A method to determine the minimal null control time of 1D linear hyperbolic systems

Guillaume Olive

(joint work with Long Hu)

Control of PDEs and related topics

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Outline of the talk

I. Framework

II. Preliminary results

III. Presentation of the method

System description

Equations

$$\frac{\partial y}{\partial t}(t,x) + \Lambda(x)\frac{\partial y}{\partial x}(t,x) = M(x)y(t,x), \quad (t,x) \in R_T,$$

- $R_T = (0,T) \times (0,1)$ and $y : R_T \to \mathbb{R}^n$ is the state.
- $\Lambda \in C^{0,1}([0,1])^{n\times n}$ is diagonal, with

$$\lambda_1 < \cdots < \lambda_m < 0 < \lambda_{m+1} < \cdots < \lambda_{m+p}.$$

- $M \in L^\infty(0,1)^{n\times n}$ is the **internal coupling matrix.**

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$$y(0,x) = y^0(x).$$

Denoting by $y = \begin{pmatrix} y_- \\ y_+ \end{pmatrix} \in \mathbb{R}^{m+p}$,

Boundary conditions

$$y_-(t,1) = u(t), \quad y_+(t,0) = Qy_-(t,0).$$

$u : (0, T) \to \mathbb{R}^m$ is the **control.** $Q \in \mathbb{R}^{p\times m}$ is the **boundary coupling matrix.**

Minimal control time

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Remark: **(NC)** in time $T_1 \quad \Longrightarrow \quad$ **(NC)** in time $T_2 \geq T_1$.

Definition

Minimal time for **(NC)**:

$$T_{\inf} = \inf \{ T > 0, \quad \text{System is } \textbf{(NC)} \text{ in time } T \}, \qquad (\in [0, +\infty]).$$

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- $T < T_{\inf} \quad \Longrightarrow \quad$ System is not **(NC)** in time T.

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- $T < T_{\inf} \quad \Longrightarrow \quad$ System is not **(NC)** in time T.

Goal

$$T_{\inf} = ??? \qquad (Q, M \text{ are fixed}).$$

Control of a single equation

The transport equation:

$$\begin{cases} \dfrac{\partial y}{\partial t} + \lambda(x)\dfrac{\partial y}{\partial x} = 0, \\ y(0,x) = y^0(x), \\ y(t,\delta) = u(t), \end{cases} \tag{TE}$$

($\delta = 1$ if $\lambda < 0$, $\delta = 0$ if $\lambda > 0$). We have

$$T_{\inf} = \int_0^1 \frac{1}{|\lambda(\xi)|}\, d\xi.$$

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Coming back to the system,

Definition

$T_k = T_{\inf}$ of (TE) with speed λ_k.

$$\lambda_1 < \cdots < \lambda_m \quad < 0 < \quad \lambda_{m+1} < \cdots < \lambda_{m+p},$$

$$\text{implies} \quad T_1 \leq \ldots \leq T_m, \qquad T_{m+1} \geq \ldots \geq T_{m+p}.$$

Some known results

Assumptions	Results	References
$\emptyset$	**(NC)** in time $T_{m+1} + T_m$	Russell 1978; Li and Rao 2003
$M = 0$	Explicit formula for T_{inf}	Weck 1982; Hu and Olive 2022
$Q \in \mathcal{B}$	**(NC)** in any time $T > T_{[\mathrm{CN}]}$	Coron and Nguyen 2019, 2021
$\mathrm{rank}\, Q = p$	T_{inf} is the same as for $M = 0$	Hu and Olive 2021a, 2022
$m = p = 1,\ Q \neq 0$	$T_{\mathrm{inf}} = T_2 + T_1$	Coron, Vazquez, et al. 2013
$Q = 0$	Explicit formula for T_{inf}	Hu and Olive 2021b
$\emptyset$	$\min_M T_{\mathrm{inf}}$ & $\max_M T_{\mathrm{inf}}$	Hu and Olive 2022
First $\min\{m, p\}$ rows of Q lin. indep.	T_{inf} is the same as for $M = 0$	
$m = 1,\ M(x) = M,\ \mathrm{diag}\, M = 0$	Explicit formula for T_{inf}	Hu and Olive 2025b

Still, T_{inf} remains unknown in some situations.

We propose a method when Λ, M are smooth enough.

Preliminary step 1: Backstepping method for PDEs

Assume $\operatorname{diag} M = 0$ (for simplicity). Volterra transformation (Krstic and Smyshlyaev 2008):

$$\tilde{y}(t,x) = y(t,x) - \int_0^x K(x,\xi) y(t,\xi)\, d\xi,$$

where $K : \{0 < \xi < x < 1\} \to \mathbb{R}^{n \times n}$ is a kernel (to be determined).

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Equivalent system

$$\begin{cases} \dfrac{\partial \tilde{y}}{\partial t} + \Lambda(x) \dfrac{\partial \tilde{y}}{\partial x} = G(x) \tilde{y}_-(t,0), \\ \tilde{y}(0,x) = \ldots, \\ \tilde{y}_-(t,1) = \ldots, \quad \tilde{y}_+(t,0) = Q \tilde{y}_-(t,0), \end{cases}$$

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- where $G(x) = K(x,0)B, \quad B = -\Lambda(0)\begin{pmatrix} \mathrm{Id}_m \\ Q \end{pmatrix},$
- provided that K satisfies the kernel equations:

$$\begin{cases} \Lambda(x)\dfrac{\partial K}{\partial x}(x,\xi) + \dfrac{\partial K}{\partial \xi}(x,\xi)\Lambda(\xi) + K(x,\xi)(\Lambda'(\xi) + M(\xi)) = 0, \\ \Lambda(x)K(x,x) - K(x,x)\Lambda(x) = M(x). \end{cases}$$

Such K exist (Hu, Di Meglio, et al. 2016).

Preliminary step 2: Compactness-uniqueness method

Denote by $G = \begin{pmatrix} G_{--} \\ G_{+-} \end{pmatrix}$, where G_{--} = the first m rows of G.

Theorem (Hu and Olive 2022)

T_{inf} does not depend on G_{--}.

Systems with $G_{--} = 0$ will be denoted by

$$(Q, G_{+-}).$$

Remarks:

- Impossible to obtain by transformation (e.g. Backstepping).
- We are dealing with **(NC)**, not **(EC)**.

Ingredient 1: T_{inf} for a class of full rank Q

Let $n^* = \min\{m, p\}$.

Theorem (Hu and Olive 2022)

Assume that the first n^* rows of Q are lin. indep. Then,

$$T_{\mathrm{inf}}(Q, G_{+-}) = T_{\mathrm{inf}}(Q, 0).$$

Ingredient 2: Linear combination of rows

Below, $Q_j = j$-th row of Q, and $G_j = j$-th row of G_{+-}.

Lemma (Hu and Olive 2022)

Let $a_1, \ldots, a_{k-1} \in \mathbb{R}$. *We have*

$$T_{\inf}(Q, G_{+-}) = T_{\inf}(\tilde{Q}, \tilde{G}_{+-}),$$

where:

$$\tilde{Q} = \begin{pmatrix} Q_1 \\ \vdots \\ Q_{k-1} \\ \hline Q_k - \sum_{j<k} a_j Q_j \\ \hline Q_{k+1} \\ \vdots \\ Q_p \end{pmatrix}, \quad \tilde{G}_{+-}(x) = \begin{pmatrix} G_1(x) \\ \vdots \\ G_{k-1}(x) \\ \hline G_k(x) - \sum_{j<k} a_j G_j(\zeta_{kj}(x)) \\ \hline G_{k+1}(x) \\ \vdots \\ G_p(x) \end{pmatrix},$$

where ζ_{kj} *is the solution to the ODE:* $\zeta_{kj}' = \frac{\lambda_{m+j}(\zeta_{kj})}{\lambda_{m+k}}$ *with* $\zeta_{kj}(0) = 0$.

Ingredient 3: Derivative of zero rows

Lemma (Hu and Olive 2025a)

Assume that $Q_k = 0$*. Then,*

$$T_{\mathrm{inf}}(Q, G_{+-}) = T_{\mathrm{inf}}(\hat{Q}, \hat{G}_{+-}),$$

where:

$$\hat{Q} = \begin{pmatrix} Q_1 \\ \vdots \\ Q_{k-1} \\ \hline G_k(0) \\ \hline Q_{k+1} \\ \vdots \\ Q_p \end{pmatrix}, \quad \hat{G}_{+-}(x) = \begin{pmatrix} G_1(x) \\ \vdots \\ G_{k-1}(x) \\ \hline (\mathcal{D}_k G_k)(x) \\ \hline G_{k+1}(x) \\ \vdots \\ G_p(x) \end{pmatrix},$$

where $\mathcal{D}_k = \lambda_{m+k} \frac{d}{dx}$.

Presentation of the method

Main ideas

Construct systems $(Q^{[k]}, G_{+-}^{[k]})$, $k = 1, 2, \dots, n^*$, such that:

- $T_{\inf}(Q^{[k]}, G_{+-}^{[k]}) = T_{\inf}(Q^{[k-1]}, G_{+-}^{[k-1]})$.
- First k rows of $Q^{[k]}$ are lin. indep.

If we succeed,

$$T_{\inf}(Q, G_{+-}) = T_{\inf}(Q^{[n^*]}, G_{+-}^{[n^*]}) = T_{\inf}(Q^{[n^*]}, 0).$$

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Construction:

- First k rows of $Q^{[k-1]}$ are lin. indep.: OK.
- Otherwise,
  - **Step 1: Put the k-th row to zero** (ingredient 2):

$$\text{new } Q_k: \quad Q_k^{[k-1]} - \sum_{j<k} a_j Q_j^{[k-1]} = 0, \quad \text{new } G_k: \quad G_k^{[k-1]}(x) - \sum_{j<k} a_j G_j^{[k-1]}(\zeta_{kj}(x)).$$

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  - **Step 2: Take the derivative of the k-th row** (ingredient 3):

$$\text{new } Q_k: \quad G_k^{[k-1]}(0) - \sum_{j<k} a_j G_j^{[k-1]}(0), \quad \text{new } G_k: \quad (\mathcal{D}_k G_k^{[k-1]})(x) - \sum_{j<k} a_j (\mathcal{D}_k (G_j^{[k-1]} \circ \zeta_{kj}))(x).$$

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Then repeat! Main assumption: We eventually find a lin. indep. new row.

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$$\begin{cases} \Lambda(x)\dfrac{\partial K}{\partial x}(x,\xi) + \dfrac{\partial K}{\partial \xi}(x,\xi)\Lambda(\xi) + K(x,\xi)(\Lambda'(\xi) + M(\xi)) = 0, \\ \Lambda(x)K(x,x) - K(x,x)\Lambda(x) = M(x). \end{cases}$$

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Indeed ($\Lambda = cst$ for simplicity),

- 0-th order:

$$k_{ij}(0,0) = \frac{m_{ij}(0)}{\lambda_i - \lambda_j}, \quad (i \neq j).$$

- 1-st order:

$$\begin{pmatrix} \lambda_i & \lambda_j \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial k_{ij}}{\partial x}(0,0) \\ \frac{\partial k_{ij}}{\partial \xi}(0,0) \end{pmatrix} = \begin{pmatrix} -\sum_{r=1}^{n} k_{ir}(0,0)m_{rj}(0) \\ \frac{m'_{ij}(0)}{\lambda_i - \lambda_j} \end{pmatrix}, \quad (i \neq j).$$

- Etc.

A $3+2$ constant system

$$\Lambda = \mathrm{diag}\left(-2, -1, -\frac{1}{2}, 1, 2\right) \quad M = \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 & -1 \\ \hline 3 & 2 & 1 & 0 & 0 \\ 8 & 9 & -\frac{20}{3} & 0 & 0 \end{array}\right), \quad Q = \begin{pmatrix} 0 & 1 & -2 \\ 0 & 2 & -4 \end{pmatrix}.$$

We have

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Known results:

- Russell 1978: **(NC)** in time $T = T_4 + T_3 = 3$.
- Coron and Nguyen 2019, 2021: not applicable ($Q \notin \mathcal{B}$).
- Hu and Olive 2022:

$$\min_M T_{\inf} = \max\{T_4 + T_2, \quad T_3\} = 2, \quad \max_M T_{\inf} = \max\{T_4 + T_2 \quad T_5 + T_3\} = \frac{5}{2}.$$

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Thank you for your attention !

More details at:

https://doi.org/10.1016/j.jde.2025.113455