Geometric Inverse Problems for PDE's: uniqueness and reconstruction

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joint work with several people

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Control of PDEs and Related Topics

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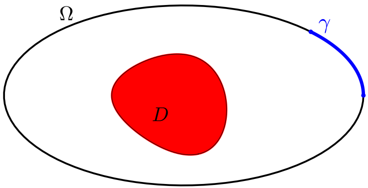
Outline

- $\Omega \subset \mathbb{R}^N$, $T > 0$, $\varphi = \varphi(x,t)$ are known
- $D \subset\subset \Omega$ is unknown



$$\begin{cases} PDE(u) = 0 & \text{in } (\Omega \setminus \overline{D}) \times (0,T) \\ u = \varphi & \text{on } \partial\Omega \times (0,T) \\ u = 0 & \text{on } \partial D \times (0,T) \\ \text{Initial cond.} & \text{in } \Omega \setminus \overline{D} \end{cases}$$

- $\gamma \subset \partial\Omega$ and $\beta = \left.\dfrac{\partial u}{\partial n}\right|_{\gamma \times (0,T)}$ are known

$PDE(u)$: heat, wave, elasticity, Navier-Stokes, Boussinesq …

Inverse Problem

Find D and the solution u

Main questions

$$\begin{cases} PDE(u_k) = 0 & \text{in } (\Omega \setminus \overline{D_k}) \times (0,T), \quad k = 1,2 \\ u_k = \varphi & \text{on } \partial\Omega \times (0,T) \\ u_k = 0 & \text{on } \partial D_k \times (0,T) \\ \dots \end{cases}$$

Observations (for example): $\beta_k = \dfrac{\partial u_k}{\partial n}$ on $\gamma \times (0,T)$

1. Uniqueness:

$$\beta_1 \equiv \beta_2 \text{ on } \gamma \times (0,T) \Longrightarrow D_1 = D_2 \,?$$

2. Stability:

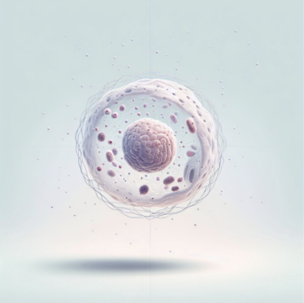
$$\text{dist.}(D_1, D_2) \leq g(\text{dist.}(\beta_1, \beta_2)) \quad \text{(at least locally) ?}$$

3. Reconstruction:

Compute (a numerical approximation of D from β ?

Motivation I

Goal in fluid mechanics: find a structure inside the fluid; free boundary problem:



Previous works, $u_0(x) \equiv 0$: Stokes, Navier-Stokes, Boussinesq

[Alvarez and al, 2005]

[Doubova, Fernández-Cara, Ortega, 2007]

[Doubova, Fernández-Cara, González-Burgos, Ortega, 2006], . . .

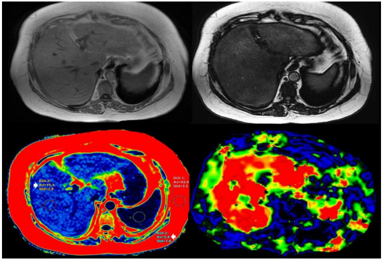
Also: nonscalar elliptic systems, chemical product, sensible to temperature effects, fills an unknown domain

[Araújo, Fernández-Cara, Souza, 2020]

Motivation II

• Elastrography: uses low-frequency vibrations (ultrasound or MRI) to analyze the stiffness of the tissue; set of noninvasive techniques

Application in Medicine: breast, liver, prostate cancers, fibrosis, etc



• To identify a rigid structure in an elastic medium

Wave equation, Elasticity system
[Doubova, Fernández-Cara, 2015, 2020]

Uniqueness

1. Classical arguments using unique continuation (zero initial data)

   Stokes, Navier-Stokes, Boussinesq, heat, wave, Lamé . . .

2. Analyze sensitivity of uniqueness to boundary or initial data: 1D models, find ℓ; positive and negative answers

   heat, wave, Burgers, fluid solid interaction model

3. Uniqueness in determining multidimensional domain with unknown initial data

   Our work: [J. Apraiz, A. Doubova, E. Fernández-Cara, M. Yamamoto, 25]

   To appear, available on arXiv: `https://arxiv.org/abs/2504.10236`.

1.- Uniqueness: by unique continuation

$$\begin{cases} \partial^2_{tt} u_k - \mu \Delta u_k - (\mu+\lambda)\nabla(\nabla\cdot u_k) = 0 & \text{in } \Omega\setminus\overline{D_k}\times(0,T), \quad k=1,2 \\ u_k = \varphi & \text{on } \partial\Omega\times(0,T) \\ u_k = 0 & \text{on } \partial D_k\times(0,T) \\ u_k(x,0) = u_0(x), \quad \partial_t u_k(x,0) = 0 & \text{in } \Omega\setminus\overline{D_k} \end{cases}$$

$u = u(x,t)$: displacement vector, $\lambda, \mu > 0$: Lamé coefficients

$\sigma(u)\cdot n = (\mu(\nabla u + \nabla u^t) + \lambda(\nabla\cdot u)\mathbf{Id})\cdot n$: normal stress tensor

$\beta_k = \sigma(u_k)\cdot n$ on $\gamma\times(0,T)$: observations

Theorem 1 (uniqueness for Lamé system) [Doubova, Fernández-Cara, 20]

Assume: $u_0(x) = 0$, $D^i \Subset \Omega$ non-empty and convex, $T > T_*(\Omega,\gamma)$, $\varphi \neq 0$ and

$$\beta_1 = \beta_2 \quad \text{on} \quad \gamma\times(0,T) \quad \Longrightarrow \quad D_1 = D_2$$

- Key point: a unique continuation (L. Hörmander) + arguments [O. Kavian, 2003]

Ideas (simplified): $G := \Omega\setminus(\overline{D_1\cup D_2})$, $u := u_1 - u_2$ in $G\times(0,T)$. Unique continuation $\Rightarrow u = 0$ in $G\times(0,T)$. Assume $D_2\setminus\overline{D_1}\neq\emptyset$. Consider u_1 in $D_2\setminus\overline{D_1}$. We have $u_1 = 0$ in $D_2\setminus\overline{D_1}\times(0,T)$. Again, unique continuation $\Rightarrow u_1 = 0$ in $\Omega\setminus\overline{D_1}\times(0,T)$, but $u_1 = \varphi \neq 0 \Rightarrow$ contradiction

Positive and negative answers

Previous works:

- 1D heat and wave equations

  [Apraiz, Cheng, Doubova, Fernández-Cara, Yamamoto, 2022]

- 1D viscous Burgers equation and related systems (heat effects; variable density)

  [Apraiz, Doubova, Fernández-Cara, Yamamoto, 2022]

- 1D Fluid solid interaction model

  [Apraiz, Doubova, Fernández-Cara, Yamamoto, 2024]

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Uniqueness and numerical reconstruction for inverse problems dealing with interval size search

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Some inverse problems for the Burgers equation and related systems

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ORIGINAL PAPER

Inverse Problems for One-Dimensional Fluid-Solid Interaction Models

J. Apraiz[1] · A. Doubova[2] · E. Fernández-Cara[2] · M. Yamamoto[3]

$\Omega \subset \mathbb{R}^d$, $d \geq 1$ bounded connected open set

$$\begin{cases} \partial_t u_k - \mathcal{A} u_k = F(x,t) & \text{in } (\Omega \setminus \overline{D_k}) \times (0,T), \quad k = 1,2 \\ u_k = 0 & \text{on } \partial(\Omega \setminus \overline{D_k}) \times (0,T) \end{cases}$$

- Attention: no information on initial values for u_1 and u_2 (incomplete direct problem)
- $F(x,t) = \mu(t) f(x)$, $f \not\equiv 0$ is an input amplitude, $\mu \not\equiv 0$ in $(0,T)$
- $\mathcal{A} v(x) := \sum_{i,j=1}^{d} \partial_i (a_{ij}(x) \partial_j v(x)) - \sum_{j=1}^{d} b_j(x) \partial_j v(x) - c(x) v(x)$
- $a_{ij} = a_{ji} \in C^1(\overline{\Omega})$, $b_j, c \in L^\infty(\Omega)$, $\exists\, \alpha > 0$: $\sum_{i,j=1}^{d} a_{ij}(x) \xi_i \xi_j \geq \alpha |\xi|^2 \forall\, \xi \in \mathbb{R}^d$, a.e. in Ω.
- $A_k : \mathcal{D}(A_k) \to L^2(\Omega \setminus \overline{D_k})$, $\quad \mathcal{D}(A_k) = \{ v \in H_0^1(\Omega \setminus \overline{D_k}) : \mathcal{A} v \in L^2(\Omega \setminus \overline{D_k}) \}$

$$(A_k v)(x) := \mathcal{A} v(x) \text{ a.e. in } \Omega \setminus \overline{D_k} \quad \forall\, v \in \mathcal{D}(A_k)$$

- Given $\frac{\partial v}{\partial \nu_A} := \sum_{i,j=1}^{d} a_{ij} \partial_i v \, \nu_j$ on $\gamma \times (0,T), \gamma \subset \partial\Omega$ (conormal derivative)

Inverse problem: Find D from $\dfrac{\partial v}{\partial \nu_A}$ on $\gamma \times (0,T)$

Theorem 2: Uniqueness [Apraiz, Doubova, Frnández-Cara, Yamamoto, 25]

Let u_1, u_2 be solutions corresponding to the simply connected open sets D_1 and D_2.

Assume: $f \in H^{2\varepsilon}(\Omega)$ for some $\varepsilon > 0$

- $f = 0$ in $D_1 \cup D_2$, but $f \not\equiv 0$
- $\mu(t)$ is piecewise polynomial, with $\mu \notin C^m([0, T])$ for some $m \geq 1$

Then:

$$\frac{\partial u_1}{\partial \nu_A} = \frac{\partial u_2}{\partial \nu_A} \quad \text{on} \quad \gamma \times (0, T) \quad \Longrightarrow \quad D_1 = D_2$$

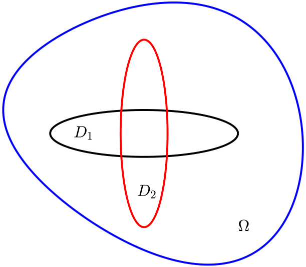
Moreover: $u_1(\cdot\,, 0) = u_2(\cdot\,, 0)$.

Idea of the proof I

Assume: $D_1 \neq D_2$. Argue by contradiction

Step 1: Let $G := \Omega \setminus (\overline{D_1 \cup D_2})$ and $u := u_1 - u_2$ in $G \times (0, T)$:

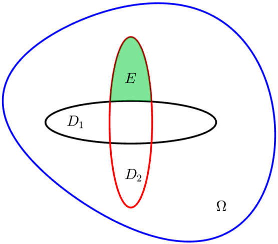
$$\begin{cases} \partial_t u - \mathcal{A}u = 0 & \text{in } G \times (0, T) \\ u = 0 & \text{on } \partial\Omega \times (0, T) \\ \dfrac{\partial u}{\partial \nu_A} = 0 & \text{on } \gamma \times (0, T) \end{cases}$$



Unique continuation $\Rightarrow$ $u = 0$ in $G \times (0, T)$ $\Rightarrow$ $u_1 = u_2$ in $G \times (0, T)$
(no initial condition is needed)

Step 2: $D_1 \neq D_2 \quad \Rightarrow \quad \exists\, E \neq \emptyset$ open:

$$E \subset (\Omega \cap \overline{D_1}) \cap D_2, \quad \partial E \subset \partial D_1 \cup \partial D_2$$



• $f = 0$ in $D_1 \cup D_2$, $z := \partial_t^m u_1$, $m \geq 1$ satisfies

$$\begin{cases} \partial_t z - A_1 z = 0 & \text{in } E \times (0, T), \\ z = 0 & \text{on } \partial E \times (0, T). \end{cases}$$

$(A_1 v)(x) := \mathcal{A}v(x)$ a.e in $\Omega \backslash \overline{D_1}$

• Introduce $E_0 \subset\subset E$ new nonempty open set and $h(t) := z(\cdot, t)\big|_{E_0}$ for all t.
Semigroup theory implies that $t \mapsto h(t)$ is analytical
Computations and properties of $\mu \quad \Rightarrow$

$$(\mathrm{Id.} - e^{-tA_1})A_1^{-1} f \equiv 0 \ \text{ in } \ E_0 \quad \forall\, t$$

$\Rightarrow f = 0$ in $\Omega \setminus \overline{D_1}$, contradiction with $f \not\equiv 0$ in $\Omega \setminus \overline{D_1}$, then $D_1 = D_2$.

$D_1 = D_2$ (UC) $\quad \Rightarrow \quad u_1 = u_2 \quad \Rightarrow \quad u_1(\cdot\,, 0) = u_2(\cdot\,, 0)$ (Backwards uniqueness) $\blacksquare$

Comments and extensions I

1. More realistic assumption for f: $\exists\, \Omega_0 \subset\subset \Omega$, $D_1, D_2 \subset \Omega_0$ such that $f = 0$ in Ω_0 (since D_1 and D_2 are unknown).
2. Positive result if $\mu(t)$ vanishes in some initial time interval: $\mu(t) = 0$ a.e. in $(0, t_*)$ for some $t_* \in (0, T)$ but $\mu \not\equiv 0$.
3. Positive result: uniqueness with different coefficients
4. A variant: boundary source and internal observation, also OK
5. Possible extension to more complex linear parabolic systems: Stokes, linearized Boussinesq, linearized Oldroyd, etc.

1. Stability of D and $u\big|_{t=0}$ w.r.t $\frac{\partial u}{\partial \nu_A}$ on $\gamma \times (0, T)$?

2. Can we also get uniqueness for more general inputs of the form $F = F(x, t)$?

3. We can get uniqueness for the Stokes systems even with $d - 1$ scalar observations (unique continuation is true in this case).

   However, for $d = 3$, it is unknown whether the result holds for $d - 2$ components, i.e. when only one component coincides.

Reconstruction

Two methods of reconstruction of D

1. Method 1 (based on FEM): new mesh at each iteration

   Lamé system

2. Method 2: Method of Fundamental Solutions (MFS), meshless

   Elliptic equation

Other possible meshless methods:

- Model order reduction techniques (Reduced Basis Methods)

  Look for a few basis functions that represent the behavior of the problem

- Physics Informed Neural Networks (PINN's)

  Use the information provided by the problem and progress of Artificial Intelligence

Geometric IP, Lamé system: reconstruction I

Goal: compute D from β on $\gamma \times (0, T)$

Technique: solve an optimization problem

Optimization problem

Assume: $d = 3$, D is a sphere and $\beta = \beta(x, t)$ is given

Find $(x_0, y_0, z_0, r) \in \mathcal{D}_{ad}$ such that

$$\begin{cases} \text{Minimize } J(x_0, y_0, z_0, r) := \dfrac{1}{2} \displaystyle\iint_{\gamma \times (0,T)} \left|\sigma(u(x_0, y_0, z_0, r)) \cdot n - \beta\right|^2 ds\, dt \\ \text{Subject to } D \in \mathcal{D}_{ad} := \{\, (x_0, y_0, z_0, r) \in \mathbb{R}^4 \;:\; \overline{B}(x_0, y_0, z_0; r) \subset \Omega,\ r > 0 \,\} \end{cases}$$

$$\sigma(u) \cdot n = \left(\mu(\nabla u + \nabla u^t) + \lambda(\nabla \cdot u)\mathbf{Id.}\right) \cdot n \quad \text{on } \gamma \times (0, T)$$

Geometric IP, Lamé system: reconstruction II

Method 1: based on Finite Element Method, new mesh at each iteration

Test 1 : $T = 5$

$u_0 = (10x, 10y, 10z), \quad u_1 = (0, 0, 0)$

$\eta = (10x, 10y, 10z)$

```
(xd, yd, zd, rd) = (-2, -2, -2, 1)
(xi, yi, zi, ri) = (0, 0, 0, 0.6)
NLopt (AUGLAG + DIRECTNoScal)
```

No Iter = 1005, FreeFem++ :

```
xc  =  -1.981405274
yc  =  -2.225232904
zc  =  -2.148084171
rc  =   0.9504115226
```

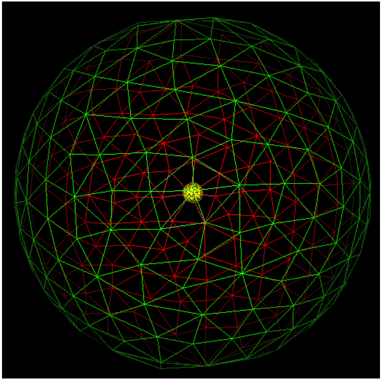


Figure: Initial mesh. Points: 829, tetrahedra: 4023, faces: 8406, edges: 5210, boundary faces: 720, boundary edges: 1080

Geometric IP, Lamé system: reconstruction III

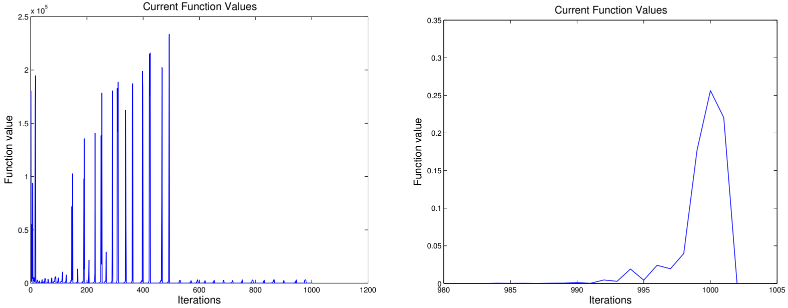


Figure: Evolution of the cost (left) and detail (right). Cost $< 10^{-3}$.

More details [Doubova, Fernández-Cara, 2020]

Method 1 (based on FEM): difficult, expensive (new mesh at each iteration) ...

Method of Fundamental Solutions (MFS)

Method 2: Method of Fundamental Solutions (MFS), meshless

- Developed by [V. Kupradze & D. Alexidze, 1960] for a direct problem
- Proposed as a computational technique by [Mathon & Johnston, 1977]
- Non-homogeneous problems: combined with Method of Particular Solutions (MPS) [Golberg, 1996]
- A collection of points is required
- Key idea: to use a basis formed by fundamental solutions

Advantages: meshless, high computational speed, exponential convergence properties

MFS for an elliptic problem

The problem:

$$\begin{cases} -\Delta u + au = h & \text{in } \Omega \setminus \overline{D} \\ u = \varphi & \text{on } \partial\Omega \\ u = 0 & \text{on } \partial D \end{cases}$$

$a = a(x)$, $h = h(x)$, $\varphi = \varphi(x)$ appropriate

Observation: $\beta := \dfrac{\partial u}{\partial n}$ on γ

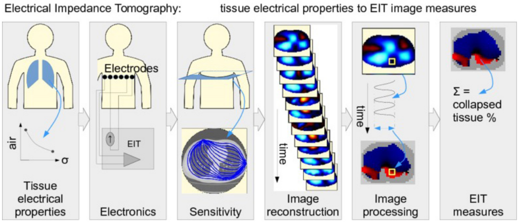


Figure: Electrical impedance tomography procedure.

1. Uniqueness: OK. Ideas from [O. Kavian, 2003]
2. Stability: OK. [A.L. Bukhgeim, J. Cheng, M. Yamamoto, 1999]
3. **Reconstruction:** OK. Several methods: optimization + least square + FEM
   Here: **MFS (Method of Fundamental Solutions)**

$\Omega = B(0;R)$. Goal: find $D = B(x_0, y_0; r)$ s.t.

$$\begin{cases} -\Delta u + au = h & \text{in} \quad \Omega \setminus \overline{D} & \text{(PDE)} \\ u = \varphi & \text{on} \quad \partial\Omega & \text{(BC on } \partial\Omega\text{)} \\ u = 0 & \text{on} \quad \partial D & \text{(BC on } \partial D\text{)} \end{cases}$$

$$\frac{\partial u}{\partial n} = \beta \quad \text{on} \quad \gamma \quad \text{(BC on } \gamma\text{)}$$

Step 1 (MFS-MPS): $-\Delta u = -au + h$

$$u(x) \approx \sum_{j=1}^{Nf} \delta_j F(||x - \eta_j||) + \sum_{k=1}^{Nb} \alpha_k G(||x - \xi_k||)$$

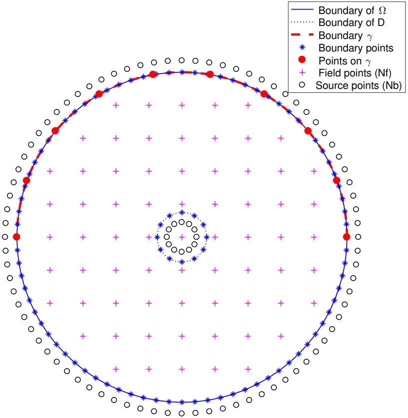


Figure: $(+, \eta_j)$, $(\circ, \xi_k)$

Nf : field points, η_j

Nb : source points, ξ_k

F : integrated radial basis, $\Delta F(r) = f(r)$, $f(r)$ a compactly supported radial basis

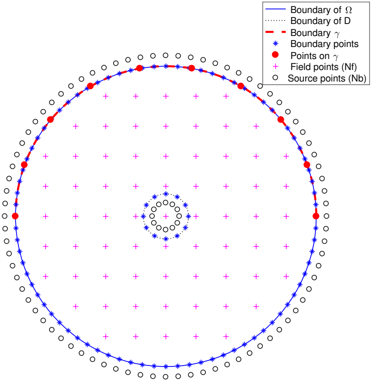
G : fundamental solution of the Laplace equation

Step 2 (Reduction to a nonlinear algebraic system):

- PDE at the field points η_i
- Dirichlet BC at the boundary points $\in \partial\Omega$
- Neumann BC at the boundary points $\in \gamma$
- Dirichlet BC at the boundary points $\in \partial D$ (unknown)
- Nonlinear system of equations:

  Find $(\delta, \alpha) \in \mathbb{R}^{Nf} \times \mathbb{R}^{Nb}$, $(x_0, y_0, r) \in X_b$:

$$M(x_0, y_0, r) \begin{bmatrix} \delta \\ \alpha \end{bmatrix} = Z$$



Step 3 (Least square): $Nf + Nb + 3$ unknowns at each iteration of optim. algorithm

$$J(\delta, \alpha, x_0, y_0, r) := \frac{1}{2} \left\| M(x_0, y_0, r) \begin{bmatrix} \delta \\ \alpha \end{bmatrix} - Z \right\|^2$$

Test 2 - MFS: D a circle I

Test 2: $N = 2, \quad \Omega = B(0,0;10), \quad a = 0.2\sqrt{x^2+y^2}, \quad h = 0.3x, \quad \varphi = 10x$

- $Nom = 60$: points on $\partial\Omega$
- $Ng = 10$: points on γ
- $Nd = 12$: points on ∂D
- $Nb = Nom + Nd$: source points ξ_k

```
xd = 2,  yd = 4,  rd = 1
xi = 0,  yi = 0,  ri = 1.5
```

```
MATLAB + fmincon
```

```
xc = 2.000274
yc = 4.000057
rc = 0.999658
```

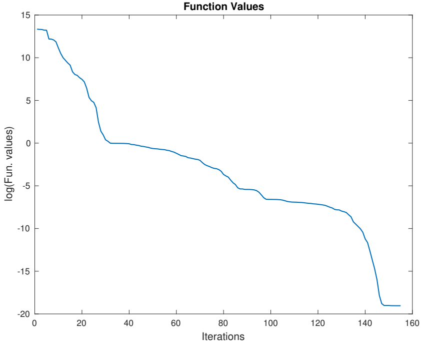


Figure: Evolution of the cost, 146 iterations. Cost $J(\delta, \alpha, x_0, y_0, r) < 10^{-9}$.

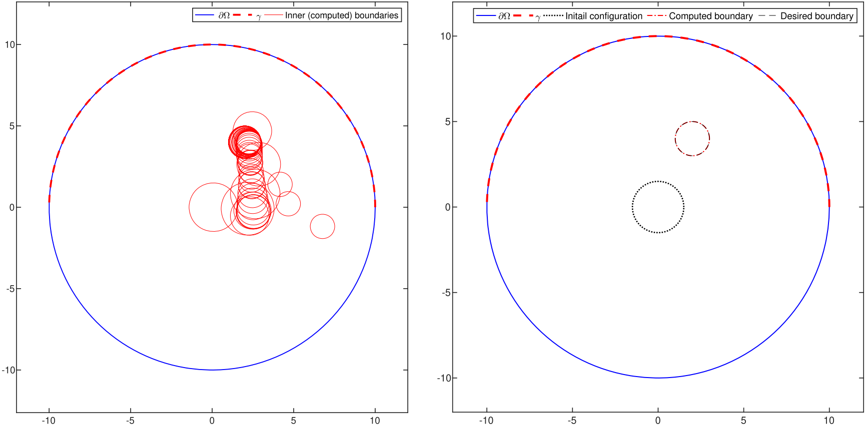


Figure: Iterations of optimization algorithm (left). Initial, desired and computed configuration (right)

With Method 1 (FEM, `FreeFem++`): **1000 iterates** to get a Cost $< 10^{-7}$

With MFS-MPS: **146 iterates** to get a Cost $< 10^{-9}$

Some open questions:

1. Similar numerical analysis for other more complex geometries
   (polyhedral unknown D in 3D, the case of three or more balls, . . .)

2. MFS-MPS for Inverse Problems evolution system
   (wave equation, Lamé, Stokes, Navier-Stokes, Boussinesq, free boundary . . .)