Domain invariance for nonlinear diffusion models

Piermarco Cannarsa

University of Rome Tor Vergata

2023-2026: Research secondment at the Interdisciplinary Center "B. Segre" Accademia Nazionale dei Lincei

Control of PDEs and related topics
Institut de Mathématiques de Toulouse
June 30 – July 4, 2025

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Outline

1. A bilinear control problem

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2. Domain invariance for evolution equations

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Bilinear control of the heat flow

$\Omega \subset \mathbb{R}^d$ bounded with smooth boundary

Problem

For any $u_0 \in H^1_0(\Omega) \setminus \{0\}$ find $f : [0, \infty) \to \mathbb{R}$ such that the solution u^f of

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + f(t)u(t,x) & \mathbb{R}_+ \times \Omega \\ u = 0 & \mathbb{R}_+ \times \partial\Omega \\ u(0,x) = u_0(x) & x \in \Omega \end{cases} \qquad (LH)$$

satisfies

$$\int_\Omega |u^f(t,x)|^2 dx = \int_\Omega |u_0(x)|^2 dx \qquad \forall t \geqslant 0$$

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Solution

L. Caffarelli and F. Lin, *Nonlocal heat flows preserving the L^2 energy*, DCDS 2009

If f satisfies

$$\int_\Omega |u^f(t,x)|^2 dx = \int_\Omega |u_0(x)|^2 dx \qquad \forall t \geqslant 0$$

then

$$0 = \int_\Omega u^f(t,x)\,\frac{\partial u^f}{\partial t}(t,x)dx = \int_\Omega u^f(t,x)\,\Delta u^f(t,x)dx + f(t)\int_\Omega |u_0(x)|^2 dx$$

So

$$f(t) = \frac{1}{\int_\Omega |u_0(x)|^2 dx}\int_\Omega |\nabla u^f(t,x)|^2 dx \qquad \forall t \geqslant 0$$

Abstract formulation

The problem reduces to the invariance of the set

$$K_{u_0} := \left\{ u \in H_0^1(\Omega) \ : \ \int_\Omega |u(x)|^2 dx = \int_\Omega |u_0(x)|^2 dx \right\}$$

under the nonlocal heat flow

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + \frac{\int_\Omega |\nabla u^f(t,x)|^2 dx}{\int_\Omega |u_0(x)|^2 dx}\, u(t,x) & \mathbb{R}_+ \times \Omega \\ u = 0 & \mathbb{R}_+ \times \partial\Omega \end{cases}$$

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under the nonlocal heat flow

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + \frac{\int_\Omega |\nabla u^f(t,x)|^2 dx}{\int_\Omega |u_0(x)|^2 dx}\, u(t,x) & \mathbb{R}_+ \times \Omega \\ u = 0 & \mathbb{R}_+ \times \partial\Omega \end{cases}$$

The above equation can be recast in abstract form in $H = L^2(\Omega)$ as

$$u'(t) = Au(t) + B\big(u(t)\big) \qquad (t \geqslant 0)$$

for suitable operators

- $A =$ Dirichlet Laplacian
- $B(u) = \frac{\int_\Omega |\nabla u(x)|^2 dx}{\int_\Omega |u_0(x)|^2 dx}\, u$

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Semilinear evolution equations

Consider the Cauchy problem

$$\begin{cases} u'(t) = Au(t) + B(u(t)) & t > 0 \\ u(0) = u_0 \in H \end{cases} \tag{P_{u_0}}$$

where

(H1) $H, \langle \cdot, \cdot \rangle_H, \| \cdot \|_H$ Hilbert space

(H2) $A : D(A) \subset H \to H$ infinitesimal generator of a strongly continuous semigroup of contractions on H, denoted by e^{tA}

(H3) $V, \langle \cdot, \cdot \rangle_V, \| \cdot \|_V$ Hilbert space such that $D(A) \subset V \subset H$ and

$$e^{tA} V \subset V \quad \text{with} \quad \|e^{tA} u\|_V \leqslant \|u\|_V$$

(H4) $B : V \to H$ locally Lipschitz continuous

We denote by $u(t, u_0)$ the maximal solution of (P_{u_0}) defined for $t \in [0, \tau_{u_0})$

Invariant sets

A set $K \subset V$ is invariant under

$$u'(t) = Au(t) + B(u(t)) \qquad (t \geqslant 0)$$

if

$$u_0 \in K \quad \Longrightarrow \quad u(t, u_0) \in K \qquad \forall t \in [0, \tau_{u_0})$$

M. Nagumo.
Über die Lage der Integralkurven gewöhnlicher Differentialgleichungen.
Proc. Phys.-Math. Soc. Japan, III. Ser., 24:551–559, 1942.

was the first to give necessary and sufficient conditions for invariance (for ODEs)

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Comm. Pure Appl. Math., 23:261–263, 1970.

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On invariant sets and on a theorem of Ważewski.
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Viability theorems for a class of differential-operator inclusions.
J. Differ. Equations, 79(2):232–257, 1989.

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über die positive Invarianz einer abgeschlossenen Teilmenge eines Banachschen Raumes bezüglich der Differentialgleichung $u' = f(t, u)$.
J. Reine Angew. Math., 285:59–65, 1976.

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In finite dimension, necessary and sufficient condition can be found in

O. Cârjă, M. Necula, and I. I. Vrabie.
Viability, invariance and applications.
Amsterdam: Elsevier, 2007.

We shall present the approach developed in

[1] P. Cannarsa, G. Da Prato, and H. Frankowska.
Invariance for quasi-dissipative systems in Banach spaces.
J. Math. Anal. Appl., 457(2):1173–1187, 2018.

[2] Piermarco Cannarsa, Giuseppe Da Prato, and Hélène Frankowska.
Domain invariance for local solutions of semilinear evolution equations in Hilbert spaces.
J. Lond. Math. Soc., II. Ser., 102(1):287–318, 2020.

and later extended to Banach spaces in

[3] Aleksander Ćwiszewski, Grzegorz Gabor, and Wojciech Kryszewski.
Invariance and strict invariance for nonlinear evolution problems with applications.
Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods, 218:32, 2022.
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Proximally smooth sets

Let $K \subset H$ be a closed set. Define $K_\delta = \left\{ u \in H \setminus K \ : \ d_K(u) < \delta \right\}$

Definition

K is proximally smooth if $\exists \delta > 0$ such that every $u \in K_\delta$ has a unique projection onto K

We denote by $\Pi_K(u)$ such a projection

Necessary and sufficient conditions for invariance

Theorem

Let the closed set $K \subset H$ be proximally smooth and suppose that

$$\exists \delta > 0 \quad \textit{such that} \quad \Pi_K\big(D(A) \cap K_\delta\big) \subset D(A)$$

Then $K \cap V$ is invariant under $u' = Au + B(u)$ if and only if

$$\langle p, Au + B(u)\rangle_H \leqslant 0 \qquad \forall u \in \partial K \cap D(A)\,,\ \forall p \in N_K^P(u) \cap D(A)$$

Here $N_K^P(u)$ stands for the proximal normal cone to K at u, given by

$$N_K^P(u) = \left\{ p \in H \;\middle|\; \exists \rho > 0 \ :\ \langle p, v - u\rangle_H \leqslant \frac{1}{2\rho}\,\|v - u\|_H^2 \right\}$$

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Application to the heat equation with nonlocal term

Let $\|u_0\|_H = 1$ and consider the Cauchy problem

$$\begin{cases} u'(t) = Au(t) + \|u(t)\|_V^2 u(t) & t > 0 \\ u(0) = u_0 \in H \end{cases} \qquad (\star)$$

where A is the Dirichlet Laplacian in $H = L^2(\Omega)$. Define

$$K = \{u \in H \ : \ \|u\|_H = 1\}$$

Then

- K is (proximally) smooth
- $\Pi_K\big(D(A) \setminus \{0\}\big) \subset D(A)$ because $\Pi_K(u) = u/\|u\|_H$
- for all $u \in \partial K (= K)$ we have that $N_K^P(u) = \mathbb{R}u$

Since

$$\langle u\,,\, Au + \|u\|_V^2 u \rangle_H = \langle u\,,\, Au \rangle_H + \|u\|_V^2 = 0 \qquad \forall u \in D(A) \cap K$$

we conclude that $K \cap V$ is invariant under $(\star)$

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Invariance as a way to obtain global existence

Let $u_0 \in V$ be sucht that $\|u_0\|_H = 1$. By the above invariance result we that that the maximal solution of the Cauchy problem

$$\begin{cases} u'(t) = Au(t) + \|u(t)\|_V^2 u(t) & t > 0 \\ u(0) = u_0 \in H \end{cases} \qquad (\star)$$

satisfies

$$\|u(t)\|_H = 1 \qquad \forall t \in [0, \tau_{u_0}) \qquad (\star\star)$$

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Invariance as a way to obtain global existence

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satisfies

$$\|u(t)\|_H = 1 \qquad \forall t \in [0, \tau_{u_0}) \tag{$\star\star$}$$

Notice that $(\star\star)$ is not enough to conclude that $\tau_{u_0} = \infty$. However, together with $(\star)$ it yields

$$\|u'(t)\|_H^2 + \frac{1}{2}\frac{d}{dt}\|u(t)\|_V^2 = \frac{1}{2}\|u(t)\|_V^2 \frac{d}{dt}\|u(t)\|_H^2 = 0 \qquad \forall t \in [0, \tau_{u_0})$$

Therefore $\|u(t)\|_V \leqslant \|u_0\|_V$ and $\tau_{u_0} = \infty$

Energy preserving nonlinear heat flows

We now study the nonlinear heat equation perturbed by a nonlocal term in a bounded domain $\Omega \subset \mathbb{R}^d$ with smooth boundary

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + g|u(t,x)|^{2\sigma} u(t,x) + \mu[u(t,\cdot)]u(t,x) & \mathbb{R}_+ \times \Omega \\ u = 0 & \mathbb{R}_+ \times \partial\Omega \\ u(0,x) = u_0(x) & x \in \Omega \end{cases}$$

where $g \in \mathbb{R}, \sigma > 0$ and

$$\mu[u] = \frac{\|u\|_V^2 - g\|u\|_{L^{2\sigma+2}(\Omega)}^{2\sigma+2}}{\|u\|_H^2} \quad (1)$$

(1) $u \in H_0^1(\Omega) \Rightarrow u \in L^{\frac{2d}{(d-2)^+}}(\Omega) \Rightarrow u \in L^{2\sigma+2}(\Omega)\ \forall \sigma < \frac{2}{(d-2)^+}$

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where $g \in \mathbb{R}, \sigma > 0$ and

$$\mu[u] = \frac{\|u\|_V^2 - g\|u\|_{L^{2\sigma+2}(\Omega)}^{2\sigma+2}}{\|u\|_H^2} \text{ (1)}$$

Formally, μ preserves the energy:

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_H^2 = -\|u(t)\|_V^2 + g\|u(t)\|_{L^{2\sigma+2}(\Omega)}^{2\sigma+2} + \mu[u(t)]\|u(t)\|_H^2 = 0$$

(1) $u \in H_0^1(\Omega) \Rightarrow u \in L^{\frac{2d}{(d-2)^+}}(\Omega) \Rightarrow u \in L^{2\sigma+2}(\Omega)\ \forall \sigma < \frac{2}{(d-2)^+}$

Energy

Consider the nonlinear heat equation perturbed by a nonlocal term

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + g|u|^{2\sigma} u + \mu[u]u & \text{in } \mathbb{R}_+ \times \Omega \\ u(t,\cdot)_{|\partial\Omega} = 0 & u(0) = u_0 \in H_0^1(\Omega) \end{cases} \tag{NLH}$$

Define the energy of a solution as

$$E[u(t)] = \frac{1}{2}\|u(t)\|_V^2 - \frac{g}{2\sigma+2}\|u(t)\|_{L^{2\sigma+2}(\Omega)}^{2\sigma+2} \begin{cases} \geqslant 0 & g \leqslant 0 \\ \text{indefinite} & g > 0 \end{cases}$$

Energy

Consider the nonlinear heat equation perturbed by a nonlocal term

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Then

$$E[u(t)] - E[u_0] = -\int_0^t ds \int_\Omega \left|\frac{\partial u}{\partial t}(s,x)\right|^2 dx \leqslant 0$$

Motivations

1. (NLH) is L^2 gradient flow constrained on a manifold

Thierry Aubin.
Some nonlinear problems in Riemannian geometry.
Springer Monogr. Math. Berlin: Springer, 1998.

Michael Struwe.
Variational methods. Applications to nonlinear partial differential equations and Hamiltonian systems., volume 34 of *Ergeb. Math. Grenzgeb., 3. Folge.*
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Weizhu Bao and Qiang Du.
Computing the ground state solution of Bose-Einstein condensates by a normalized gradient flow.
SIAM J. Sci. Comput., 25(5):1674–1697, 2004.

Qiang Du.
Numerical computation of quantized vortices in the Bose-Einstein condensate.
In *Recent progress in computational and applied PDEs. Conference proceedings for the international conference held in Zhangjiajie, China, July 1–7, 2001.*, pages 157–169. New York, NY: Kluwer Academic / Plenum Publishers, 2002.

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References/3

The literature on the nonlinear heat equation perturbed by a nonlocal term

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + g|u|^{2\sigma} u + \mu[u]u & \text{in } \mathbb{R}_+ \times \Omega \\ u(t,\cdot)_{|\partial\Omega} = 0 & u(0) = u_0 \in H_0^1(\Omega) \end{cases} \tag{NLH}$$

includes

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includes

- Caffarelli and Lin, *DCDS* (2009)
  - studied $g = 0$ (linear case)
  - proved global existence, convergence as $t \to \infty$, identified limit

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- Ma and Cheng, *J. Evol. Equ.* (2009)
  - studied the case of $g < 0$ (positive definite energy)
  - proved global existence and weak convergence on $t_n \to \infty$

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- Caffarelli and Lin, *DCDS* (2009)
  - studied $g = 0$ (linear case)
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- Ma and Cheng, *J. Evol. Equ.* (2009)
  - studied the case of $g < 0$ (positive definite energy)
  - proved global existence and weak convergence on $t_n \to \infty$
- Antonelli – C – Shakarov, *Calc. Var. Partial Differ. Equ.* (2024)
  - studied the case of $g \in \mathbb{R}$ for both Ω bounded and $\Omega = \mathbb{R}^d$

Stochastic wave equation: Cerrai and Xie, *Electron. J. Prabab.* (2025)

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The general case $g \in \mathbb{R}$

Antonelli – C – Shakarov, *Calc. Var. Partial Differ. Equ.* (2024)

Outline

- local well-posedness
- global solution
- strong convergence on a time sequence $t_n \to \infty$
- identification of the limit as $t \to \infty$ for $u_0 \geqslant 0$

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Local well-posedness 1

Antonelli – C – Shakarov, *Calc. Var. Partial Differ. Equ.* (2024)

Theorem

Let

$$g > 0 \quad \text{and} \quad 0 < \sigma < \frac{2}{(d-2)^+}$$

Then for any $u_0 \in H_0^1(\Omega)$ *there exists a unique mild solution*[a]

$$u \in \mathcal{C}\big([0, \tau_{u_0}); H_0^1(\Omega)\big)$$

of

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + g|u|^{2\sigma} u + \mu[u]u & \text{in } \mathbb{R}_+ \times \Omega \\ u(t,\cdot)_{|\partial\Omega} = 0 & u(0) = u_0 \end{cases} \qquad (NLH)$$

Moreover, either $\tau_{u_0} = \infty$ *or* $\tau_{u_0} < \infty$ *and* $\lim_{t \to \tau_{u_0}} \|u(t)\|_V = \infty$

[a] $u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}\Big(g|u(s)|^{2\sigma}u(s) + \mu[u(s)]u(s)\Big)ds$

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Local well-posedness 2

Difficulties

Local well-posedness 2

Difficulties

- The energy functional

$$E[u(t)] = \frac{1}{2}\|u(t)\|_V^2 - \frac{g}{2\sigma+2}\|u(t)\|_{L^{2\sigma+2}(\Omega)}^{2\sigma+2}$$

being indefinite, well-posedness cannot follow from a priori bounds

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  being indefinite, well-posedness cannot follow from a priori bounds

- The contraction mapping theorem seems hard to apply because:
  - the nonlocal term

$$\mu[u] = \frac{\|u\|_V^2 - g\|u\|_{L^{2\sigma+2}(\Omega)}^{2\sigma+2}}{\|u\|_H^2}$$

    suggests to look for a fixed point in the space $\mathcal{C}\big([0,T];V\big)$, but
  - the power-like nonlinearity $G(u) = |u|^{2\sigma}u$ fails to be locally Lipschitz in V for $0 < \sigma < 1/2$

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  - the power-like nonlinearity $G(u) = |u|^{2\sigma}u$ fails to be locally Lipschitz in V for $0 < \sigma < 1/2$

- Uniqueness has to be derived by an ad hoc method

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Local well-posedness 3

Strategies

Local well-posedness 3

Strategies

- We prove the local well-posedness of

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + g|u|^{2\sigma} u + \lambda(t) u & \text{in } \mathbb{R}_+ \times \Omega \\ u(t,\cdot)_{|\partial\Omega} = 0 & u(0) = u_0 \in V \end{cases}$$

with $\lambda \in L^\infty(\mathbb{R})$

Local well-posedness 3

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with $\lambda \in L^\infty(\mathbb{R})$

- We employ the Schauder fixed point theorem to construct a solution of

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + g|u|^{2\sigma} u + \mu_0[u] u & \text{in } \mathbb{R}_+ \times \Omega \\ u(t,\cdot)_{|\partial\Omega} = 0 & u(0) = u_0 \in D(A) \end{cases} \qquad (NLH_0)$$

where

$$\mu_0[u] = \frac{\|u\|_V^2 - g\|u\|_{L^{2\sigma+2}(\Omega)}^{2\sigma+2}}{\|u_0\|_H^2}$$

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Local well-posedness 3

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- We use a density argument to show the existence of solutions for $u_0 \in V$

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Local well-posedness 3

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  where

$$\mu_0[u] = \frac{\|u\|_V^2 - g\|u\|_{L^{2\sigma+2}(\Omega)}^{2\sigma+2}}{\|u_0\|_H^2}$$

- We use a density argument to show the existence of solutions for $u_0 \in V$
- We prove the equivalence of (NLH_0) and

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + g|u|^{2\sigma} u + \mu[u] u & \text{in } \mathbb{R}_+ \times \Omega \\ u(t,\cdot)_{|\partial\Omega} = 0 & u(0) = u_0 \end{cases}$$

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Global solution 1

Antonelli – C – Shakarov, *Calc. Var. Partial Differ. Equ.* (2024)

Theorem

Let

$$\begin{cases} g \leqslant 0 \\ 0 < \sigma < \frac{2}{(d-2)^+} \end{cases} \quad \text{or} \quad \begin{cases} g > 0 \\ \sigma < \frac{2}{d} \end{cases}$$

Then for any $u_0 \in H_0^1(\Omega)$ *the solution* $u \in \mathcal{C}\big([0,\tau_{u_0}); H_0^1(\Omega)\big)$ *of*

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + g|u|^{2\sigma}u + \mu[u]u & \text{in } \mathbb{R}_+ \times \Omega \\ u(t,\cdot)_{|\partial\Omega} = 0 & u(0) = u_0 \end{cases} \tag{NLH}$$

is global, that is, $\tau_{u_0} = \infty$

Global solution 2

Strategies

Global solution 2

Strategies

- If $\begin{cases} g \leqslant 0 \\ 0 < \sigma < \frac{2}{(d-2)^+} \end{cases}$ then $\|u(t)\|_V^2 \leqslant 2E[u(t)] \leqslant 2E[u_0]$

Global solution 2

Strategies

- If $\begin{cases} g \leqslant 0 \\ 0 < \sigma < \frac{2}{(d-2)^+} \end{cases}$ then $\|u(t)\|_V^2 \leqslant 2E[u(t)] \leqslant 2E[u_0]$
- If $\begin{cases} g > 0 \\ 0 < \sigma < \frac{2}{d} \end{cases}$

  then we use the Gagliardo-Nirenberg inequality

$$\|u(t)\|_V^2 = 2E[u(t)] + \frac{g}{\sigma+1} \|u(t)\|_{L^{2\sigma+2}(\Omega)}^{2\sigma+2} \lesssim E[u_0] + \|u(t)\|_V^{d\sigma}$$

  Since $d\sigma < 2$ we conclude

$$\|u(t)\|_V \lesssim \|u_0\|_V$$

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Global solution 3 ($\star$)

Potential well

We now want to address the case of $\begin{cases} g > 0 \\ 0 < \sigma < \frac{2}{(d-2)^+} \end{cases}$

Assuming $g = 1$ we have that

$$E[u(t)] = \frac{1}{2}\left(\|u(t)\|_V^2 - \frac{1}{\sigma+1}\,\|u(t)\|_{L^{2\sigma+2}(\Omega)}^{2\sigma+2}\right)$$

Define $I[u] = \|u\|_V^2 - \|u\|_{L^{2\sigma+2}(\Omega)}^{2\sigma+2}$ and $p = \inf\left\{E[u]\ :\ u \in V \setminus \{0\}\,,\ I[u] = 0\right\}$

Potential Well [ref: Payne-Sattinger (1975), Quittner-Souplet (2019)]

$$\mathcal{W} = \left\{u \in V\ :\ E[u] < p\,,\ I[u] > 0\right\} \cup \{0\}$$

Notice that $\|f\|_V < \sqrt{2p} \ \Rightarrow\ f \in \mathcal{W}$

Theorem

- *$\mathcal{W}$ is invariant under* $\frac{\partial u}{\partial t} = \Delta u + g|u|^{2\sigma}u + \mu[u]u$
- *If $u_0 \in \mathcal{W}$ then* $\tau_{u_0} = \infty$

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Long time behaviour of solutions

Antonelli – C – Shakarov, *Calc. Var. Partial Differ. Equ.* (2024)

Let

$$\begin{cases} g \leqslant 0 \\ 0 < \sigma < \frac{2}{(d-2)^+} \end{cases} \qquad \text{or} \qquad \begin{cases} g > 0 \\ \sigma < \frac{2}{d} \end{cases}$$

and consider the solution $u \in \mathcal{C}\big([0,\infty); V\big)$ of

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + g|u|^{2\sigma} u + \mu[u]u & \text{in } \mathbb{R}_+ \times \Omega \\ u(t,\cdot)_{|\partial\Omega} = 0 & u(0) = u_0 \end{cases} \qquad (NLH)$$

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Theorem

$\exists\, t_n \uparrow \infty$ and $u_\infty \in V$ such that

$$u(t_n) \xrightarrow{V} u_\infty \quad \text{and} \quad \mu[u(t_n)] \to \mu[u_\infty] \quad \text{as} \quad n \to \infty$$

Moreover

$$\Delta u_\infty + g|u_\infty|^{2\sigma} u_\infty + \mu[u_\infty]u_\infty = 0 \quad \text{and} \quad \|u_\infty\|_H = \|u_0\|_H$$

A result from elliptic theory

Let $\Omega = B_R = \{x \in \mathbb{R}^d \ : \ |x| < R\}$ and let

$$\begin{cases} g \leqslant 0 \\ 0 < \sigma < \frac{2}{(d-2)^+} \end{cases} \qquad \text{or} \qquad \begin{cases} g > 0 \\ \sigma < \frac{2}{d} \end{cases}$$

Ground state

For any $u_0 \in V \setminus \{0\}$ there exists a unique positive (in B_R) minimizer $\underline{u} \in V$ of the constrained energy

$$\underline{u} \to \min \left\{ E[u] \ : \ u \in V \, , \ \|u\|_H = \|u_0\|_H \right\} \tag{$\star$}$$

Moreover, the ground state $\underline{u}$ is radially symmetric and satisfies

$$\Delta \underline{u} + g|\underline{u}|^{2\sigma}\underline{u} + \mu[\underline{u}]\underline{u} = 0 \quad \text{and} \quad \|\underline{u}\|_H = \|u_0\|_H \tag{$\star\star$}$$

Problem ($\star$) and equation ($\star\star$) have been extensively studied, see, e.g., Gidas-Ni-Nirenberg (1979), Stuart (1982), P-L Lions (1984), McLeod-Serrin (1987), Kwong (1989)

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Identification of the limit as $t \to \infty$

Antonelli – C – Shakarov, *Calc. Var. Partial Differ. Equ.* (2024)

Let $\Omega = B_R$ and let $\begin{cases} g \leqslant 0 \\ 0 < \sigma < \frac{2}{(d-2)^+} \end{cases}$ or $\begin{cases} g > 0 \\ \sigma < \frac{2}{d} \end{cases}$

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Theorem

For any $u_0 \in V \setminus \{0\}$, with $u_0 \geqslant 0$, the solution $u \in \mathcal{C}\big([0,\infty); V\big)$ of

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + g|u|^{2\sigma} u + \mu[u]u & \text{in } \mathbb{R}_+ \times \Omega \\ u(t,\cdot)_{|\partial\Omega} = 0 & u(0) = u_0 \end{cases} \qquad (NLH)$$

converges to the ground state $\underline{u}$ in V as $t \to \infty$

Identification of the limit as $t \to \infty$

Steps of the proof

Identification of the limit as $t \to \infty$

Steps of the proof

1. Let $t_n \uparrow \infty$ and $u_\infty \in V$ be such that $u(t_n) \stackrel{V}{\to} u_\infty$, with

$$\Delta u_\infty + g|u_\infty|^{2\sigma} u_\infty + \mu[u_\infty] u_\infty = 0 \quad \& \quad \|u_\infty\|_H = \|u_0\|_H$$

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$$\Delta u_\infty + g|u_\infty|^{2\sigma} u_\infty + \mu[u_\infty]u_\infty = 0 \quad \& \quad \|u_\infty\|_H = \|u_0\|_H$$
2. By the Maximum Principle
$$u(t_n) \geqslant 0 \quad \Longrightarrow \quad u_\infty \geqslant 0 \quad \& \quad u_\infty \not\equiv 0$$
3. Since $\Delta u + g|u|^{2\sigma} u + \mu[u]u = 0$ has a unique positive solution, $u_\infty = \underline{u}$

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4. Since $t \mapsto E[u(t)]$ is continuous and decreasing,
$$\lim_{t\to\infty} E[u(t)] = \lim_{n\to\infty} E[u(t_n)] = E[u_\infty] = E[\underline{u}]$$

   Then, one shows that $u(t) \stackrel{V}{\to} \underline{u}$ as $t \to \infty$ by contradiction

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Remark

- *The proof applies to any bounded* Ω *such that* $\Delta u + g|u|^{2\sigma} u + \mu[u]u = 0$ *has a unique positive solution (which minimizes the constrained energy)*

Identification of the limit as $t \to \infty$

Steps of the proof

1. Let $t_n \uparrow \infty$ and $u_\infty \in V$ be such that $u(t_n) \xrightarrow{V} u_\infty$, with
$$\Delta u_\infty + g|u_\infty|^{2\sigma} u_\infty + \mu[u_\infty] u_\infty = 0 \quad \& \quad \|u_\infty\|_H = \|u_0\|_H$$
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Remark

- *The proof applies to any bounded Ω such that $\Delta u + g|u|^{2\sigma} u + \mu[u]u = 0$ has a unique positive solution (which minimizes the constrained energy)*
- *For general u_0, the solution may not converge to the ground state*

Navier-Stokes equations on $\mathbb{T}^2$

Similar problems were studied for the Navier-Stokes equations on the 2D torus

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) + \big(u(t,x)\cdot\nabla\big)u(t,x) - \Delta u(t,x) + \nabla p(t,x) = f(t)u(t,x) & t>0\,,\ x\in\mathbb{T}^2 \\ u(0,x) = u_0(x) & x\in\mathbb{T}^2 \end{cases}$$

in

E. Caglioti, M. Pulvirenti, and F. Rousset.
On a constrained 2-D Navier-Stokes equation.
Commun. Math. Phys., 290(2):651–677, 2009.

where the N-S equations were considered in vorticity form with two kinds of constraint—constant energy and moment of inertia—proving the existence of a unique global solution for a special family of initial data, and in

Zdzisław Brzeźniak, Gaurav Dhariwal, and Mauro Mariani.
2d constrained Navier-Stokes equations.
J. Differ. Equations, 264(4):2833–2864, 2018.

where a global existence result for the nonlocal N-S system was deduced from the invariance of the unit sphere

Abstract form of the N-S system

Let us rewrite the Navier-Stokes system in abstract form as follows

$$\begin{cases} u'(t) = Au(t) + B(u(t)) + f(t)u(t) & 0 < t < T \\ u(0) = u_0 \in H \end{cases} \qquad (NS_f)$$

where:

- $u_0 = u_0(x)$ belongs to the space H of vector-valued functions in $L^2(\mathbb{T}^2; \mathbb{R}^2)$ which are divergence free
- $Au = \mathcal{P}(\Delta u - u)$, with $\mathcal{P}$ the orthogonal projector of $L^2(\mathbb{T}^2; \mathbb{R}^2)$ onto H
- $B(u) = \mathcal{P}\big((u \cdot \nabla)u\big)$
- $f : [0, T) \to \mathbb{R}$ is a locally bounded function

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Denote by V the subspace of H consisting of all vectors in $H^1(\mathbb{T}^2; \mathbb{R}^2)$ which are divergence free. The following trilinear form on V is of common use

$$b(u, v, w) = \int_{\mathbb{T}^2} \langle (u \cdot \nabla)v, w \rangle dx = \sum_{H,K=1}^{2} u_k(x) \frac{\partial v_h}{\partial x_k}(x)\, w_h(x)\, dx$$

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Hyperplane-constrained evolution

For the Navier-Stokes system

$$\begin{cases} u'(t) = Au(t) + B(u(t)) + f(t)u(t) & 0 < t < T \\ u(0) = u_0 \end{cases} \qquad (NS_f)$$

consider the following bilinear control problem

Problem

Given $u_0, v_0 \in V$ with $\langle u_0, v_0 \rangle_H \neq 0$, find $f : [0, T) \to \mathbb{R}$ such that

$$\langle u^f(t), v_0 \rangle_H = \langle u_0, v_0 \rangle_H \qquad \forall t \in [0, T)$$

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Let f be a solution. Then

$$\begin{aligned} f(t) &= -\frac{1}{\langle u_0, v_0 \rangle_H} \Big\{ \langle Au^f(t), v_0 \rangle_H + b(u^f(t), u^f(t), v_0) \Big\} \\ &= \frac{1}{\langle u_0, v_0 \rangle_H} \Big\{ \langle u^f(t), v_0 \rangle_V + b(u^f(t), v_0, u^f(t)) \Big\} \end{aligned}$$

Recasting as an invariance problem

C – Frankowska, *Nonlinear Differ. Equ. Appl.* (2025), dedicated to G. Da Prato

The above bilinear control problem can be recast as an invariance problem for

$$\begin{cases} u'(t) = Au(t) + B(u(t)) + F(u(t)) & 0 < t < T \\ u(0) = u_0 \end{cases} \qquad (NS_F)$$

with $F : V \to H$ given by

$$F(u) = \frac{\langle u, v_0 \rangle_V + b(u, v_0, u)}{\langle u_0, v_0 \rangle_H} \, u \qquad \forall u \in V$$

($u_0, v_0 \in V$ are such that $\langle u_0, v_0 \rangle_H \neq 0$ and $\|v_0\|_V = 1$)

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($u_0, v_0 \in V$ are such that $\langle u_0, v_0\rangle_H \neq 0$ and $\|v_0\|_V = 1$)

Theorem

Problem (NS_F) *has a unique maximal solution* $u(t, u_0)$ *defined for* $0 \leqslant t < \tau_{u_0} \leqslant \infty$

Crucial property:
$\langle B(u), Au\rangle_H = 0$ for all $u \in D(A)$ by periodic boundary conditions in 2D

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Invariance of hyperplanes

Let $v_0 \in V$ be such that $\|v_0\|_V = 1$ and let $c \neq 0$ be a real number

Theorem

If $u \in V_c(v_0) := \{u \in V \ : \ \langle u, v_0 \rangle_H = c\}$*, then the maximal solution of*

$$\begin{cases} u'(t) = Au(t) + B(u(t)) + F(u(t)) & 0 < t \\ u(0) = u_0 \end{cases} \tag{NS_F}$$

belongs to $V_c(v_0)$ *for all* $t \in [0, \tau_{u_0})$

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Proof.

Since $F(u) = \frac{\phi(u)}{\langle u_0, v_0\rangle_H} u$ with $\phi(u) = \langle u, v_0\rangle_V + b(u, v_0, u)$, we have that

$$\begin{aligned} \langle u'(t), v_0\rangle_H &= \langle Au(t) + B(u(t)), v_0\rangle_H + \frac{\phi(u(t))}{c} \langle u(t), v_0\rangle_H \\ &= -\phi(u(t)) + \frac{\phi(u(t))}{c} \langle u(t), v_0\rangle_H \end{aligned}$$

So, $\frac{d}{dt}\big(\langle u(t), v_0\rangle_H - c\big) = \frac{\phi(u(t))}{c} \big(\langle u(t), v_0\rangle_H - c\big)$ forcing $\langle u(t), v_0\rangle_H - c = 0$ □

Application to bilinear control of N-S

Let $u_0, v_0 \in V$ be such that $\langle u_0, v_0 \rangle_H \neq 0$ and $\|v_0\|_V = 1$

Then there exists a unique control $f : [0, \tau_{u_0}) \to \mathbb{R}$ such that the solution u^f of

$$\begin{cases} u'(t) = Au(t) + B(u(t)) + f(t)u(t) & 0 < t < T \\ u(0) = u_0 \end{cases} \tag{NS_f}$$

satisfies

$$\langle u^f(t), v_0 \rangle_H = \langle u_0, v_0 \rangle_H \qquad \forall t \in [0, \tau_{u_0})$$

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$$\langle u^f(t), v_0\rangle_H = \langle u_0, v_0\rangle_H \qquad \forall t \in [0, \tau_{u_0})$$

Moreover

$$f(t) = \frac{\phi(\overline{u}(t))}{\langle u_0, v_0\rangle_H} \quad \big(t \in [0, \tau_{u_0})\big)$$

where $\phi(u) = \langle u, v_0\rangle_V + b(u, v_0, u)$ and $\overline{u} : [0, \tau_{u_0}) \to V$ is the maximal solution of

$$\begin{cases} u'(t) = Au(t) + B(u(t)) + \frac{\phi(u(t))}{\langle u_0, v_0\rangle_H}\, u(t) & (t > 0) \\ u(0) = u_0 \end{cases}$$

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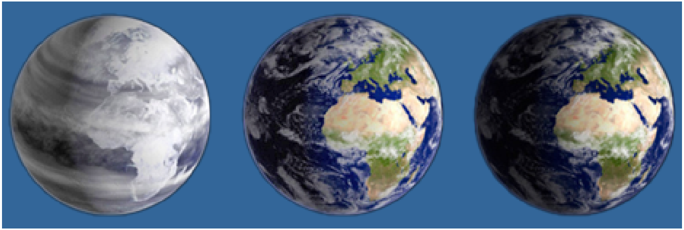
Some mathematical approaches to climate change and its impacts

NextGenerationEU



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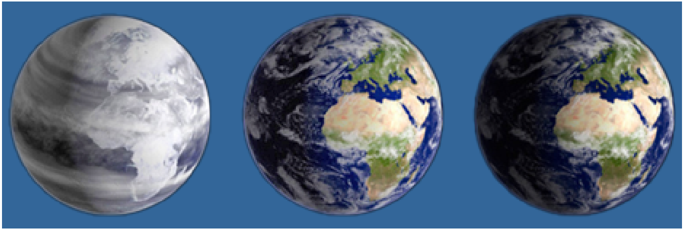

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