: For $T > 0, \omega$ open. $\exists C_{T,\omega,\mathbf{A}} > 0$, s.t. for any $u_0 \in L^2$,

$$\|u_0\|_{L^2}^2 \leq C_{T,\omega,\mathbf{A}} \int_0^T \|e^{-itH_{\mathbf{A},V}} u_0\|_{L^2(\omega)}^2 dt$$

• Lebeau '92 : Geometric control condition (GCC) is sufficient : (GCC) allows to observe $h-$oscillating high-frequency wave packets at the semi-classical time scale $(s = t/h)$ $O(1)$.

• When $\mathbf{A} \equiv 0$, due to the instability of the geodesic flow on $\mathbb{T}^d$, delocalization happens at semi-classical time scale $O(1/h^2)$, resulting $(\mathrm{Obs})_{T,\omega}$ for any $T > 0$ and any non-empty open set $\omega \subset \mathbb{T}^d$:

- Jaffard '90 (Fourier series approach) :$d = 2, V \equiv 0$.
- Burq-Zworski '12, '19, Bourgain-Burq-Zworski '14 (semiclassical analysis+dispersive tools):
  $d = 2, V \in L^2, \omega$ open; and $d = 2, V \equiv 0, \omega$ measurable and $|\omega| > 0$.
- Anatharaman-Macià '14 (2nd semiclassical measures) :
  $d \geq 2, V \in C^0, \omega$ open.
- Burq-Zhu ('25) (dispersive tools) : any d, rough space-time observation region and rough V.

The case $\mathbf{A} \neq 0$? Some notations

The first order perturbation will influence the long-time semiclassical Schrödinger dynamics (Wunsch '12, Rivière-Macià '18). In our context of observability, new geometric conditions appear.

We need some notions :

- For any $f \in L^1(\mathbb{T}^2; \mathbb{R}^m), \vec{e} \in \mathbb{R}^2, |\vec{e}| = 1$,

$$\langle f \rangle_{\vec{e}}(z) := \lim_{T \to \infty} \frac{1}{T} \int_0^T f(z + t\vec{e}) dt.$$

On $\mathbb{T}^2$, we distinguish $\vec{e}$ as

- Periodic : If $\vec{e}$ generates a closed geodesic, i.e. $\vec{e} = \frac{(p,q)}{\sqrt{p^2+q^2}}, gcd(p,q) = 1$, a rational direction
- Ergodic : If not, i.e. $\vec{e}$ is a irrational direction, generating a dense orbit.

In particular, if $\vec{e}$ ergodic,

$$\langle f \rangle_{\vec{e}} = \fint_{\mathbb{T}^2} f,$$

while if $\vec{e}$ periodic,

$$\langle f \rangle_{\vec{e}}(z) = \fint_{\gamma_{\vec{e}}} f,$$

where $\gamma_{\vec{e}}$ is the closed geodesic generated by $\vec{e}$.

Condition (MGCC) and main results

Let ω be an open set of $\mathbb{T}^2$, γ the closed geodesic generated by the periodic $\vec{\gamma}$. Denote $\omega_{\vec{\gamma}^\perp}$ the projection of ω on the direction of $\vec{\gamma}$.

Definition (MGCC)

We say that ω satisfies the magnetic geometric control condition (MGCC), if for any periodic direction $\vec{\gamma}$, $\omega_{\vec{\gamma}^\perp}$ contains all the zeros of $\langle B \rangle_{\vec{\gamma}}$.

- Since $B = \nabla \wedge \mathbf{A}$, the (MGCC) is equivalent to: for any periodic $\vec{\gamma}$, $\omega_{\vec{\gamma}^\perp}$ contains all the critical points of the function $A_\gamma := \langle \mathbf{A} \rangle_\gamma \cdot \vec{\gamma}^\perp$.

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Theorem (Le Balc'h-Niu-S. arXiv Today)

Let ω be an open subset of $\mathbb{T}^2$.

- *If ω satisfies (MGCC), then $(\mathrm{Obs})_{T,\omega}$ holds for any $T > 0$.*
- *If for some periodic $\vec{\gamma}$, $\langle B \rangle_{\vec{\gamma}}$ has a non-degenerate zero outside $\overline{\omega}_{\vec{\gamma}^\perp}$, then $(\mathrm{Obs})_{T,\omega}$ cannot hold for any $T > 0$.*

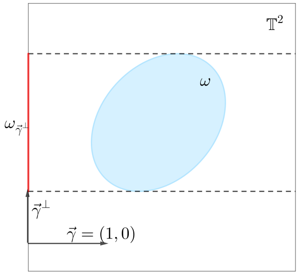
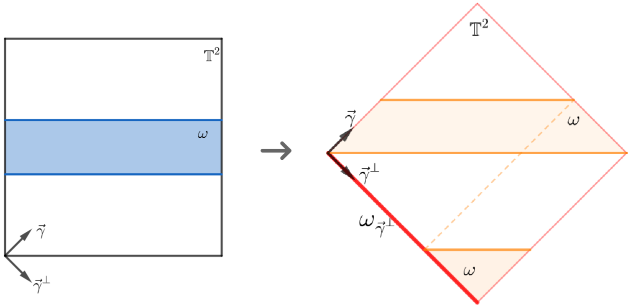


Figure: Control region projection



Main results, sequel

Theorem (Le Balc'h-Niu-S.)

Under (MGCC), we have proved the following resolvent estimate :

$$\|u\|_{L^2(\mathbb{T}^2)} \leq \frac{C}{1+|\lambda|^{1/4}} \|(H_{\mathbf{A},V}+\lambda)u\|_{L^2(\mathbb{T}^2)} + \|u_\lambda\|_{L^2(\omega)}, \forall \lambda \in \mathbb{R}.$$

Corollary

Under (MGCC), internal exact controllability holds.

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Corollary

Under (MGCC), internal exact controllability holds.

- (MGCC) is sufficient, but not necessary with following missing cases :
  - $B \equiv 0$: There exists a gauge χ such that $\mathbf{A}_\chi := \mathbf{A} + \nabla g = const$. Following the work Le Balc'h-Martin '23, the observability of (1) holds for any $T > 0$ and any non-empty open set $\omega \subset \mathbb{T}^2$.
  - $B \neq 0$ and there are finite order of zeros of $\langle B \rangle_\gamma$ on $\partial\omega_{\vec{\gamma}^\perp}$?
  - $\langle B \rangle_\gamma$ has infinite order of zeros ?

A closely related work

• Morin-Rivière'24 prove the Quantum Unique Ergodicity for the magnetic Laplacian on $\mathbb{T}^2$ under the condition $\langle B \rangle_\gamma > 0$ everywhere, for all $\vec{\gamma}$. In their setup, B cannot be derived from a magnetic potential $\mathbf{A}$. $\widehat{B}_0 = \fint_{\mathbb{T}^2} B$ is the total flux satisfying the quantization condition $\widehat{B}_0 \in 2\pi\mathbb{Z}$. Their arugment leads to the same resolvent estimate as ours in the non-zero flux case $\widehat{B}_0 \neq 0$.

• In the non-zero flux case $\widehat{B}_0 \neq 0$, Morin-Rivière used magnetic Weyl-quantization and the second semiclassical measure approach in the spirit of Anatharaman-Macià.

• In the zero flux case $\widehat{B}_0 = 0$, the standard Weyl-quantization is sufficient. Our argument is based on the normal form approach in the spirit of Burq-Zworski.

Part II : Sketch of the Proof

A model example

Consider the model case $\mathbf{A} = (A_1(y), A_2 = 0)$, $V = -|A_1|^2$. Then the magnetic Schrödinger equation writes

$$i\partial_t u + \Delta u - 2iA_1(y)\partial_x u = 0.$$

Taking the Fourier transform in x :

$$i\partial_t u_k + (\partial_y^2 - k^2)u_k + 2A_1(y)ku_k = 0.$$

Around a non-degenerate critical point y_0 of A_1 with $A_1'(y_0) = 0, A_1''(y_0) = -\omega_0^2 < 0$,

$$A_1(y) \approx A_1(y_0) - \omega_0^2 \frac{(y - y_0)^2}{2}.$$

Consider $u_k \mapsto v_k := u_k e^{-it(k^2 - A_1(y_0)k)}$, then

$$i\partial_t v_k + \partial_y^2 v_k - k\omega_0^2 (y - y_0)^2 v_k = 0.$$

A model example, sequel

For $k \gg 1$, take

$$v_k(0) = c_k e^{-\frac{\sqrt{k}\omega_0(y-y_0)^2}{2}}$$

to be the ground state of the harmonic oscillator $-\partial_y^2 + k\omega_0^2(y-y_0)^2$. Then

$$v_k(t,y) = c(k)e^{-\frac{\sqrt{k}\omega_0(y-y_0)^2}{2} - itk}$$

which concentrates around $y = y_0$ for all $t \in \mathbb{R}$. So we cannot have observability if a horizontal observation region ω does not contain the line $y = y_0$.

Proof under (MGCC) I: High-energy observability

By the standard compactness-uniqueness argument of Lebeau and the unique continuation property w.r.t. $H_{\mathbf{A},V}$, it suffices to prove the high-energy observability. More precisely, Denote

$$\Pi_{h,\rho} u := \chi\left(\frac{h^2 H_{\mathbf{A},V} - 1}{\rho}\right) u, \qquad u \in L^2(\mathbb{T}^2).$$

We need to prove for any $T > 0$, and sufficiently small $0 < h, \rho \ll 1$,

$$\|\Pi_{h,\rho} u_0\|_{L^2(\mathbb{T}^2)}^2 \le C \int_0^T \int_\omega \left| e^{-itH_{\mathbf{A},V}} \Pi_{h,\rho} u_0(z) \right|^2 dzdt \qquad \forall u_0 \in L^2(\mathbb{T}^2).$$

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Under (MGCC), we will indeed prove a stronger high-energy observability result in much shorter time, equivalent to our resolvent estimate :

Proposition

There exists a numerical constant $T_0 > 0$ such that for any $T \geq T_0$, there exist constants $\rho_0 > 0$, $h_0 > 0$ and $C > 0$ such that for any $\rho \in (0, \rho_0)$, $h \in (0, h_0)$, we have $(\mathrm{Obs})_{h,T,\omega}$:

$$\|\Pi_{h,\rho} u_0\|^2_{L^2(\mathbb{T}^2)} \leq C \int_0^T \int_\omega \left|e^{-ith^{\frac{1}{2}} H_{\mathbf{A},V}} \Pi_{h,\rho} u_0(z)\right|^2 dzdt \qquad \forall u_0 \in L^2(\mathbb{T}^2). \tag{4}$$

Proof under (MGCC) II: Semiclassical measures

Though a quantitative argument is possible, we argue by contradiction for clarity. If $(\mathrm{Obs})_{h,T,\omega}$ is untrue, there is a sequence $(u_h)_{0<h\ll 1}$ such that

$$\|u_h\|_{L^2} = 1, \quad \int_0^T \|e^{-ith^{\frac{1}{2}} H_{\mathbf{A},V}} u_h\|^2_{L^2(\omega)} dt = o(1), \; h \to 0+ .$$

Up to extracting a subsequence, there exists a semiclassical defect measure μ on $\mathbb{R}_t \times T^*\mathbb{T}^2_z$ such that for any function $\psi \in C^0_0(\mathbb{R}_t)$ and any $a \in C^\infty_c(T^*\mathbb{T}^2_z)$, we have

$$\langle \mu, \psi(t) a(z,\zeta) \rangle = \lim_{n \to +\infty} \int_{\mathbb{R}_t \times \mathbb{T}^2_z} \psi(t) (\mathrm{Op}^w_h(a) u_h)(t,z) \overline{u_h(t,z)} dz dt.$$

- The measure μ is supported in $S^*\mathbb{T}^2$, i.e.

$$\mathrm{supp}(\mu) \subset \{(t,z,\zeta) \in \mathbb{R}_t \times T^*\mathbb{T}^2 : |\zeta| = 1\},$$

- For any $t_0 < t_1$, we have

$$\mu((t_0,t_1) \times T^*\mathbb{T}^2) = t_1 - t_0, \quad \mu|_{(0,T_0) \times \omega \times \mathbb{S}^1} = 0.$$

- For a.e. $t \in \mathbb{R}$,

$$\zeta \cdot \nabla_z \mu(t,\cdot) = 0.$$

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- Thanks to the invariant property and the fact that ω is open, we have

  $$\mu = \sum_{\zeta_0 \text{periodic}} \mu|_{\mathbb{R}\times\mathbb{T}^2\times\{\zeta_0\}},$$

  with only finitely-many periodic directions ζ_0. We only need to show that $\mu|_{\mathbb{R}\times\mathbb{T}^2\times\{\zeta_0\}} = 0$ for any periodic ζ_0.
- Up to changing coordinate, we may assume for $\zeta_0 = (1,0)$

2nd semiclassical scale and 1d reduction

- Gage : $\mathbf{A} = (A_1(y), A_2(x,y))$. Our equation becomes

$$ih^{3/2}\partial_t u_h - P_h u_h = 0,$$

where

$$P_h = p_0^w(hD) + hp_1^w(z, hD) + h^2 p_2^w(z, hD)$$

with symbols

$$\begin{aligned} p_0 &= |\zeta|^2 = \xi^2 + \eta^2, \\ p_1 &= 2A_1(y)\xi + 2A_2(x,y)\eta, \\ p_2 &= V + A_1^2 + A_2^2 \end{aligned}$$

- Need to perform a second microlocalization near the coisotropic subspace $\{\eta = 0\}$. This could be realized simply by normal-form reduction + positive commutator method. It turns out that the second-semiclassical scale can be chosen as $|\eta| \sim h^{\frac{1}{4}+}$

Normal form reduction

We search for $Q_h = \mathrm{Op}_h^w(q(x,y,\xi)\eta)$ to average the potential $A_2(x,y)$ through conjugation :

$$\begin{aligned} e^{Q_h} P_h e^{-Q_h} =& P_h + [Q_h, P_h] + \mathcal{O}(h^2) \\ =& \mathrm{Op}_h^w\left(p_0 + 2hA_1(y)\xi\right) + \mathrm{Op}_h^w\left(2hA_2(x,y)\eta + \frac{h}{i}\{q\eta, \xi^2 + \eta^2\}\right) + \mathcal{O}(h^2) \\ =& \underbrace{\mathrm{Op}_h^w\left(p_0 + 2hA_1(y)\xi\right)}_{\text{principal}} + \mathrm{Op}_h^w\left(2h(A_2(x,y) + i\xi\partial_x q)\eta\right) \\ &+ \underbrace{2ih\mathrm{Op}_h^w((\partial_y q)\eta^2) + \mathcal{O}(h^2)}_{\text{remainder}=o(h^{3/2})}. \end{aligned}$$

- To average A_2, we choose q by solving

$$\partial_x q(x,y,\xi) = -\frac{1}{i\xi}(A_2 - \langle A_2\rangle_{(1,0)}(y)).$$

• The operator $2ih\mathrm{Op}_h^w(\partial_y q\eta^2)$ can be viewed as remainder only if $\eta = o(h^{1/4})$, this explains the choice of the second semiclassical scale. For wave packets oscillating at scale $\eta \gtrsim h^{1/4}$, we detect it transversal propagation via the multiplier $\varphi(y)y\partial_y$.

Key 1d analysis

We now prove the "1d" observability of the equation

$$ih^{3/2}\partial_t u_h + h^2\Delta_{x,y}u_h + 2ihA_1(y)h\partial_x u_h + 2ihA_2(y)h\partial_y u_h = o_{L^2_{t,x,y}}(h^{\frac{3}{2}})$$

on finite union of blue horizonal strips containing all critical points of A_1 in the interior :

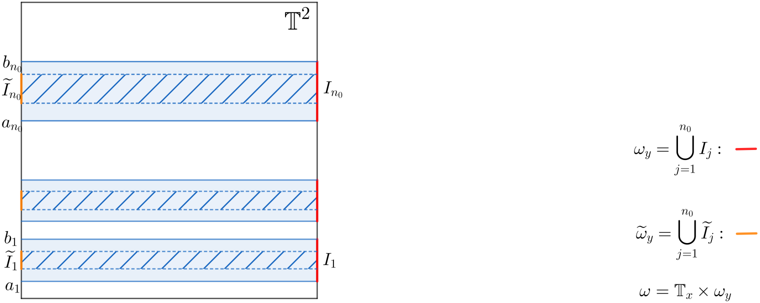

Figure: Multiple strips

Key 1d analysis

• On a gap (b_j, a_{j+1}) of blue strips, $A_1'(y) \geq c_0 > 0$ (or uniformly negative). We use the localized multiplier $\theta(\frac{t}{T})\chi(y)(y - b_j + \epsilon_0)\partial_y$. Thinking $h\partial_x = 1$, then the positive commutator comes from

$$-[ih^{\frac{3}{2}}\partial_t + h^2\partial_y^2 + 2hA_1(y), \chi(y)(y - b_j + \epsilon_0)\partial_y]$$
$$= \underbrace{-2\chi(y)h^2\partial_y^2 + 2h\chi(y)A_1'(y)(y - a + \epsilon_0)}_{\text{positive operators}} + \underbrace{l.o.t.}_{\text{higher power in } h \,+\, \text{terms with } \partial\chi} .$$

• The positive commutator will essentially control

$$\|h\partial_y u_h\|_{L^2_{x,y}}^2 + \underbrace{\|h^{1/2}u_h\|_{L^2_{x,y}}^2}_{\text{principal thanks to (MGCC)}}$$

• On the other hand, the commutator involving $[ih^{3/2}\partial_t, \theta(t/T)\cdots h^{-1}h\partial_y]$ will finally contribute a main term in the remainder

$$\frac{O(h^{1/2})}{T}\|u_h\|_{L^2}\|h\partial_y u_h\|_{L^2},$$

hence we need $T \geq T_0 \gg 1$ (but independent of h).

About the optimality

Assume for some periodic $\vec{\gamma}$, $\langle B \rangle_\gamma$ has a zero outside $\overline{\omega}_{\vec{\gamma}^\perp}$.
To disprove $(\mathrm{Obs})_{T,\omega}$:

- By changing coordinate, translation and gauge transform, we may assume that $\overline{\omega}_{\vec{\gamma}^\perp}$ are horizontal strips and $\mathbf{A} = (A_1(y), A_2(x,y))$ such that a critical point $y_0 = 0$ of $A_1(y)$ is outside $\overline{\omega}_{\vec{\gamma}^\perp}$ and $A_1''(0) \neq 0$.
- Well-prepared modes : Preparing the highly-concentrated sequences as in the model example (there $A_2 = \langle A_2 \rangle_{(1,0)}(y)$).
- Since the normal form transform is invertible, we do the inverse normal form transform (de-average A_2) as in the previous proof to transfer the well-prepared modes in the model example to get the desired modes.
- Additional point : To disprove the observability $(\mathrm{Obs})_{T,\omega}$, only $o_{L^2}(h^2)$ terms can be viewed as remainders (comparing to $o_{L^2}(h^{3/2})$). We need to do one step further normal form to average symbols that are $O_{L^2}(h^2)$ in a priori.

Perspectives

- Our result could be generalized to the case with non-zero flux $\widehat{B}_0 \neq 0$ on $\mathbb{T}^2$, under (MGCC).
- In terms observability, $H_{\mathbf{A},V}$ is clearly not a perturbation of $-\Delta$, comparing to $-\Delta + V$. It is challenging to study the question of rough potential or rough control, as in the context of $-\Delta + V$ (talk of Nicolas). It is possible to relax the regularity of the electronic potential V.
- The case $d \geq 3$? Description of the semiclassical measures for the magnetic Schrödinger equations ? Delocalization ? Concentration ?

Thank you !