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Neutrinos and Dark Matter
14 May 2025, IHP Paris

ihp Institut Henri Poincaré

Conditions for Instability in Collective Neutrino Oscillations

Basudeb Dasgupta
Tata Institute, Mumbai

The Basic Idea

$$i\partial_t|\nu_i\rangle = \left(\sum_{j=1}^{N}(1-\hat{p}_j\cdot\hat{p}_i)\underbrace{|\nu_j\rangle\langle\nu_j|}_{\rho_j} + \ldots\right)|\nu_i\rangle$$

Collective effect

Usual terms

Neutrinos give a phase-shift to other neutrinos.

This couples the linear equations and makes them “nonlinear”.

Can linearize and ask is the “evolution frequency” complex? If yes, Instability.

Physical Origin

Fermi National Accelerator Laboratory

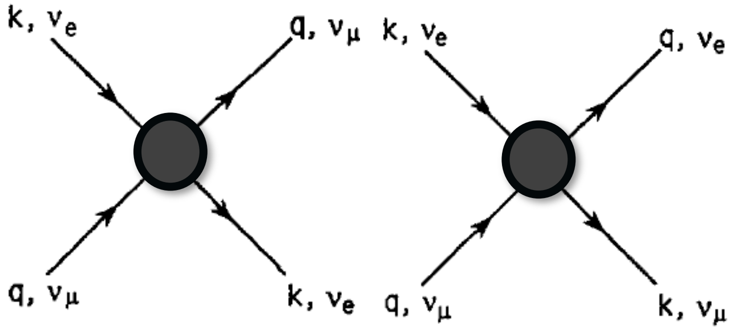
UCRHEP-T84
FERMILAB-PUB-92/18-T
January 1992

DIRAC NEUTRINOS
IN
DENSE MATTER

James Pantaleone

Fermi National Accelerator Lab
Batavia, IL 60510
and
Department of Physics
University of California
Riverside, CA 92521

In this formulation, it is apparent that basis rotations of the "propagating" neutrino cancel with those of the "background" neutrinos. Thus the U(2) flavor symmetry is maintained. *To neglect the off diagonal terms in every basis is obviously incorrect since it breaks this symmetry and then the result of the flavor evolution of a given state would be different in each basis.* The U(2) symmetry maintains the net flavor content.



Pantaleone (1992)

Forward scattering neutrinos can exchange flavor

$$
\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\begin{vmatrix} |\nu_e(\boldsymbol{k})\nu_e(\boldsymbol{q})\rangle \\ |\nu_e(\boldsymbol{k})\nu_\mu(\boldsymbol{q})\rangle \\ |\nu_\mu(\boldsymbol{k})\nu_e(\boldsymbol{q})\rangle \\ |\nu_\mu(\boldsymbol{k})\nu_\mu(\boldsymbol{q})\rangle \end{vmatrix} = V_2 \begin{vmatrix} |\nu_e(\boldsymbol{k})\nu_e(\boldsymbol{q})\rangle \\ |\nu_e(\boldsymbol{k})\nu_\mu(\boldsymbol{q})\rangle \\ |\nu_\mu(\boldsymbol{k})\nu_e(\boldsymbol{q})\rangle \\ |\nu_\mu(\boldsymbol{k})\nu_\mu(\boldsymbol{q})\rangle \end{vmatrix}
$$

where $V_{2\nu} = \sqrt{2}G_F\xi\frac{1}{V}\begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix}$

In Full Glory

$$i\left(\partial_t + \mathbf{v_p}\cdot\partial_\mathbf{p}\right)\rho_\mathbf{p} = +\left[\frac{\mathsf{M}^2}{2E},\,\rho_\mathbf{p}\right] + \sqrt{2}G_\mathrm{F}\left[\mathsf{L},\rho_\mathbf{p}\right] + \sqrt{2}G_\mathrm{F}\int\frac{d^3\mathbf{q}}{(2\pi)^3}(1-\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})\left[\rho_\mathbf{q}-\bar{\rho}_\mathbf{q},\rho_\mathbf{p}\right]$$

Vacuum oscillations depend on neutrino mass matrix M
Overall minus sign for antineutrinos

$$\omega = \frac{\Delta m^2}{2E}$$

MSW effect depends on ordinary matter density L, i.e. mainly electron density

$$\lambda = \sqrt{2}G_F\,n_e$$

Collective effects depends on the neutrino density

$$\mu = \sqrt{2}G_F\,n_\nu$$

In general, a 7 dimensional problem
3 momentum (E, θ_p, ϕ_p) + 3 space (r, θ, ϕ) + 1 time (t)

A Garden of Ideas

Spectral Swaps / Splits
Flavor Equilibrium

QFT Derivation

Applications to Early Universe

Symmetries/ Inhomogeneous /
Temporal Modes

Many-body Physics

Applications to SN Heating

Fast Depolarization /
Flavor Equilibration

Collisional Processes

Applications to NS Cooling
and Nucleosynthesis

Talks by McLaughlin, Radice, Balantekin, Ehring, Xiong, Parkkinen that touch upon many of these ideas.
See also the Reviews of Modern Physics by Volpe for a broad and detailed perspective.

Collective Oscillation: Instability

When density is high

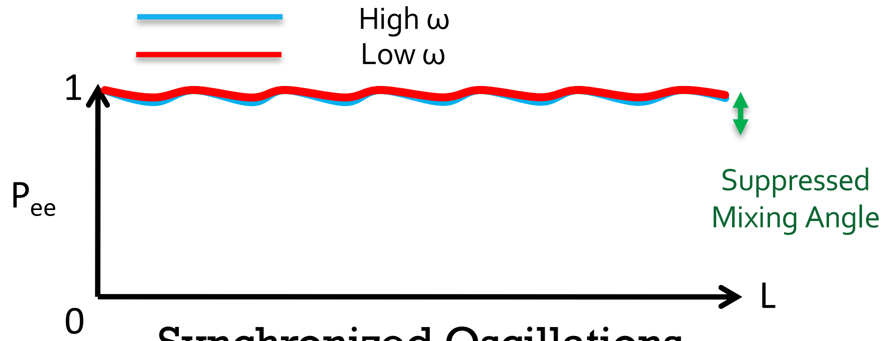
$$\mu = G_F n_\nu \gg \frac{\Delta m^2}{2E} = \omega$$

the collective oscillations have small amplitude

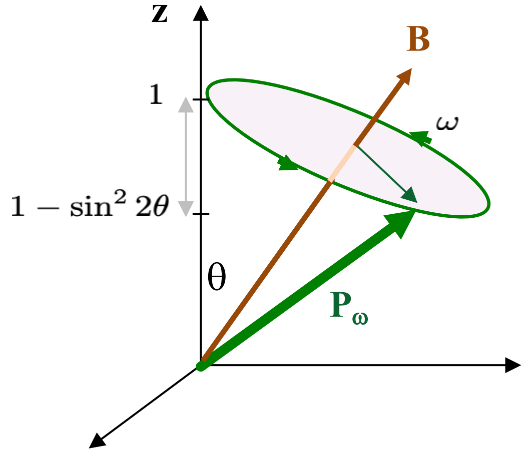
As neutrino density gets lower

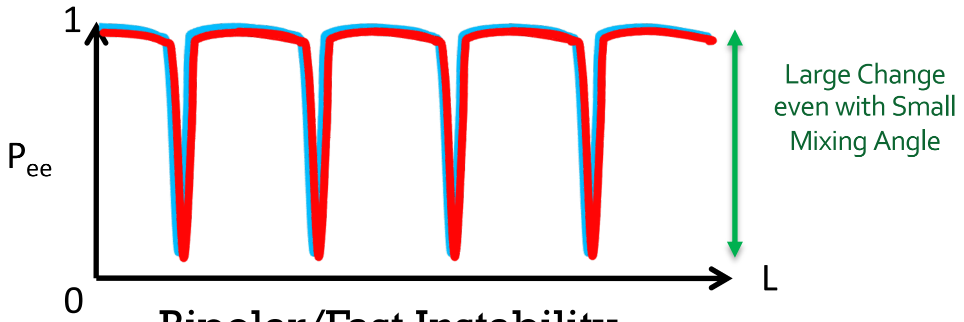
$$G_F\, n_\nu \gtrsim \frac{\Delta m^2}{2E}$$

the system can be unstable



Synchronized Oscillations



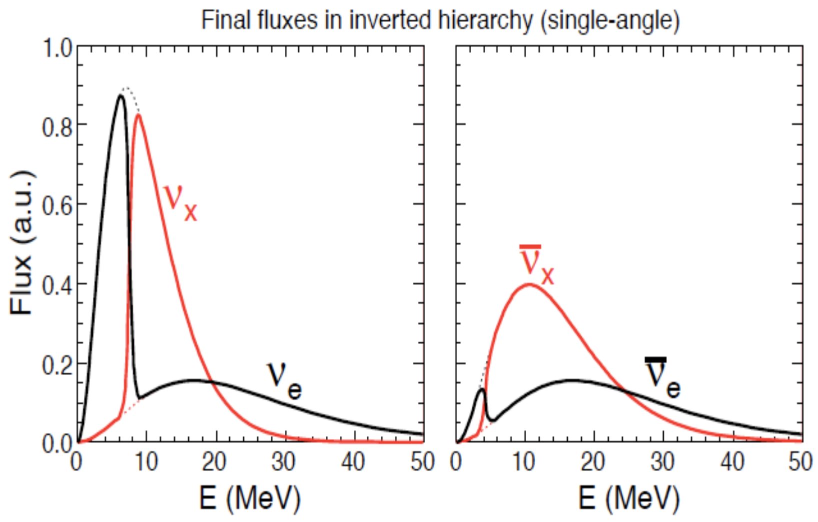


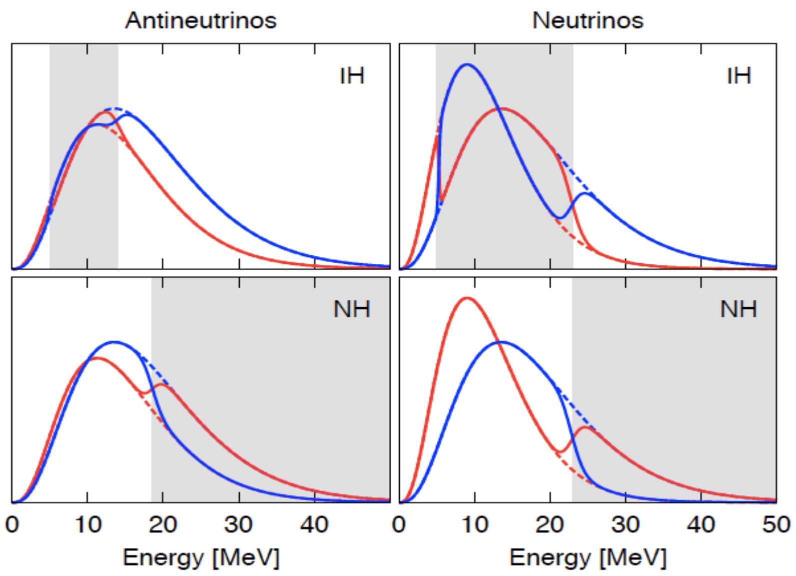
Bipolar/Fast Instability

Instability grows at rate sqrt(ωμ) (slow) or μ (fast)

Pioneering work by Pantaleone, Kostelecky, Samuel in the 90s
+ a few hundred papers
See Reviews of Mod Phys by Volpe for a good list of references

Spectral Swaps due to Slow Effects





Portions of the energy spectra get exchanged

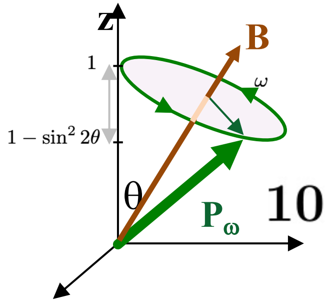
Initially thought to occur for Inverted ordering only

Later realized that this occurs for both orderings and there can be multiple spectral splits

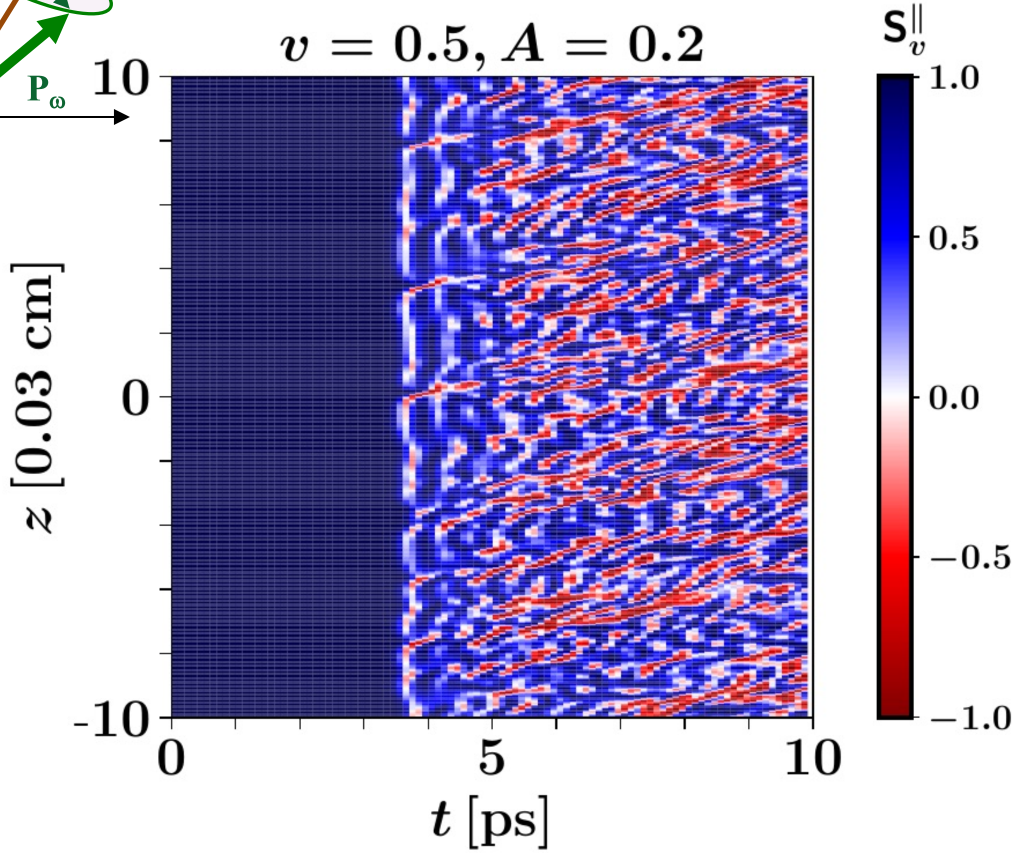
Seminal papers by Duan, Fuller, Carlson, Qian (2005, 2006, 2007)
Raffelt and Smirnov (PRD 2007, PRD 2007)

Dasgupta, Dighe, Raffelt, Smirnov (PRL, 2009)

Fast Depolarization



Numerical solution



Bhattacharyya and Dasgupta (PRD, 2020 ; PRL 2021)

Analytical Understanding via Coarse-graining

$$\partial_t \langle M_n \rangle = \frac{\langle M_1 \rangle}{2} \left(\partial_n^2 \langle M_n \rangle + \frac{1}{n} \partial_n \langle M_n \rangle \right)$$

A diffusion of the "difference of flavors" to higher multipoles of emission angle (i.e., momentum)

Several other groups have since obtained similar results

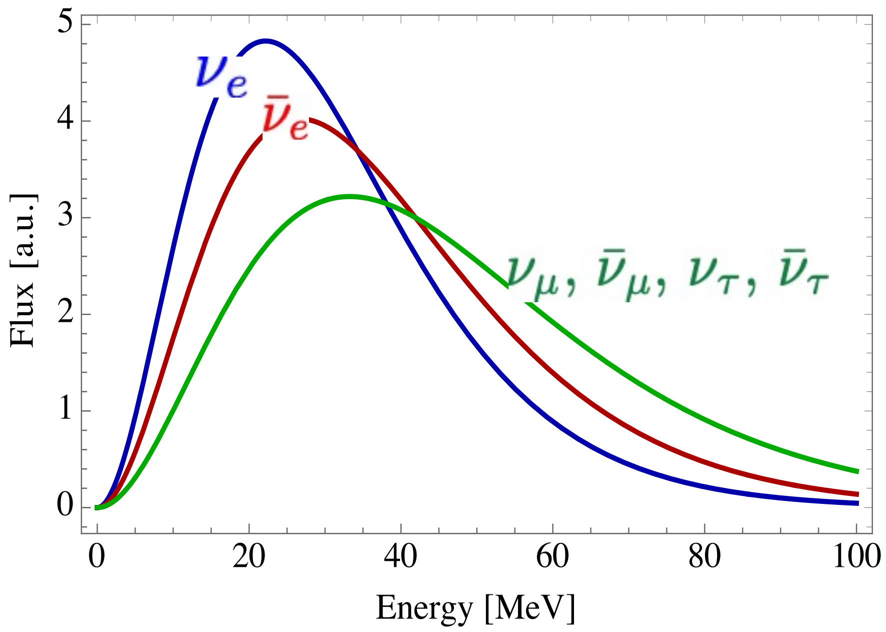
- Richers et al. @ Berkeley
- Wu et al. @ Taiwan
- Sigl @ Hamburg

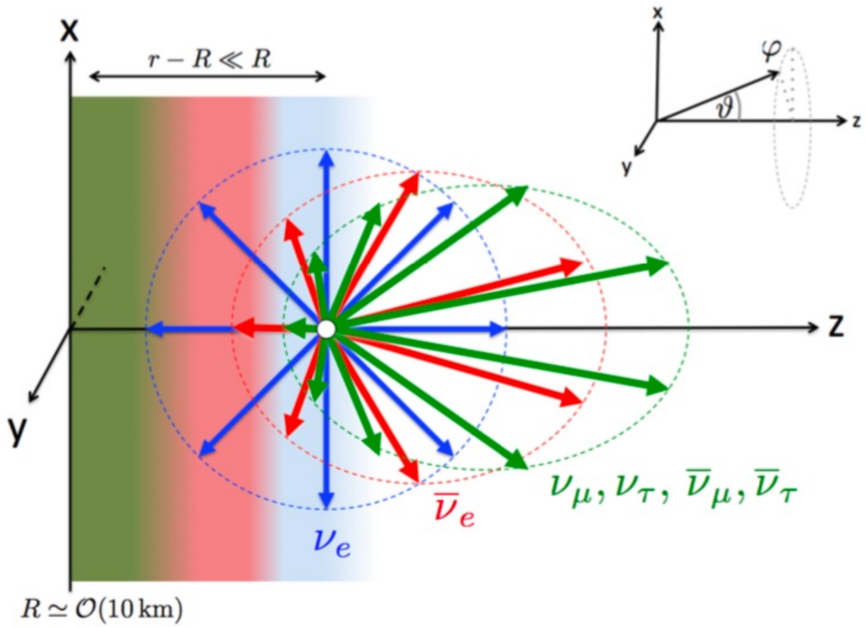
Survival Probability starts at 1

Oscillates coherently a few times

And then decoheres

Crossing: The Importance of Egalite





Different flavors have different energy spectrum

Crossing leads to (slow) instability

Different flavors have different angular distribution

Crossing leads to (fast) instability

Frontiers in Theoretical Neutrino Physics
APC Paris, 16-19 March 2009

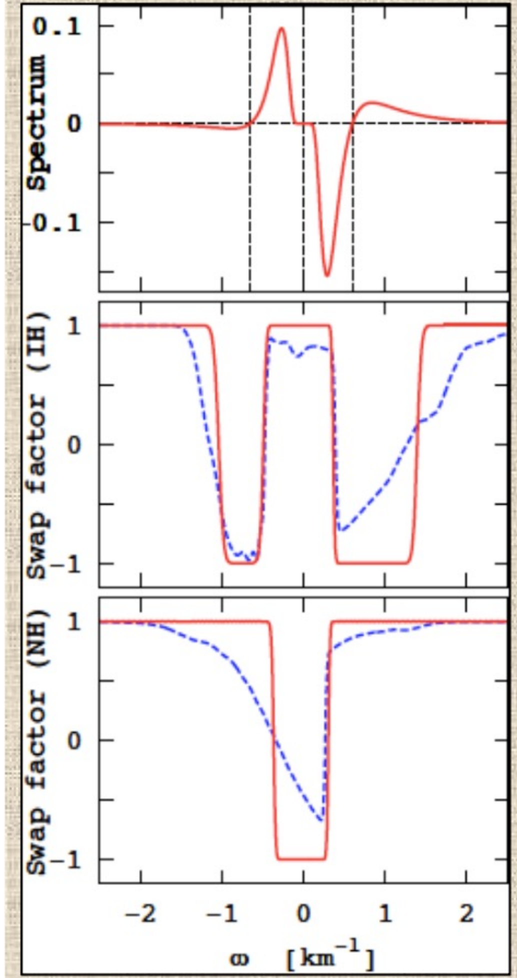
Multiple Spectral Splits of Supernova Neutrinos

Based on arXiv:0903.xxxx (to appear soon)

Basudeb Dasgupta and Georg Raffelt
Max Planck Institute for Physics, Munich

We now recognize the importance of spectral crossings as the main feature in initial conditions that dictates whether there is an instability

Our SN example again



- **Swaps around every "± crossing"**
- **Each swap flanked by two splits**
- **Splits not always washed out completely by multi-angle effects**
- **We have answered the questions…**
  - **Why are there swaps around a crossing?**
  - **Why the ± for IH/NH?**
  - **What is the width of the swap?**

Basudeb Dasgupta, Frontiers of Theoretical Neutrino Physics at APC Paris, 17 March 2009.

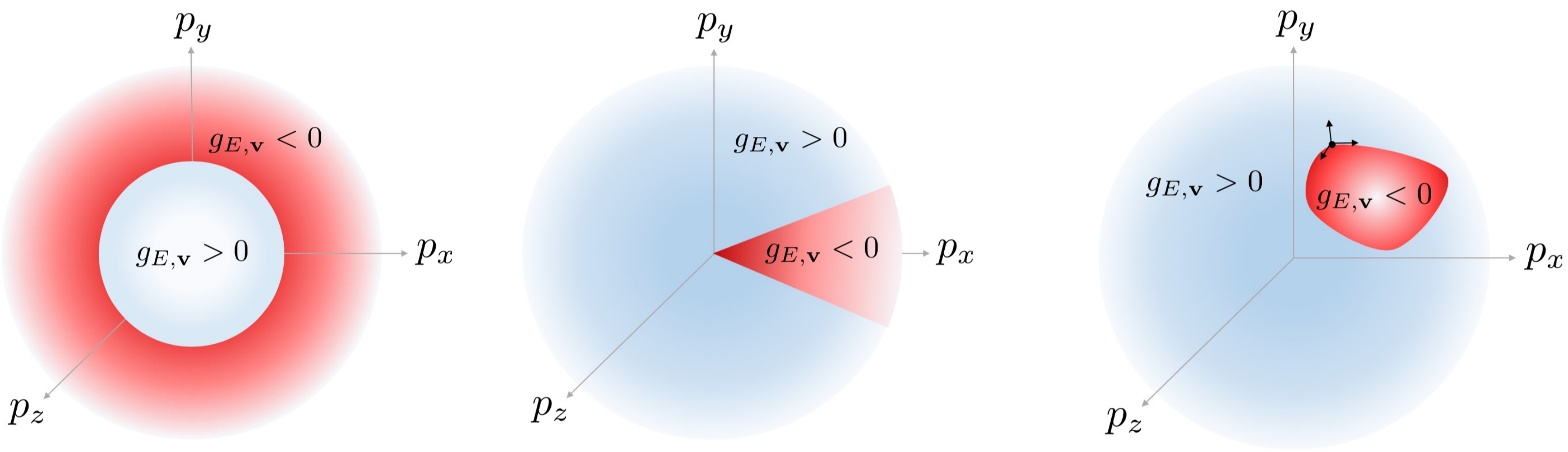
Thm: Crossings are Necessary

Collective instability occurs *only if* momentum distributions of any two flavors *cross each other* around some momentum

This is related to the positive-definiteness of a matrix

Dasgupta (PRL 2022)
Morinaga (PRD, 2022)

$$g_\Gamma = \sqrt{2}G_\mathrm{F}\begin{cases} f_{\nu_e,\mathbf{p}} - f_{\nu_\mu,\mathbf{p}} & \text{for } E > 0, \\ f_{\bar\nu_\mu,\mathbf{p}} - f_{\bar\nu_e,\mathbf{p}} & \text{for } E < 0, \end{cases}$$



Sketch of Sufficiency Proof

$$\mathcal{D}(\omega, \mathbf{k}) \equiv 1 + \int d\boldsymbol{\Gamma} \frac{g_{\boldsymbol{\Gamma}}}{\omega - \mathbf{k}\cdot\mathbf{v} - w_E} = 0$$

A. Proposition

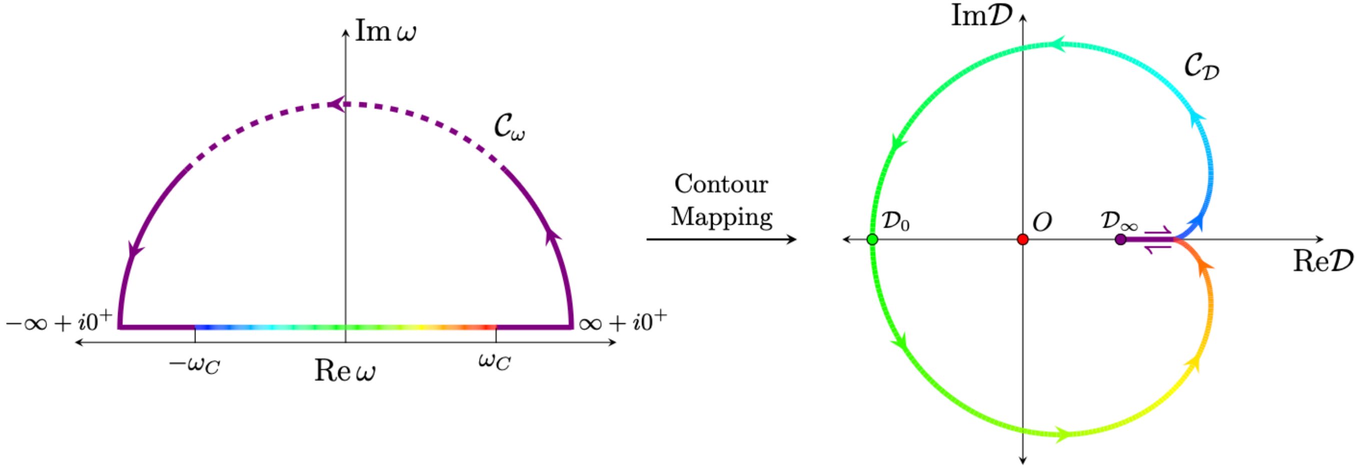
The dispersion relation $\mathcal{D}(\omega, \mathbf{k}) = 0$, admits a complex roo ω with $\mathrm{Im}\,\omega > 0$ at some wave-vector $\mathbf{k}$, *if*

1a: the distribution function $g_{\boldsymbol{\Gamma}}$ has a zero-crossing at some point $\boldsymbol{\Gamma}_0 = (w_{E_0}, \mathbf{v}_0)$, i.e., $g(w_{E_0}, \mathbf{v}_0) = 0$ and $g_{\boldsymbol{\Gamma}}$ takes both signs in the neighborhood,

1b: where the gradients of $g_{\boldsymbol{\Gamma}}$ and $h_{\boldsymbol{\Gamma}} = \omega - \mathbf{k}\cdot\mathbf{v}$ have a positive dot-product, i.e., $(\nabla_{\boldsymbol{\Gamma}} g_{\boldsymbol{\Gamma}})_0 \cdot (\nabla_{\boldsymbol{\Gamma}} h_{\boldsymbol{\Gamma}})_0 > 0$, and

2a: the principal value $\mathcal{I}_{\mathrm{PV}}(\omega_0) < -1$ at a frequency $\omega_0 = h_{\boldsymbol{\Gamma}_0}$ corresponding to the crossing at $\boldsymbol{\Gamma}_0$,

2b: while $\mathcal{I}_{\mathrm{PV}}(\omega_i) > -1$ for frequencies $\omega_i = h_{\boldsymbol{\Gamma}_i}$ corresponding to any other crossings $\boldsymbol{\Gamma}_{i\neq 0}$.



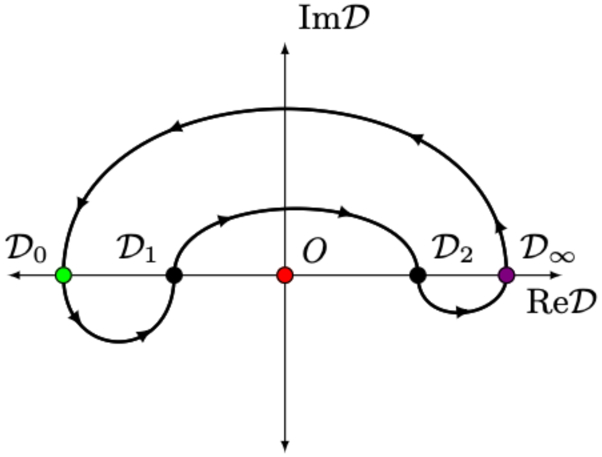
Imagine taking w on a loop around the upper-half-plane. The dispersion D traces a loop. One can show that the D-loop winds around the origin if the conditions are true. But nonzero winding ensures roots of D in upper-half-plane.

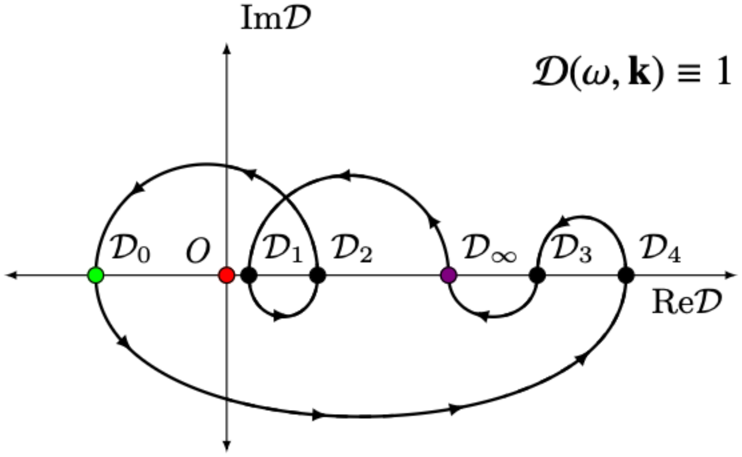
Dasgupta and Mukherjee (2505.03886)

These are essentially equivalent to the Nyquist / Penrose Stability Conditions

Why we need these conditions

- The condition 1b (gradient positivity) ensures that the D contour goes from above-to-below (at the leftmost crossing)
- The condition 2a (PV<-1) ensures that this point is to the left of origin O.
- The condition 2b (PV'>-1) ensures that other such points are to right of O.





$$\mathcal{D}(\omega, \mathbf{k}) \equiv 1 + \int d\Gamma \frac{g_\Gamma}{\omega - \mathbf{k}\cdot\mathbf{v} - w_E} = 0$$

- Given these, the winding around origin is guaranteed.

Fast Case: Pushing the Crossings

- The Principal Values can be massaged to a great extent by choosing the size of the wavevector k

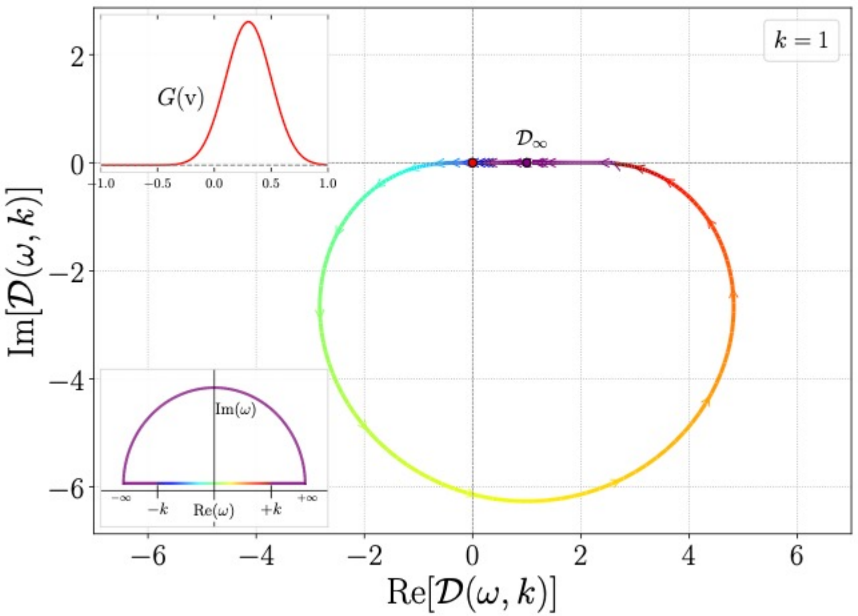
- Given $\mathcal{D}(\omega, \mathbf{k}) \equiv 1 + \int d\Gamma \frac{g_\Gamma}{\omega - \mathbf{k}\cdot\mathbf{v} - \omega_E} = 0$ one can choose $\mathbf{k} \to \lambda\mathbf{k}$,

  one has $\mathcal{I}_{\rm PV}(\omega_i) \to \frac{1}{\lambda}\mathcal{I}_{\rm PV}(\omega_i)$ for any $\omega_i = \mathbf{k}\cdot\mathbf{v}_i$

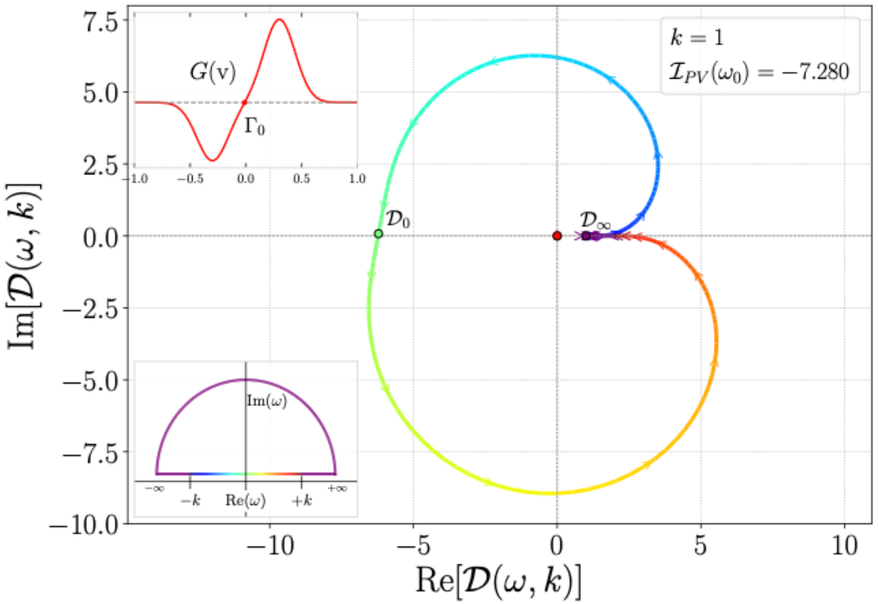
- So, for fast oscillations, one simply need PV<0 for one of the crossings. And this can be identically satisfied for $\max(0, -\kappa_1) \equiv \lambda_{\min} < \lambda < \lambda_{\max} \equiv -\min(0, \kappa_0)$

- Slope doesn't matter because the crossing points of g form closed curves and can approached from either side.

Example: Fast Oscillations

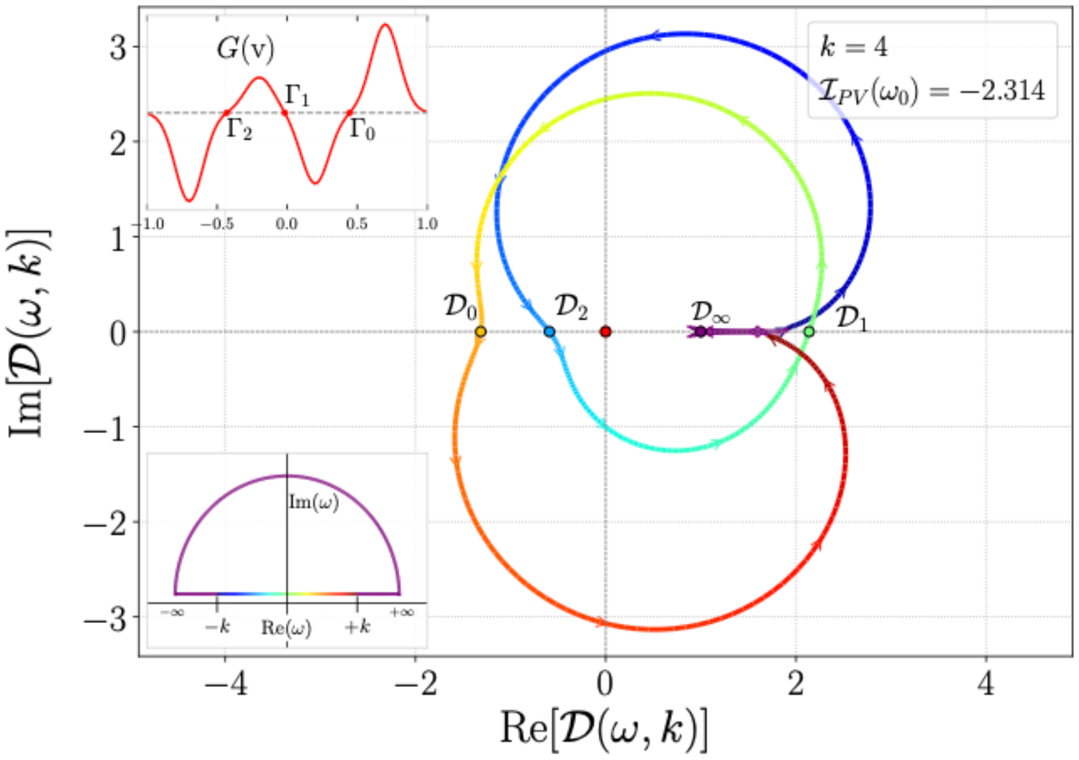


No Crossing = No Winding

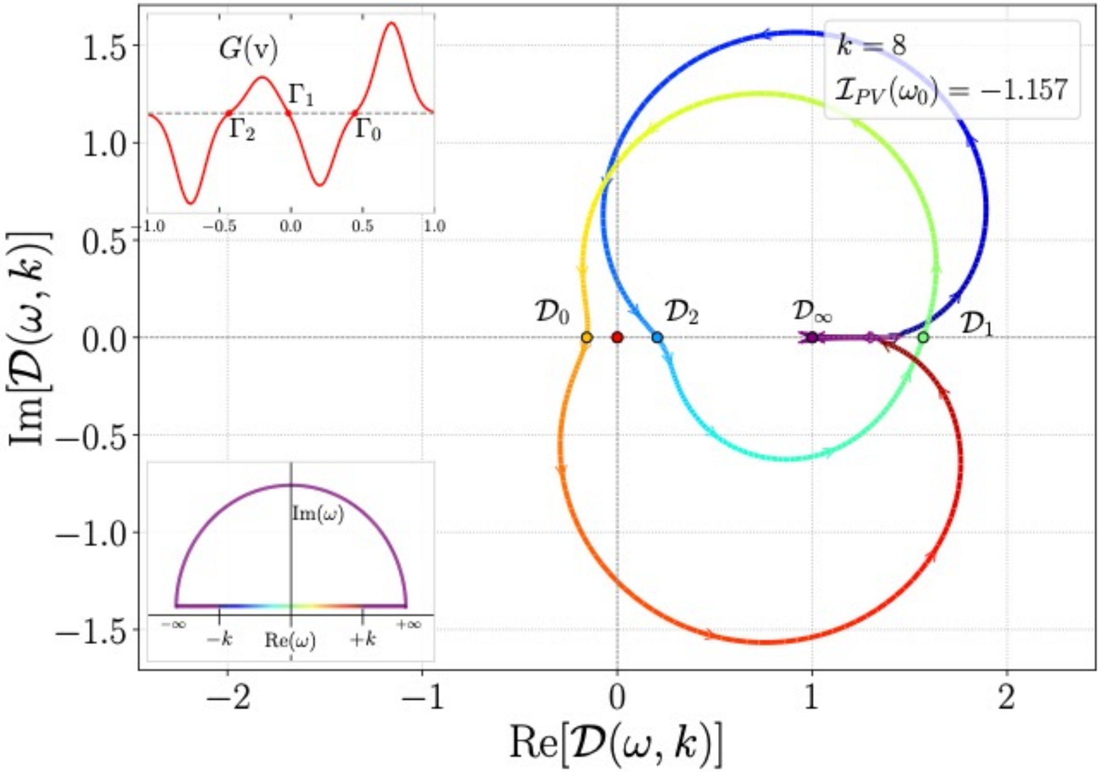


Crossing = Winding

Scaling of k



Multiple D-Crossings on Left of Origin



One D-Crossings on Left of Origin

Slow Case: Pushing the Crossings

- The Principal Values can be massaged to a lesser extent by choosing the size of the wavevector k

- Given $\mathcal{D}(\omega, \mathbf{k}) \equiv 1 + \int d\Gamma \frac{g_\Gamma}{\omega - \mathbf{k}\cdot\mathbf{v} - w_E} = 0$ one can choose $\mathbf{k} \to \lambda\mathbf{k}$,

  one does not have $\mathcal{I}_{\rm PV}(\omega_i) \to \frac{1}{\lambda}\mathcal{I}_{\rm PV}(\omega_i)$ but $\mathcal{I}_{\rm PV} \to \frac{1}{\lambda}\mathcal{I}_{\rm PV}\left[1 + \sum_{n=1}^{\infty}\frac{\langle \widetilde{w_E}^{(n)}\rangle_<}{\lambda^n} + \sum_{n=1}^{\infty}\frac{\langle \widetilde{w_E}^{(-n)}\rangle_>}{\lambda^{-n}}\right]$

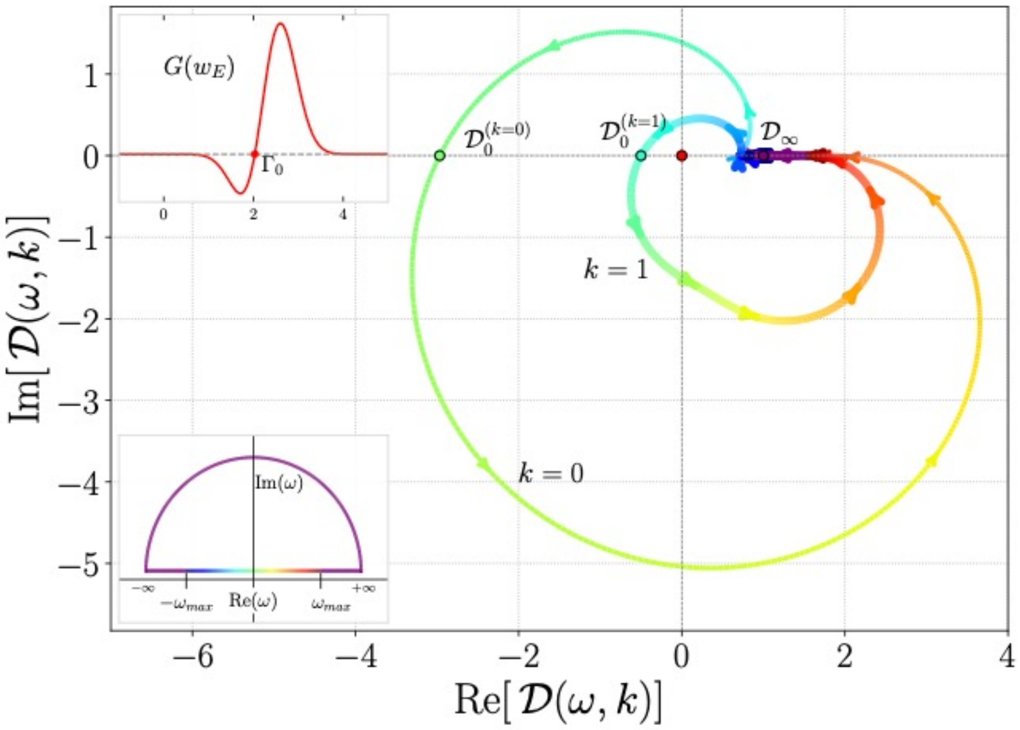
- So, for slow oscillations, one needs "consistency conditions" 2a and 2b.

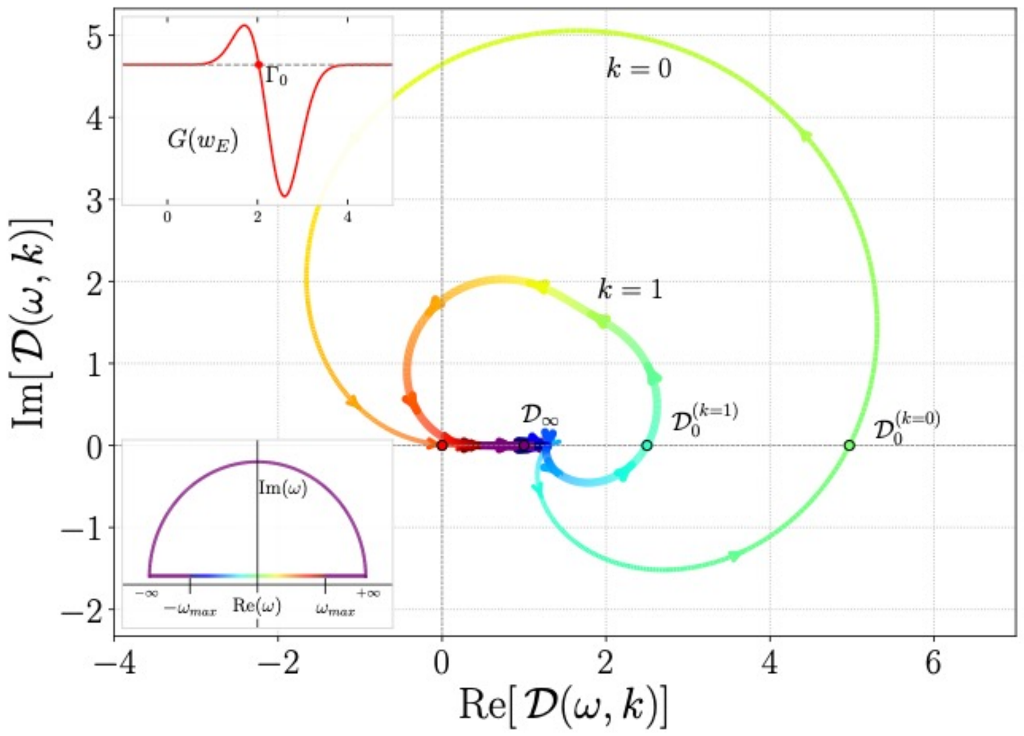
- Of course, if w_E is small, then an approximate version of the scaling works.

$$\max\left(|\langle w_E\rangle|, \lambda_{\rm min}\right) \lesssim \lambda \lesssim \min\left(|\langle w_E\rangle^{-1}|, \lambda_{\rm max}\right)$$

- But slope of g(w) must be positive [for IH, time-like ev] (condition 1b).

Example: Slow

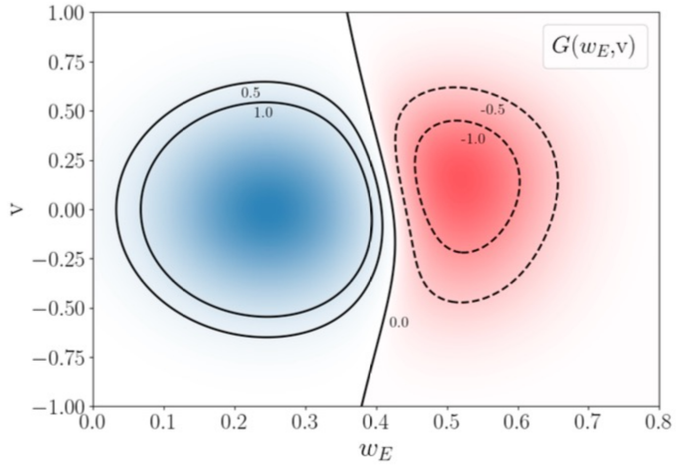


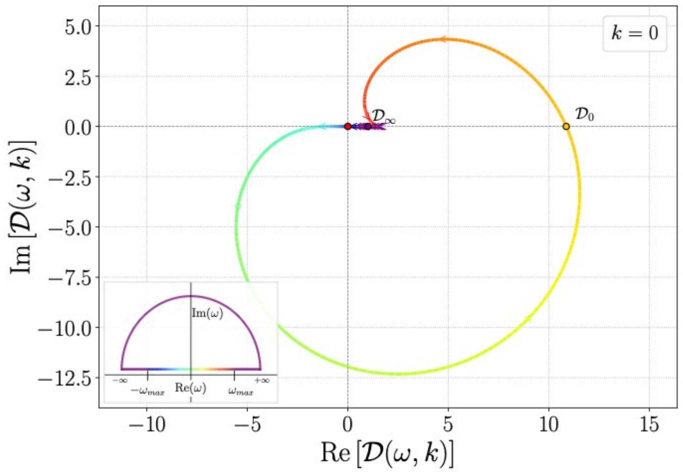


Positive Crossing = Winding

Negative Crossing = No Winding

Example: Mixed Crossing





- Consider a g that depends on both w and v, and has nontrivial crossings.
- In general, one needs to satisfy Condition 1b

$$\left(\mathbf{k}-(\mathbf{k}\cdot\mathbf{v}_0)\mathbf{v}_0\right)\cdot\left(\nabla_{\mathbf{v}} g_{\Gamma}\right)_0 > -w_{E_0}^{2}\left(\frac{\partial g_{\Gamma}}{\partial w_E}\right)_0$$

together with an acceptable k that allows Conditions 2a,b to be satisfied via scaling.

- The left example has dg/dw <0 and thus there can't be any instability (for k=0).
- If dg/dw > 0, k can be small and the only restriction comes from Conditions 2a,b

Big Picture

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- Astrophysical Importance
  - In SN and NS mergers, neutrino flavor conversions can be very important: for overall energetics, for dynamics (shock revival), for nucleosynthesis, etc.
- Oscillation theory
  - Beyond the usual MSW picture, but completely Standard Physics
  - A diversity of physical effects, special cases, etc.
- Crossing as a key feature
  - Starting from the 2009 paper on multiple splits, we understand that crossings are closely tied to the presence of instabilities
  - In 2022 it was shown by Morinaga that crossings are necessary and sufficient for fast instability. For slow and mixed it was shown to be necessary.
  - The sufficiency conditions for slow/mixed were not known. We now show that crossings are not sufficient. More detailed restrictions (on slope, PV) arise.
  - Stay tuned…

Neutrinos from a SN