Integrability in the Dynamics of Binary Black Holes

Carlos F. Sopuerta

Institute of Space Sciences
(ICE, CSIC & IEEC)
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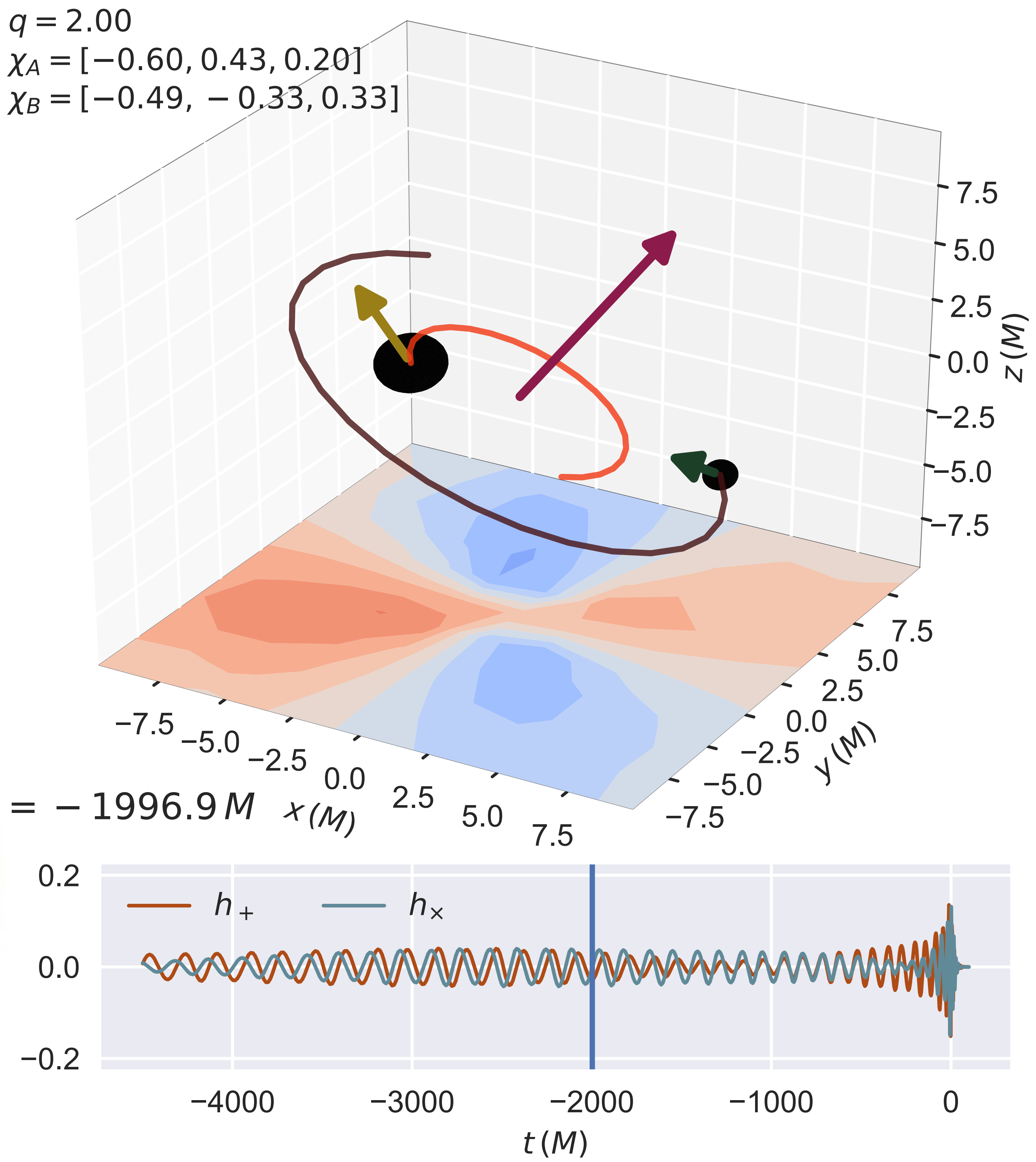

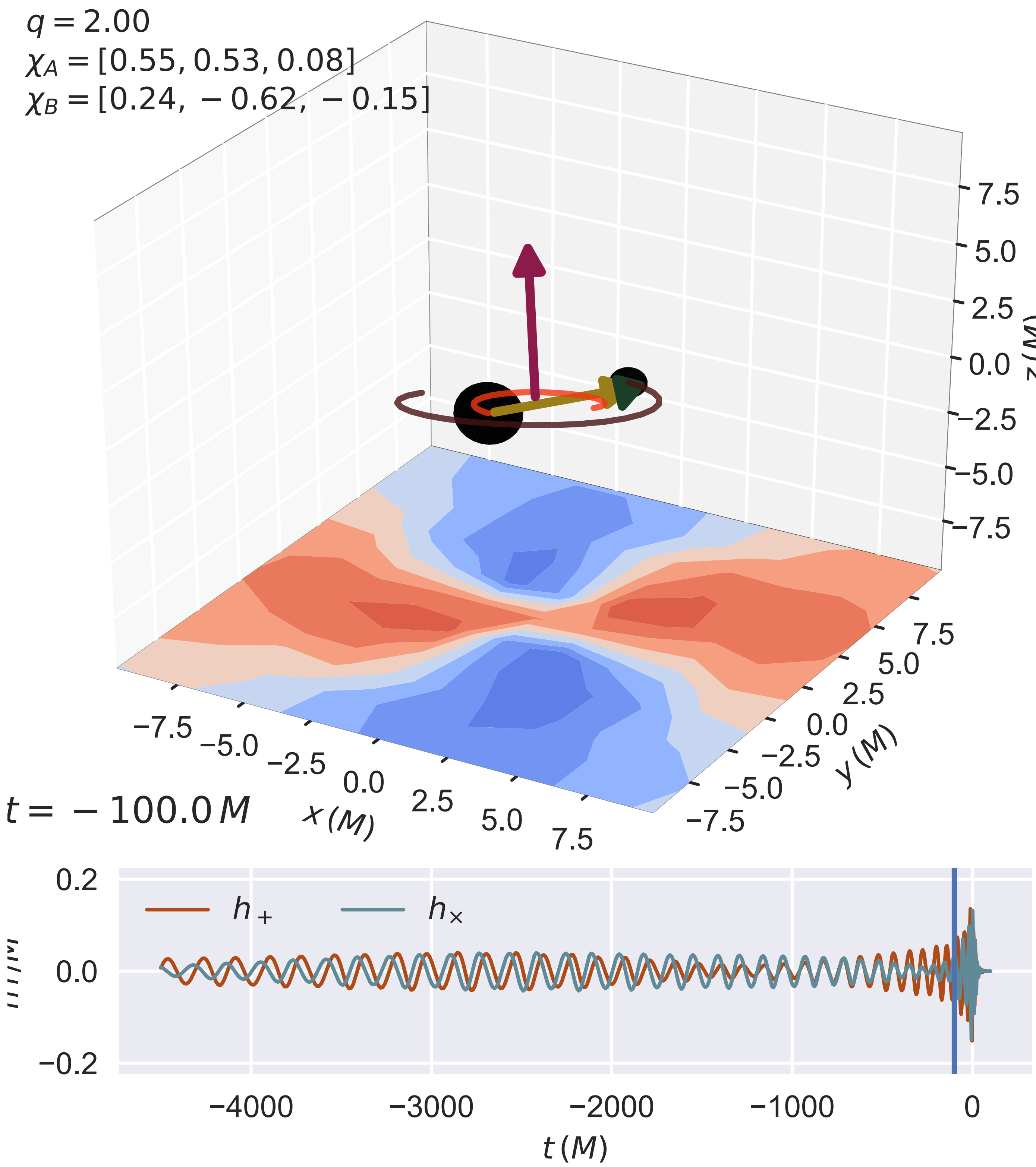

Varma, Stein & Gerosa
arXiv:1811.06552v3

Mathematical Physics of Gravity and Symmetry

Institut de Mathématiques de Bourgogne (IMB), Dijon (France)

Outline

1. Physical Motivation (Gravitational Wave Astronomy)

2. Black Holes

3. The Binary Black Hole Problem (GR two-body problem)

4. A Hierarchical Approach to the Binary Black Hole Problem

5. Remarks and Conclusions

Universality in Binary Black Hole Dynamics: An Integrability Conjecture

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[1]Institut de Mathématiques de Bourgogne (IMB), UMR 5584, CNRS, Université de Bourgogne, F-21000 Dijon, France

[2]Institute for Mathematics, Astronomy and Particle Physics Radboud University, Heyendaalseweg 135, 6525 AJ Nijmegen, The Netherland

[3]Institute of Space Sciences (ICE-CSIC and IEEC), Campus UAB, Carrer de Can Magrans s/n, 08193 Cerdanyola del Vallès, Spain

Collaborators: JL Jaramillo, M Lenzi, B Krishnan, C Vitel, U Sperhake, P Laguna, N Yunes, F Fayos

Honorable Mention Awards for Essays on Gravitation 2023

30 March 2023

arXiv:2305.08554

Abstract

The waveform of a binary black hole coalescence appears to be both simple and universal. In this essay we argue that the dynamics should admit a separation into 'fast and slow' degrees of freedom, such that the latter are described by an integrable system of equations, accounting for the simplicity and universality of the waveform.

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An Integrability Conjecture

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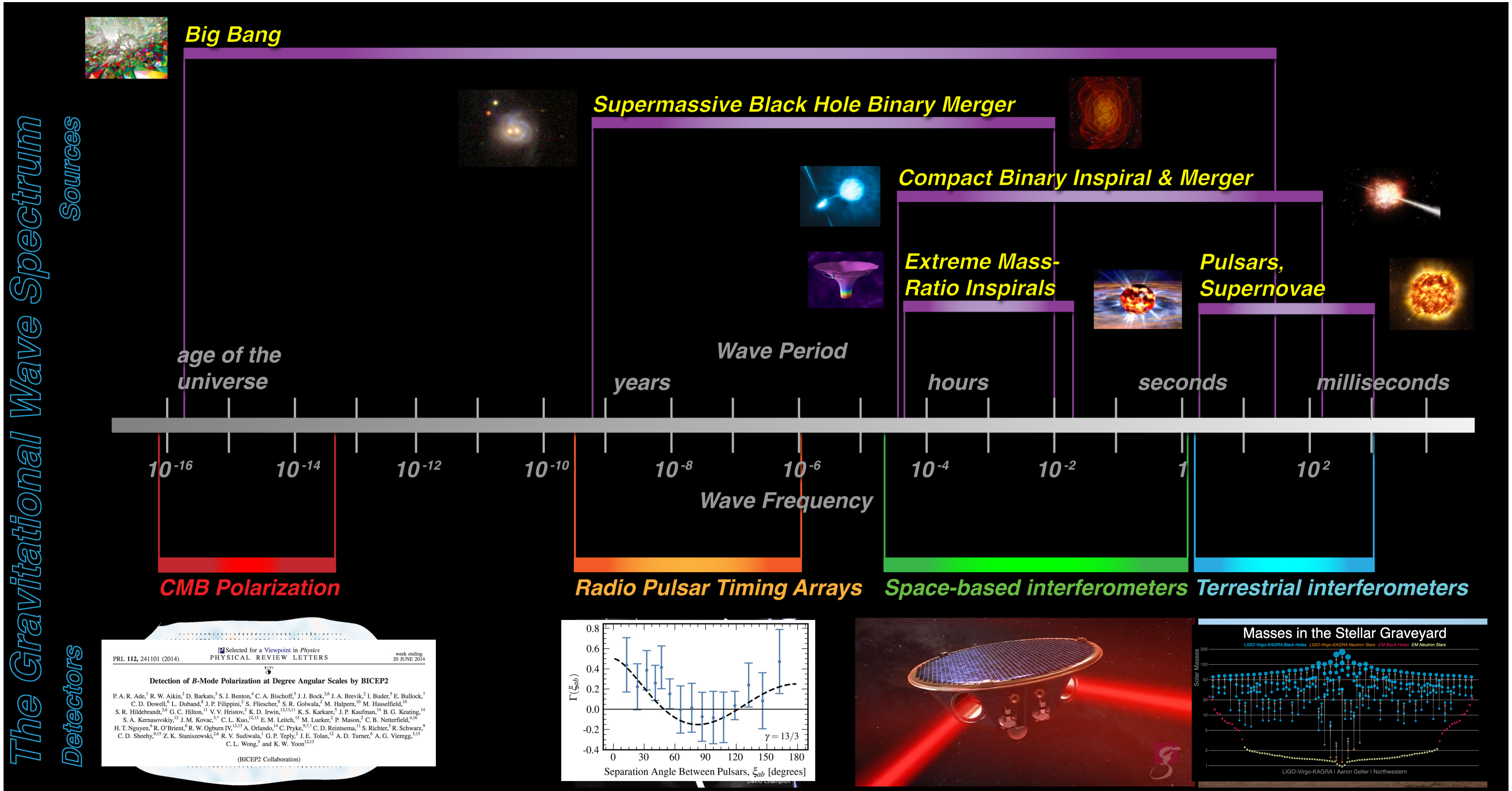
International Journal of Modern Physics D, 32, 2342005

Physical Motivation

Physical Motivation: The advent of Gravitational Wave Astronomy



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Credit: Ira Thorpe (NASA)

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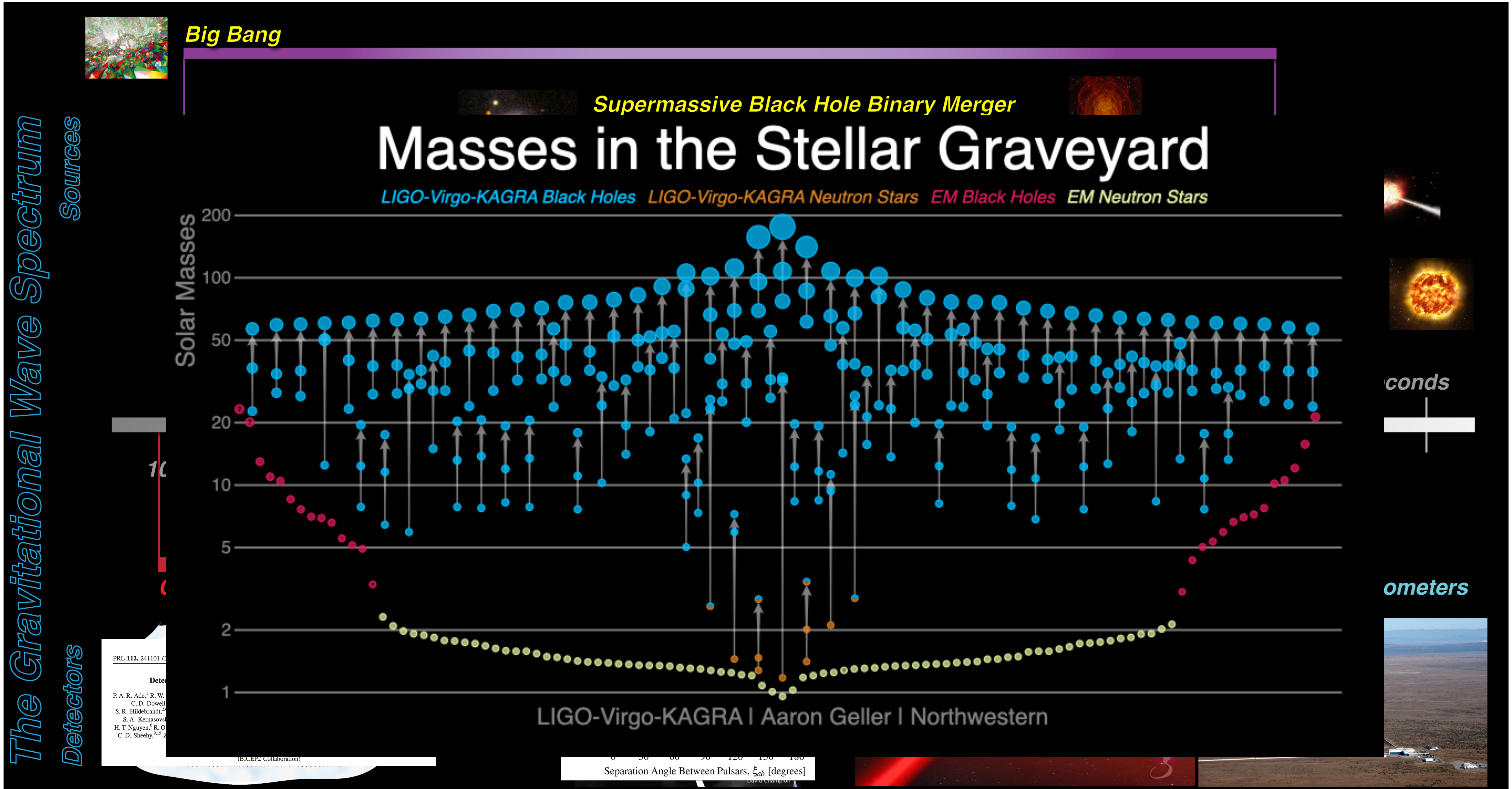

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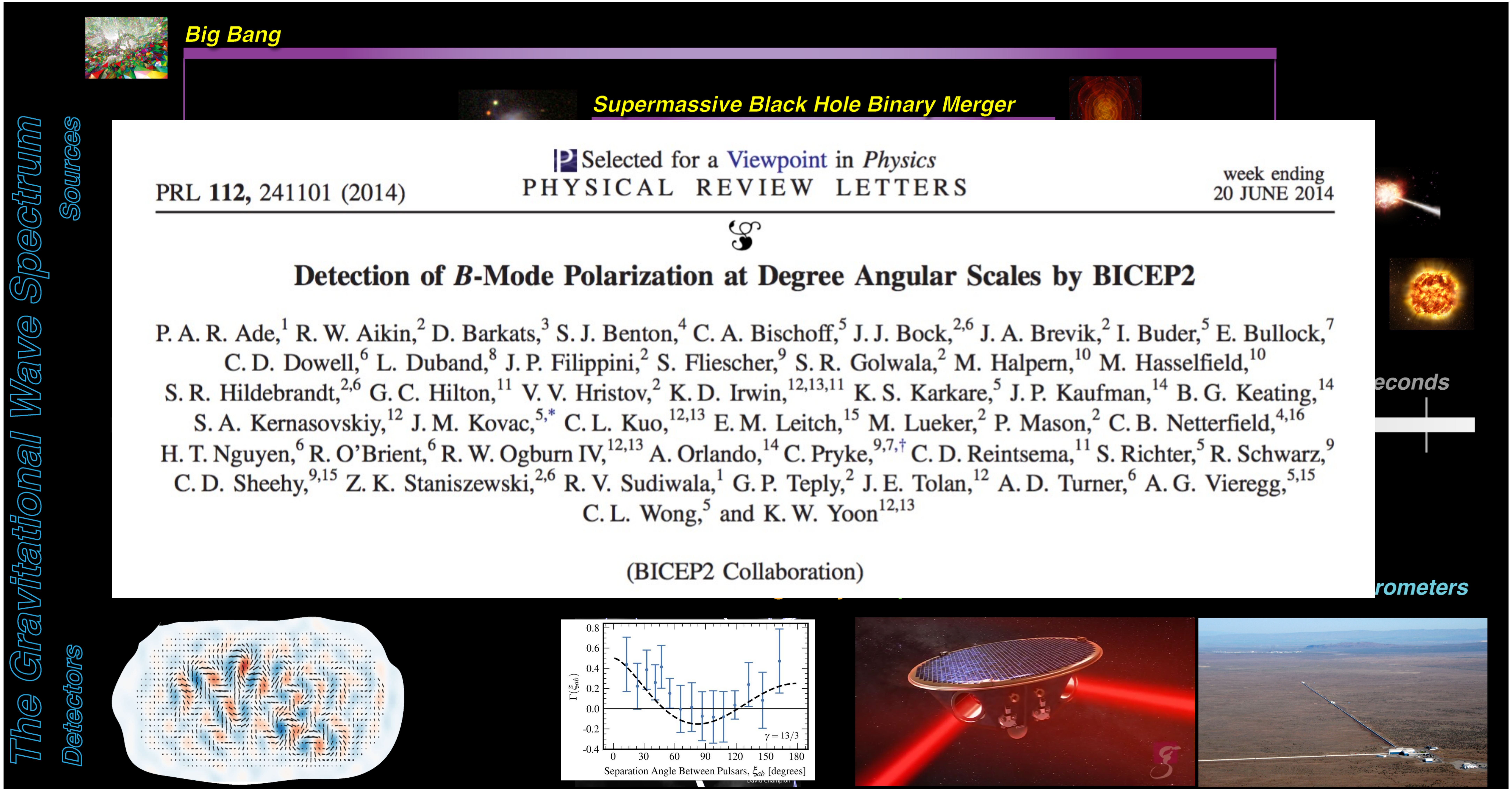

Selected for a Viewpoint in *Physics*

PRL **112,** 241101 (2014) PHYSICAL REVIEW LETTERS week ending 20 JUNE 2014

Detection of *B*-Mode Polarization at Degree Angular Scales by BICEP2

P. A. R. Ade,[1] R. W. Aikin,[2] D. Barkats,[3] S. J. Benton,[4] C. A. Bischoff,[5] J. J. Bock,[2,6] J. A. Brevik,[2] I. Buder,[5] E. Bullock,[7] C. D. Dowell,[6] L. Duband,[8] J. P. Filippini,[2] S. Fliescher,[9] S. R. Golwala,[2] M. Halpern,[10] M. Hasselfield,[10] S. R. Hildebrandt,[2,6] G. C. Hilton,[11] V. V. Hristov,[2] K. D. Irwin,[12,13,11] K. S. Karkare,[5] J. P. Kaufman,[14] B. G. Keating,[14] S. A. Kernasovskiy,[12] J. M. Kovac,[5,*] C. L. Kuo,[12,13] E. M. Leitch,[15] M. Lueker,[2] P. Mason,[2] C. B. Netterfield,[4,16] H. T. Nguyen,[6] R. O'Brient,[6] R. W. Ogburn IV,[12,13] A. Orlando,[14] C. Pryke,[9,7,†] C. D. Reintsema,[11] S. Richter,[5] R. Schwarz,[9] C. D. Sheehy,[9,15] Z. K. Staniszewski,[2,6] R. V. Sudiwala,[1] G. P. Teply,[2] J. E. Tolan,[12] A. D. Turner,[6] A. G. Vieregg,[5,15] C. L. Wong,[5] and K. W. Yoon[12,13]

(BICEP2 Collaboration)

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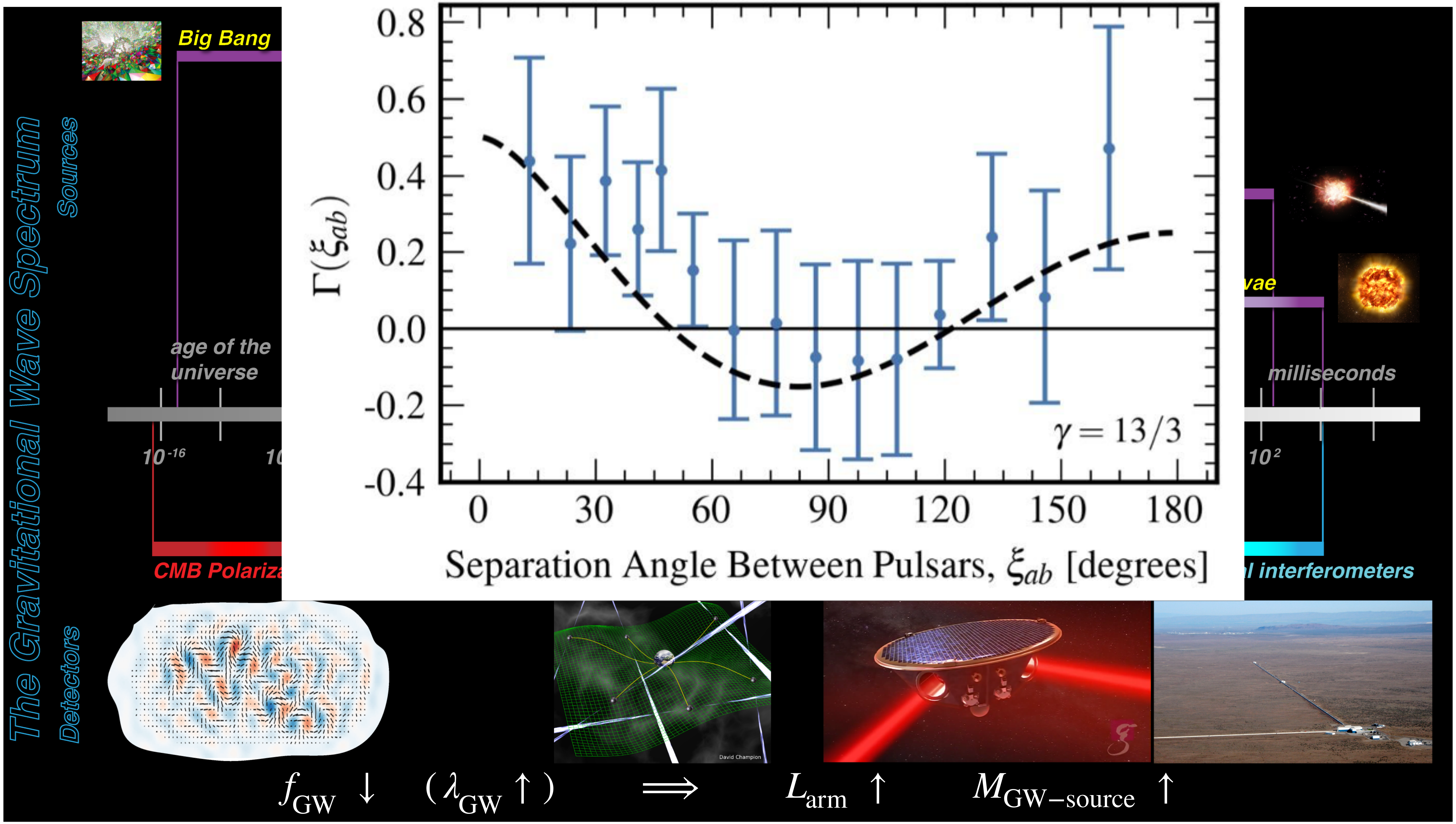

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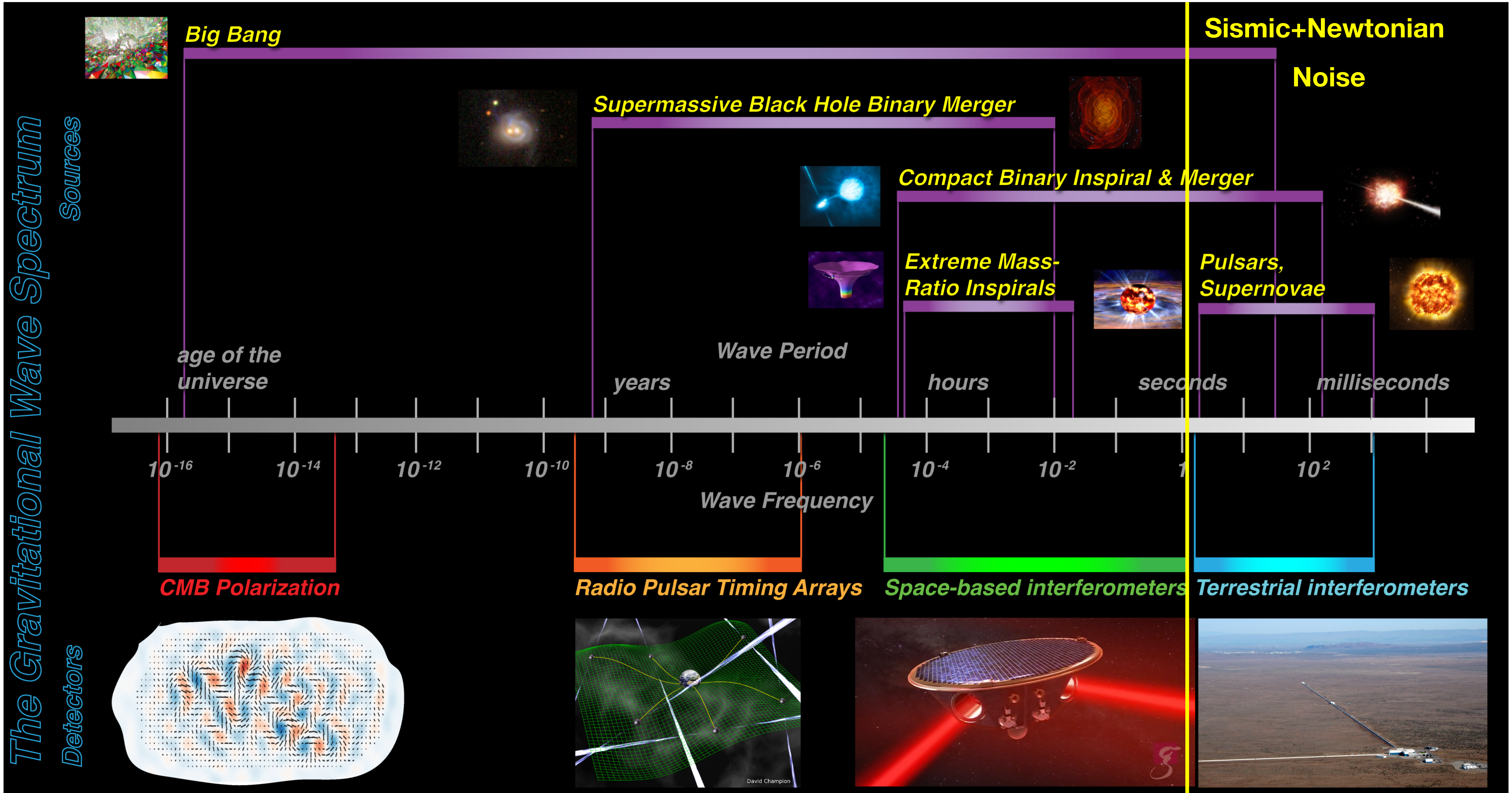

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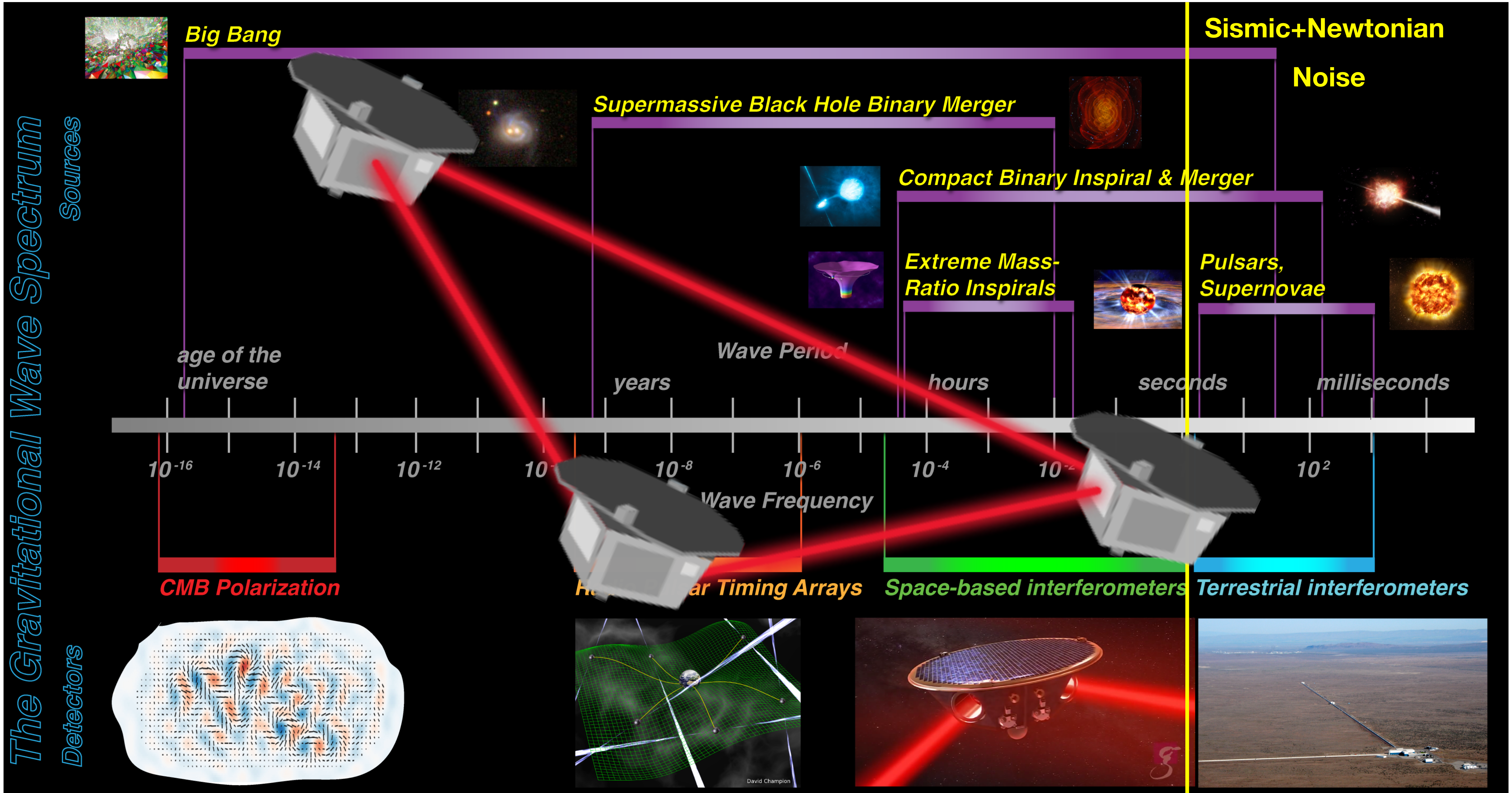

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Main GW Source Type: Compact Binary Coalescence

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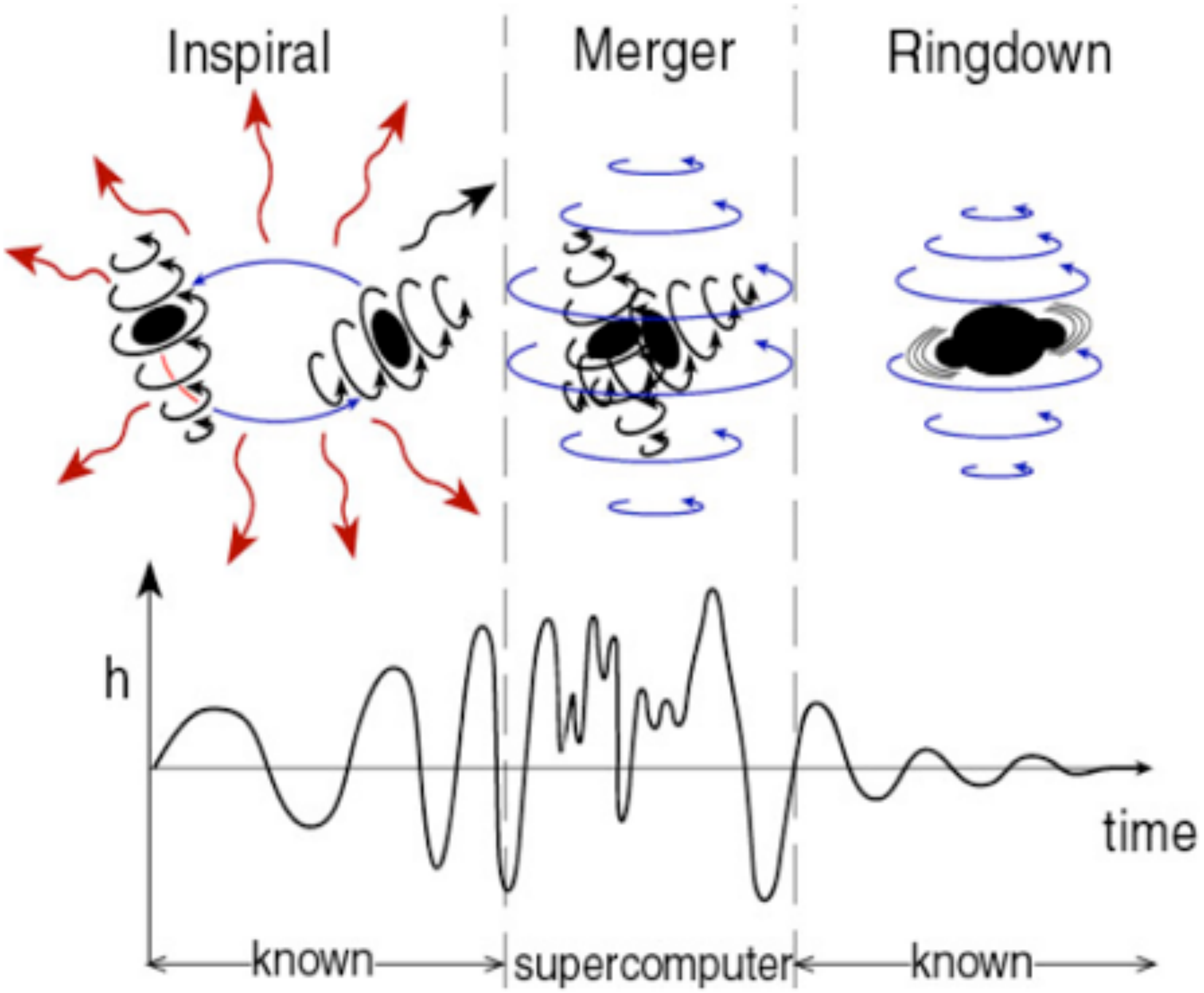
Inspiral, Merger, and Ringdown of Binary Black Holes

* When the evolution of a Black Hole Binaries are driven by gravitational-wave emission, one can distinguish three different stages: **Inspiral, Merger and Ringdown**.



From: Kip Thorne (Caltech)

Inspiral, Merger, and Ringdown of Binary Black Holes

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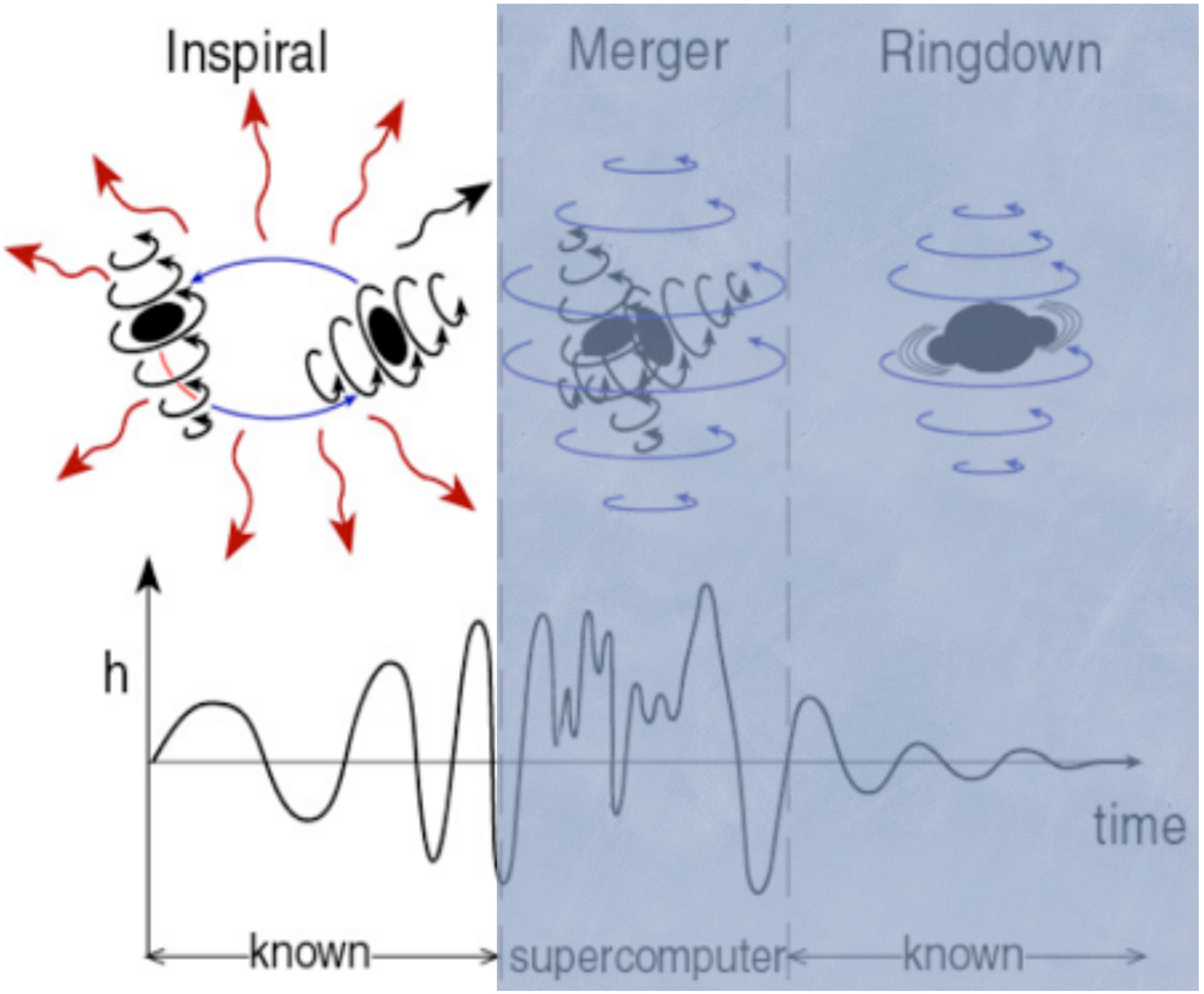
Post-Newtonian expansions are good enough to describe the slow inspiral. The Phase of the waves is known to order 3.5PN, that is, to order (v/c) ^7.



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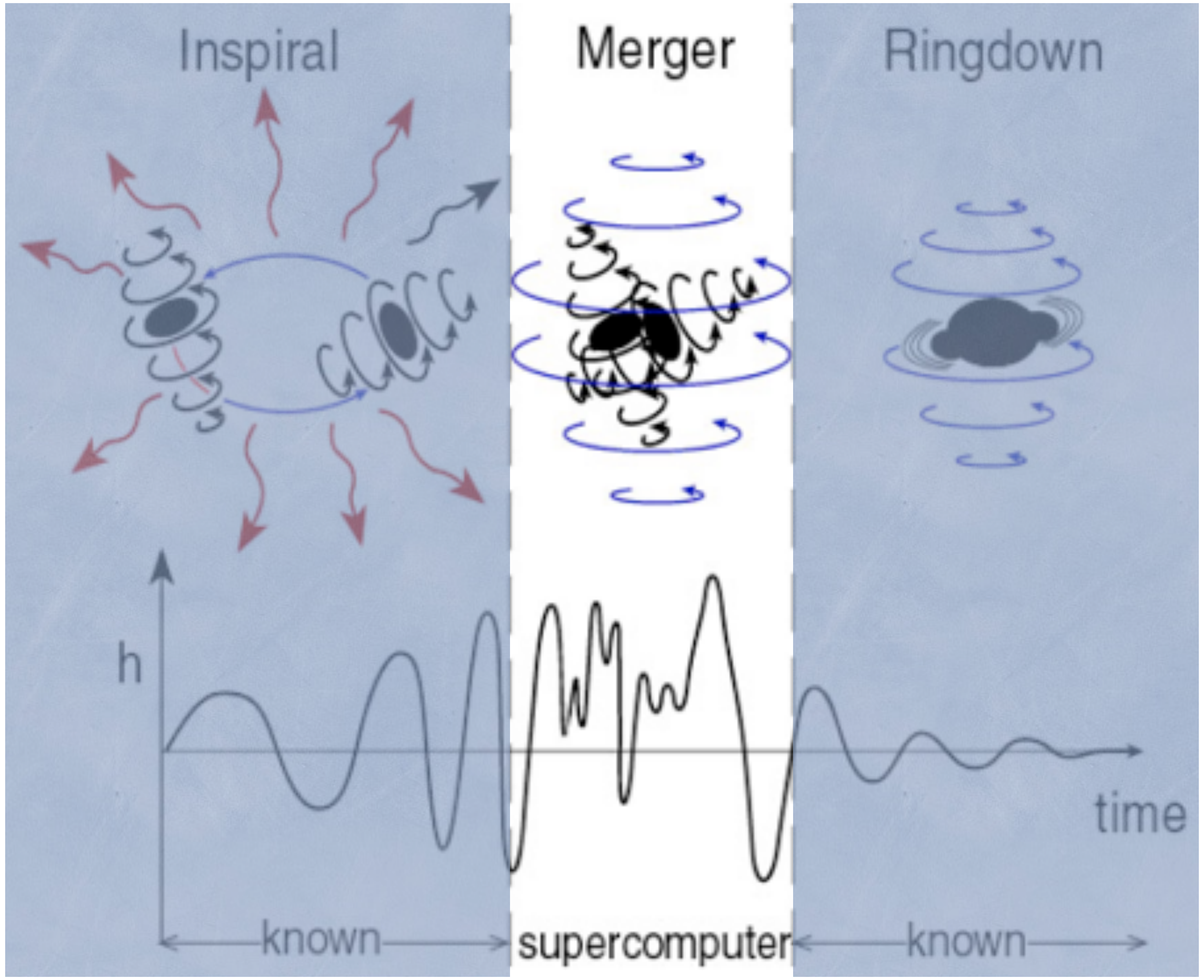
This stage has to be described using Numerical Relativity (Numerical solution of the full vacuum Einstein equations). Breakthrough in year 2005. [Pretorius, PRL, 95,121101 (2005)]



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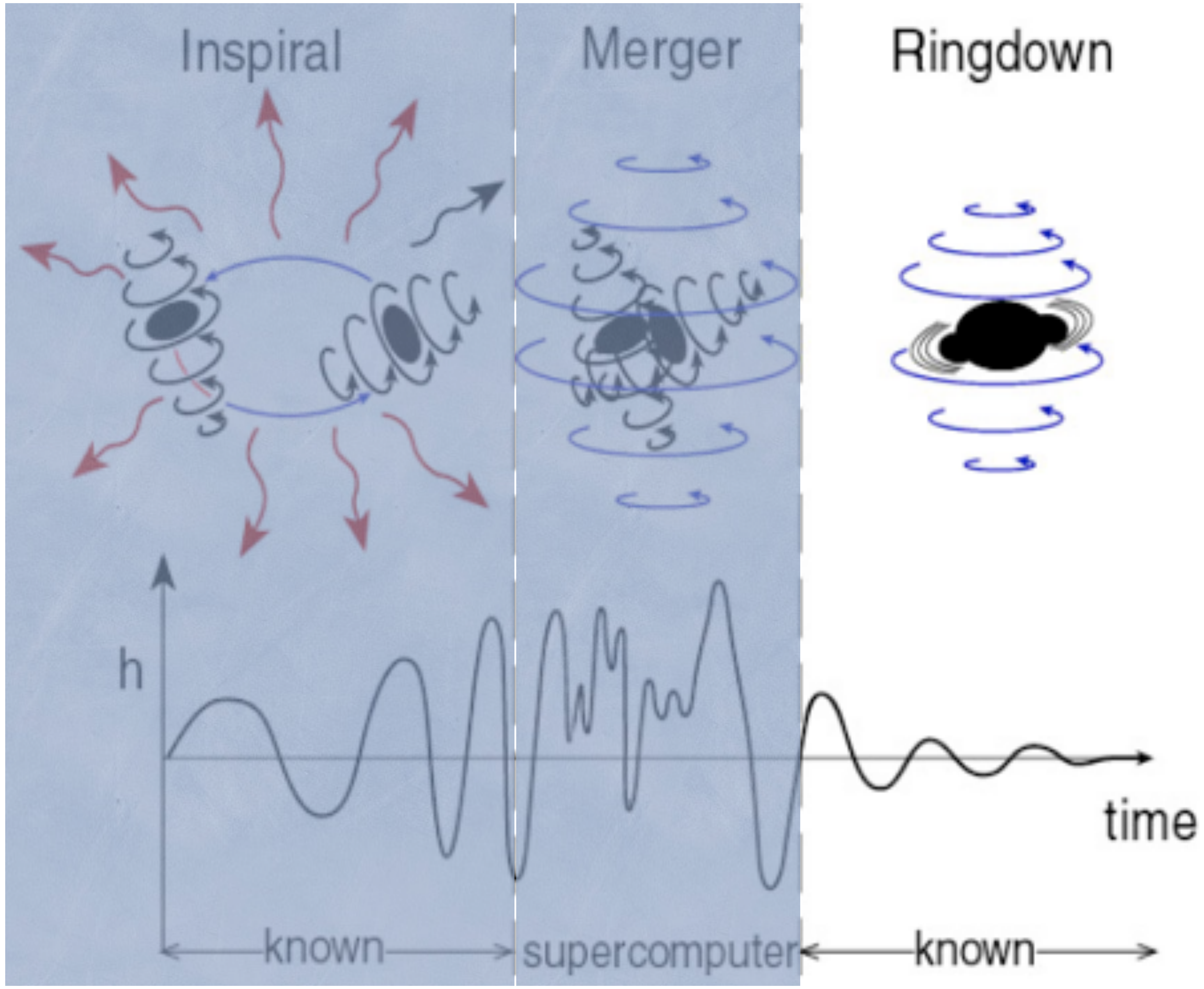
The system here resembles a perturbed single Black Hole. The evolution can be followed using BH perturbation theory (evolution of damped sinusoids, i.e. Quasi-normal modes).



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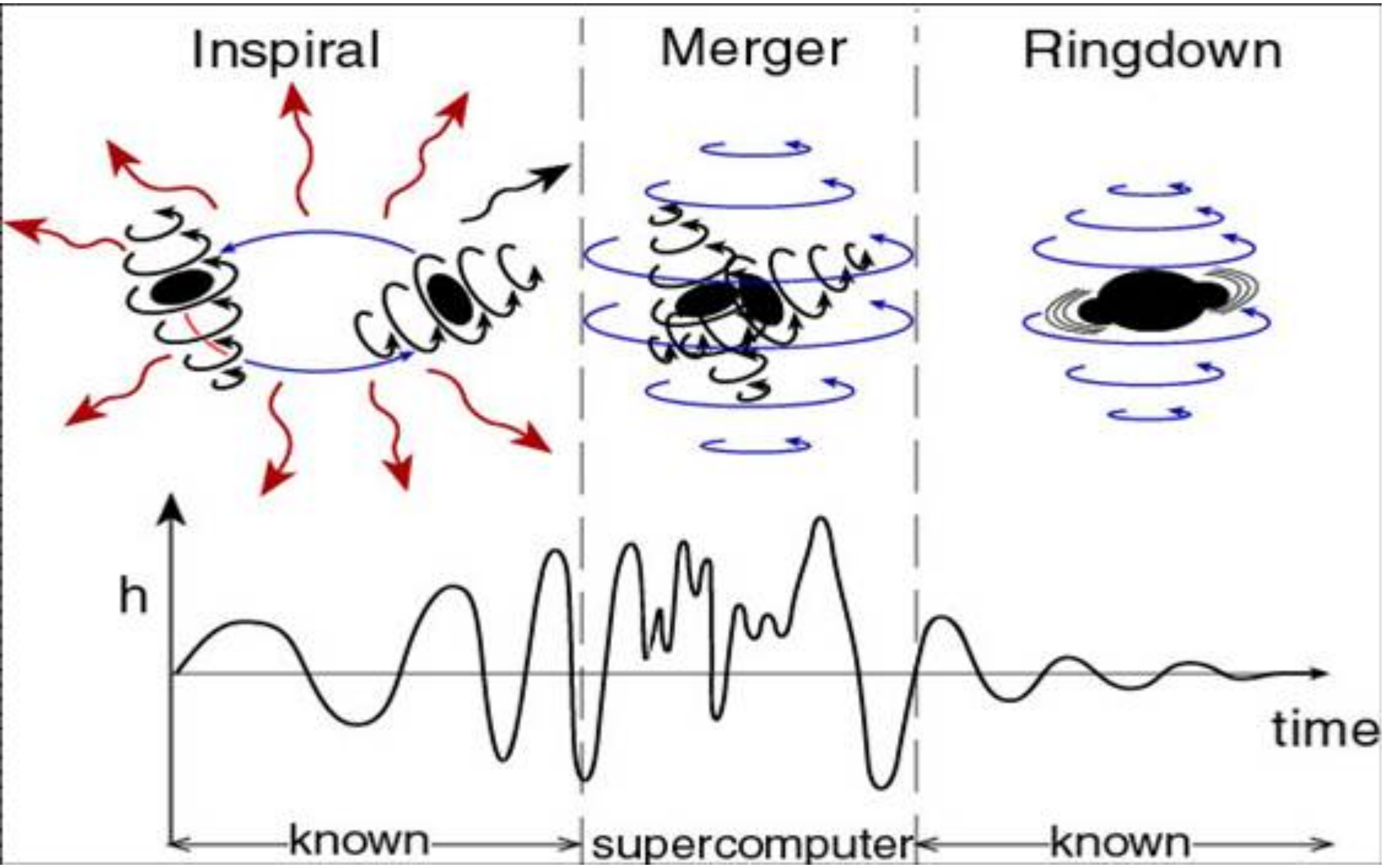
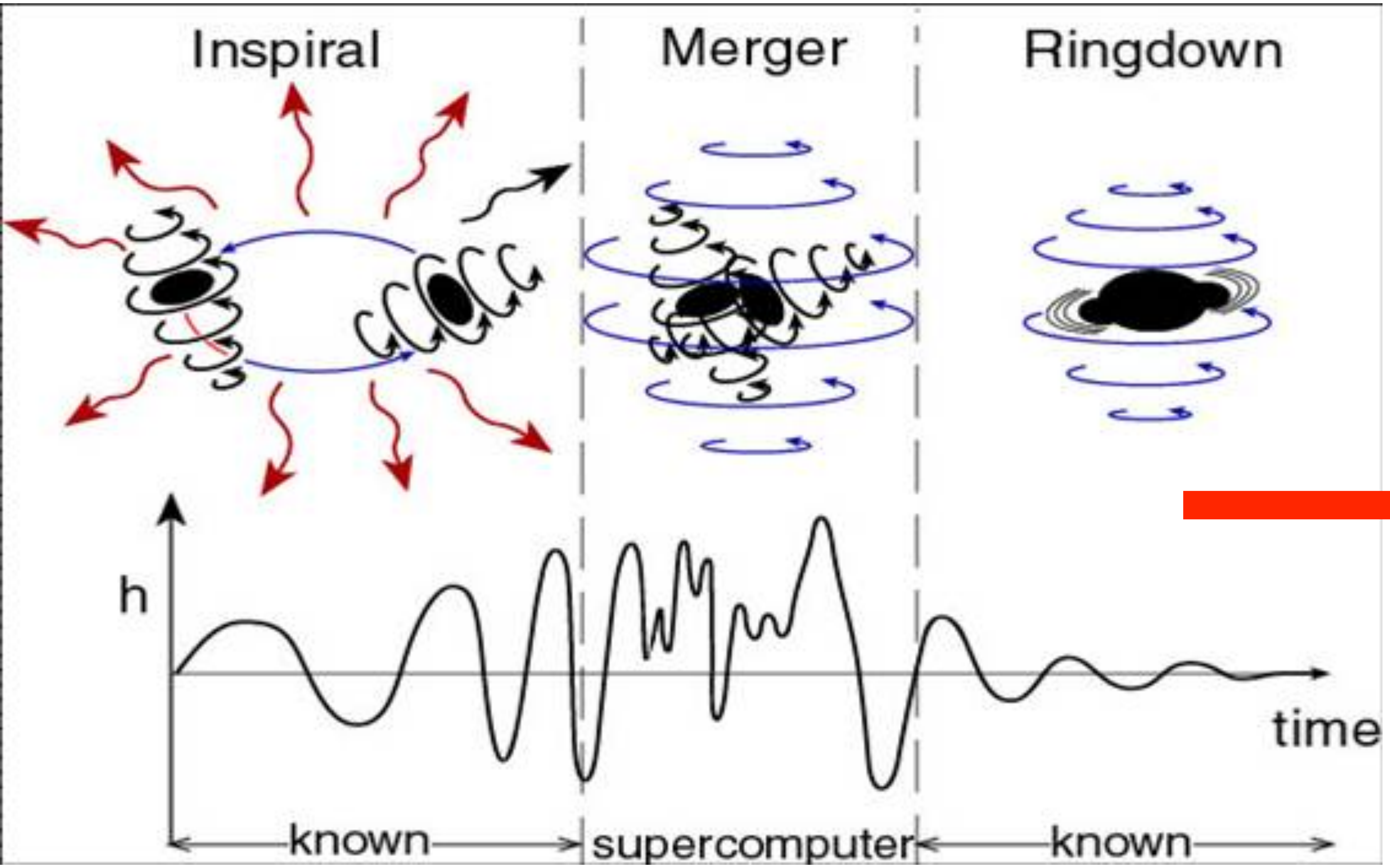
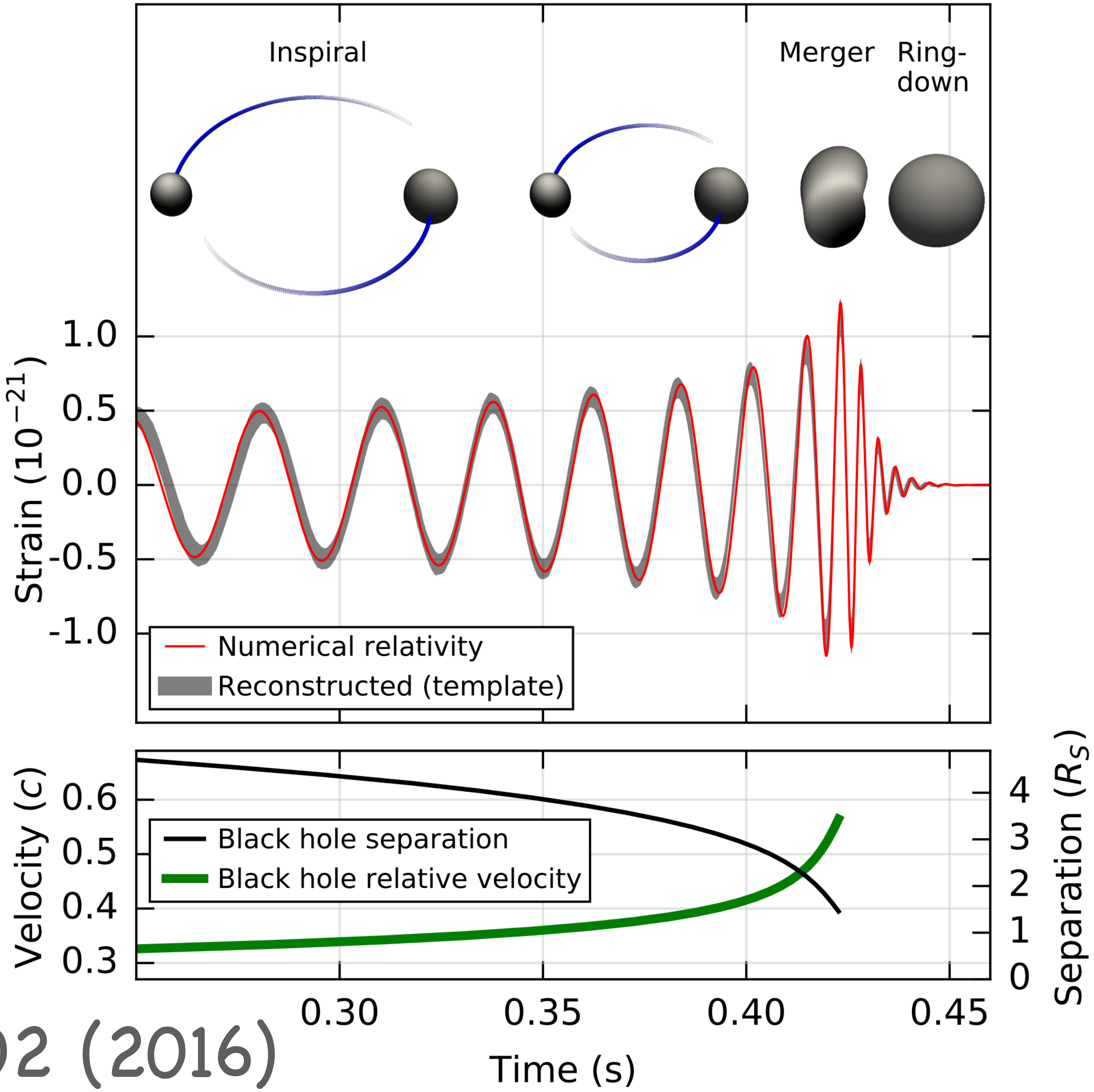
Kip Thorne prediction for BBH dynamics



Kip Thorne prediction for BBH dynamics

GW150914



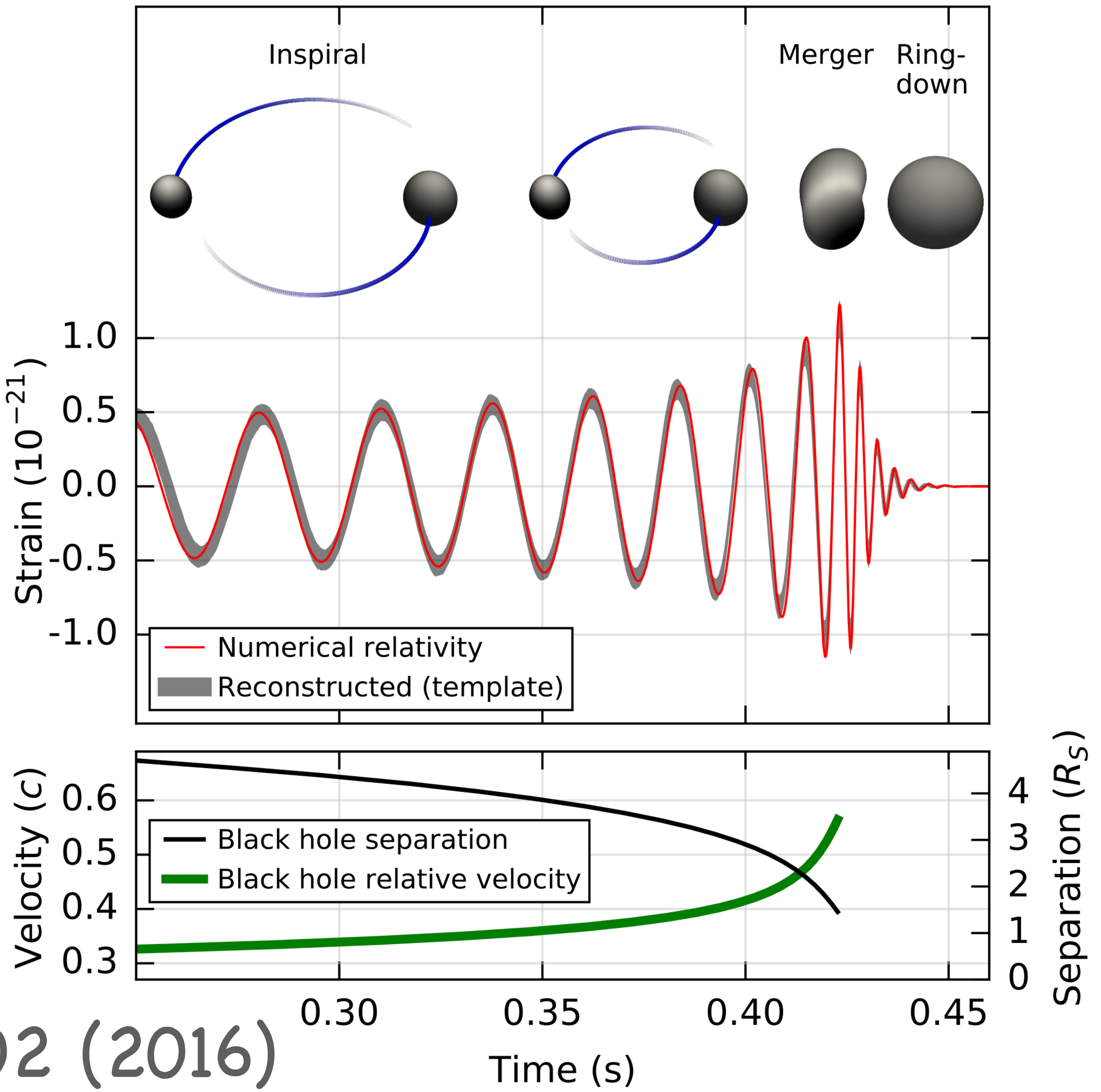


Physical Review Letters **116**, 061102 (2016)

Kip Thorne prediction for BBH dynamics

GW150914

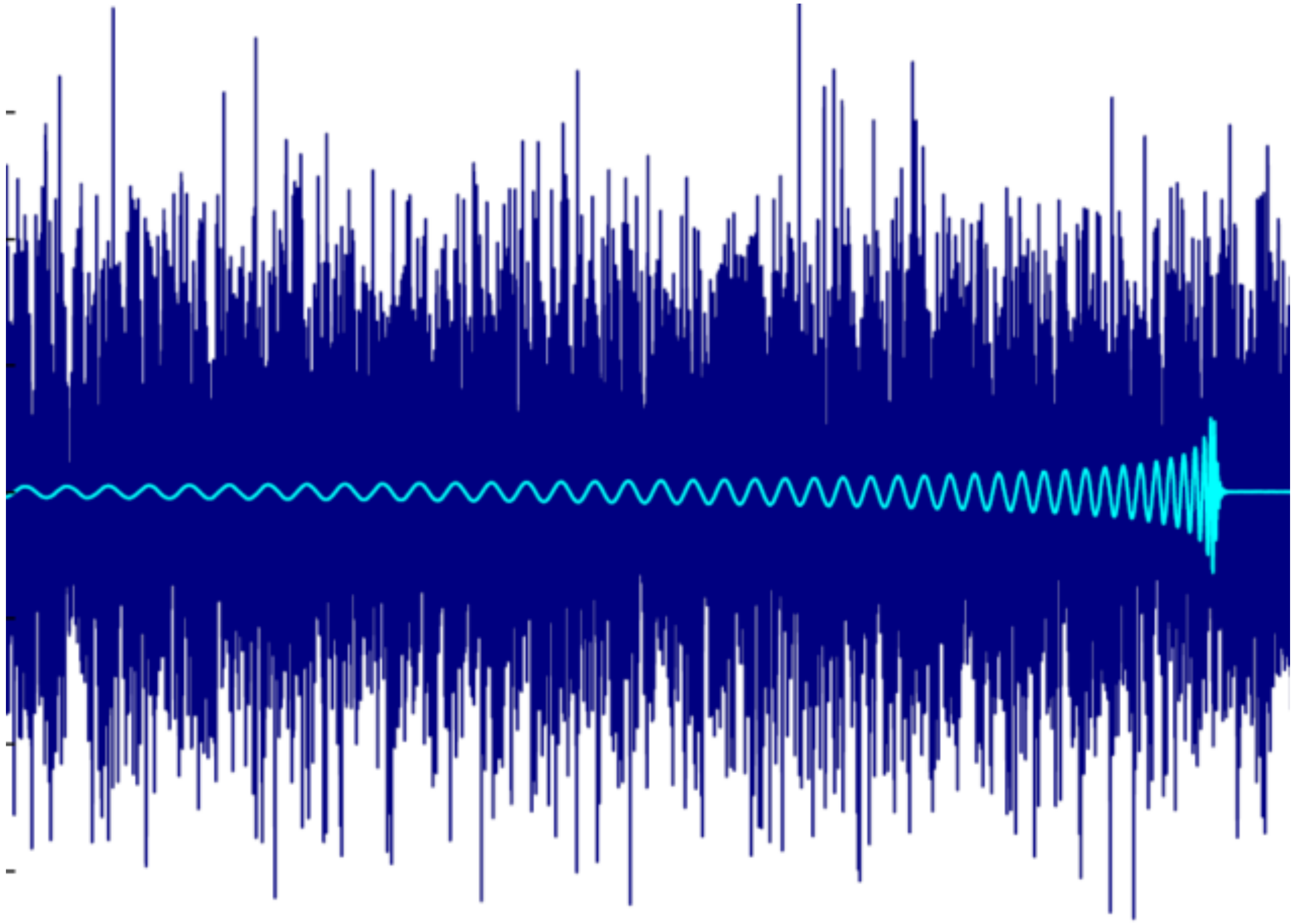
The GW signal is **universal** in the sense that it only depends on the physical parameters of the Binary Black Hole system, namely: Black Hole masses, Black Hole spins, initial orbital parameters (for LVK mostly quasicircular), sky location and luminosity distance, gravitational wave polarization.



Physical Review Letters **116**, 061102 (2016)

The relevance of waveforms for Gravitational Wave Astronomy

- In practice, it is crucial to have **theoretical models** of the waveforms $h(t, \vec{\lambda})$ to extract the Gravitational Wave signals from the data, in particular in those situations where the signal is much below the noise. It is also crucial for parameter estimation, i.e. to obtain $\vec{\lambda}$ and $P(\Delta\vec{\lambda})$.



Fundamental Physics with Massive Black Hole Mergers

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Inspiral Phase (post-Newtonian)

The phase carries information about the propagation of the Gravitational Waves

Can test theories of gravity that predict massive gravitons and/or extra polarizations, improving present bounds.

Fundamental Physics with Massive Black Hole Mergers

- High precision measurements of Strong Gravity

Merger
(Numerical Relativity)

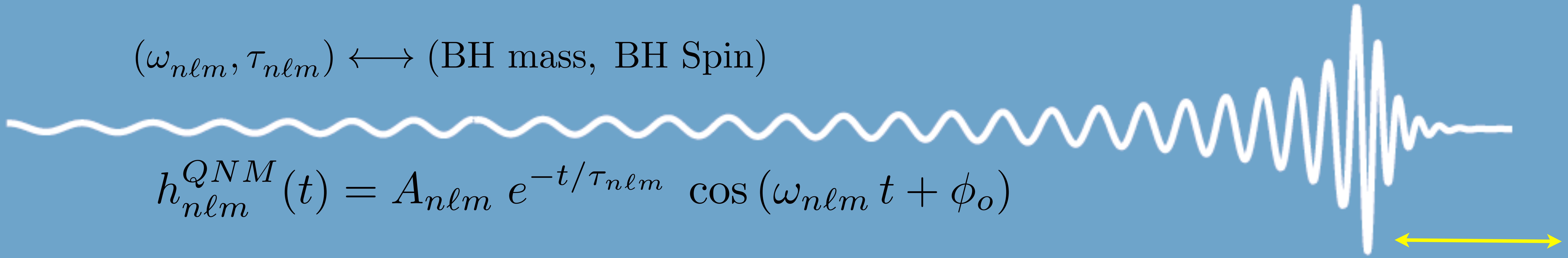
Short bursts of Gravitational Waves in the non-linear regime of General Relativity.

It is a very energetic event, concentrating most power of the gravitational radiation emission.

Fundamental Physics with Massive Black Hole Mergers

- High precision measurements of Strong Gravity

$$(\omega_{n\ell m}, \tau_{n\ell m}) \longleftrightarrow (\text{BH mass},\ \text{BH Spin})$$



$$h^{QNM}_{n\ell m}(t) = A_{n\ell m}\, e^{-t/\tau_{n\ell m}}\ \cos\left(\omega_{n\ell m}\, t + \phi_o\right)$$

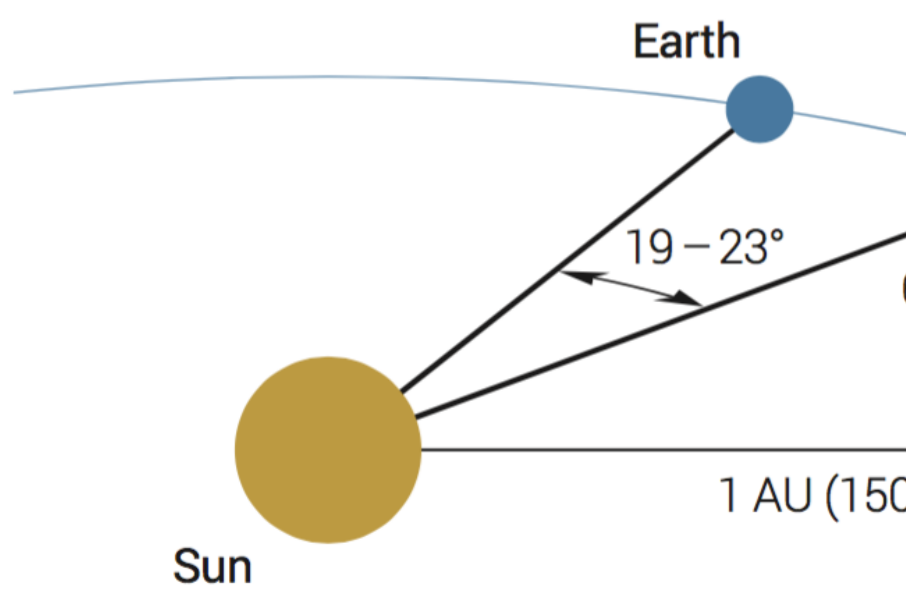
Ringdown
(Perturbation Theory)

The asymmetric remnant after the merger settles down to a single (Kerr) Black Hole. In this "relaxation" process the system emits Gravitational Waves that are combinations of the QuasiNormal Modes (QNMs) of the final Black Hole.

The QNMs, according to General Relativity, only depend on the Mass and Spin of the Black Hole (*no hair* conjecture).

The identification of two QNMs provides a test of the geometry of Black Holes (are they really Kerr Black Holes?). The QNM spectrum is sufficiently rich to allow for distinction of different object.

* How LISA would see a Massive Black Hole merger assuming the LISA Pathfinder acceleration noise?



Physical Processes around Black Holes

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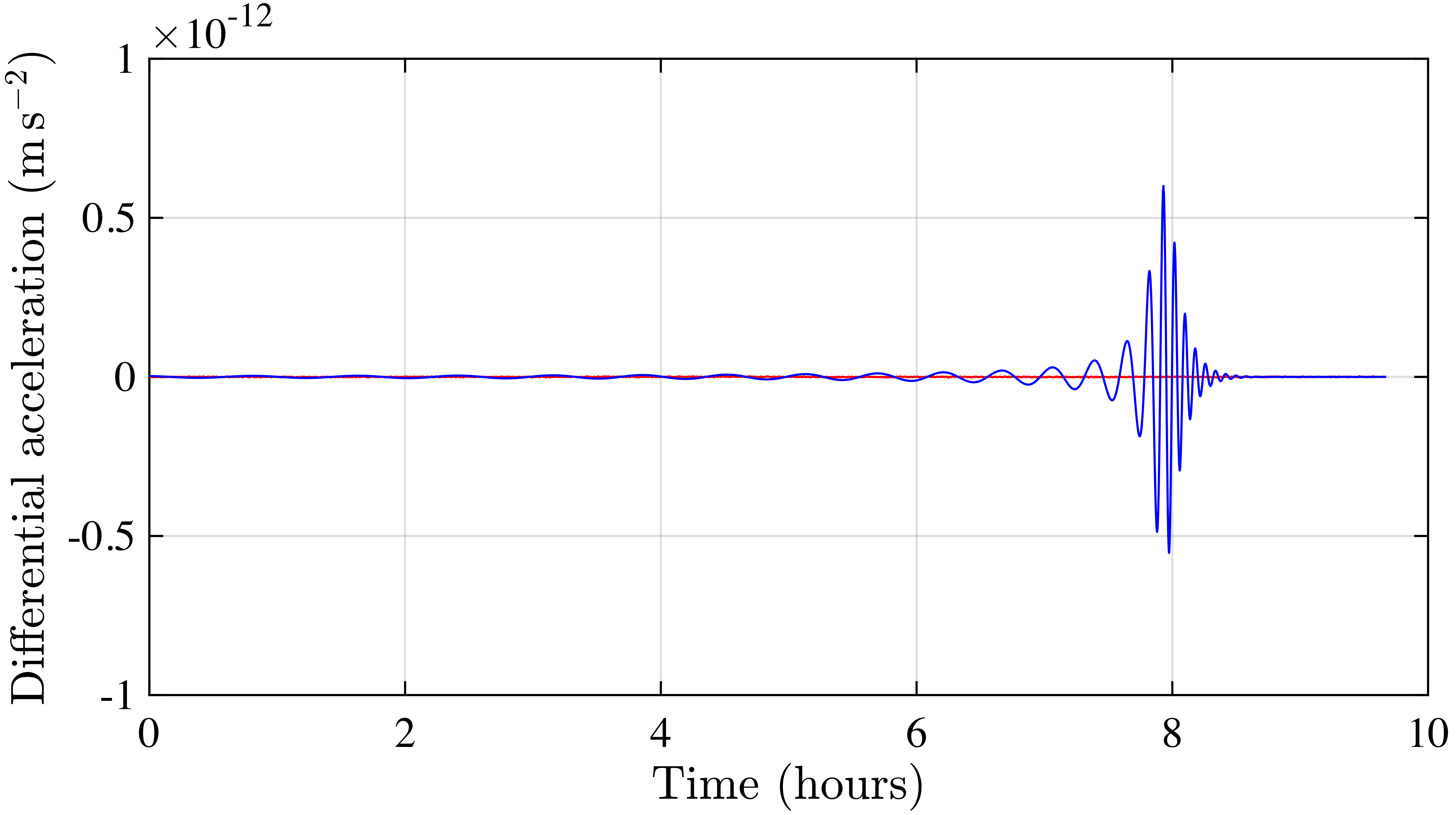

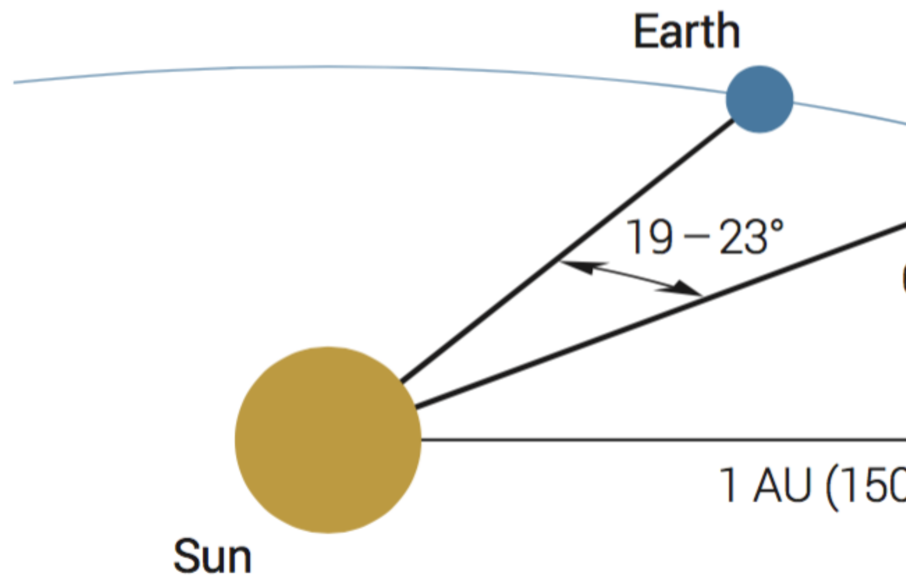

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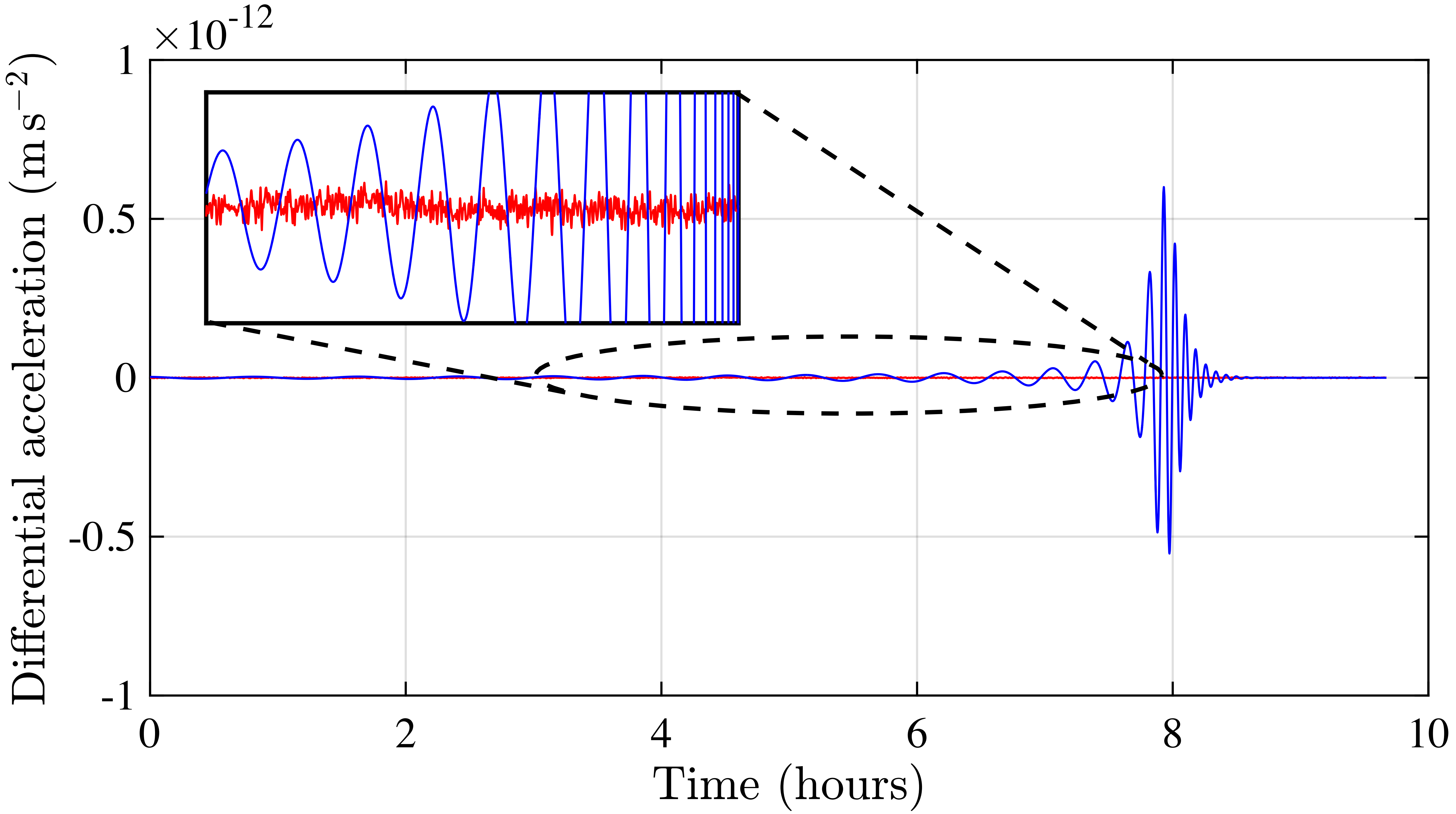

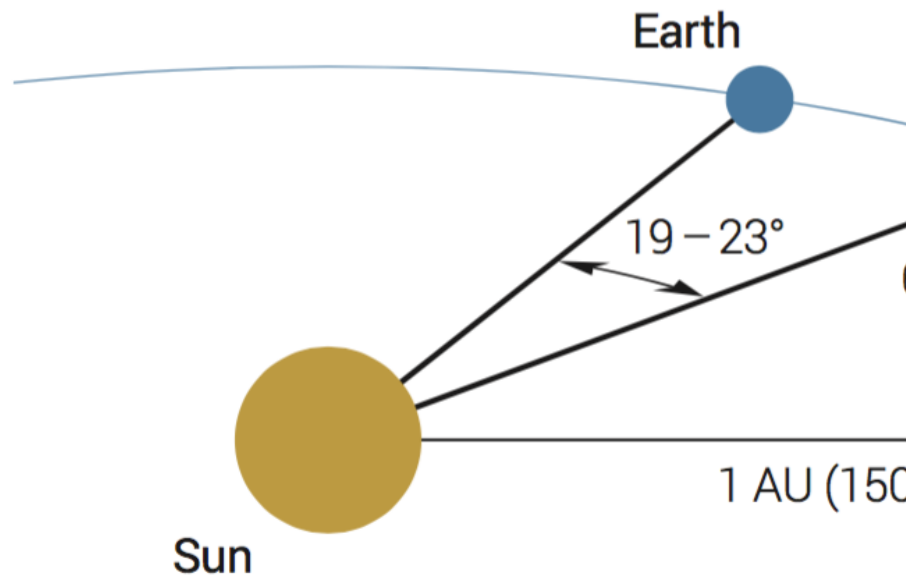

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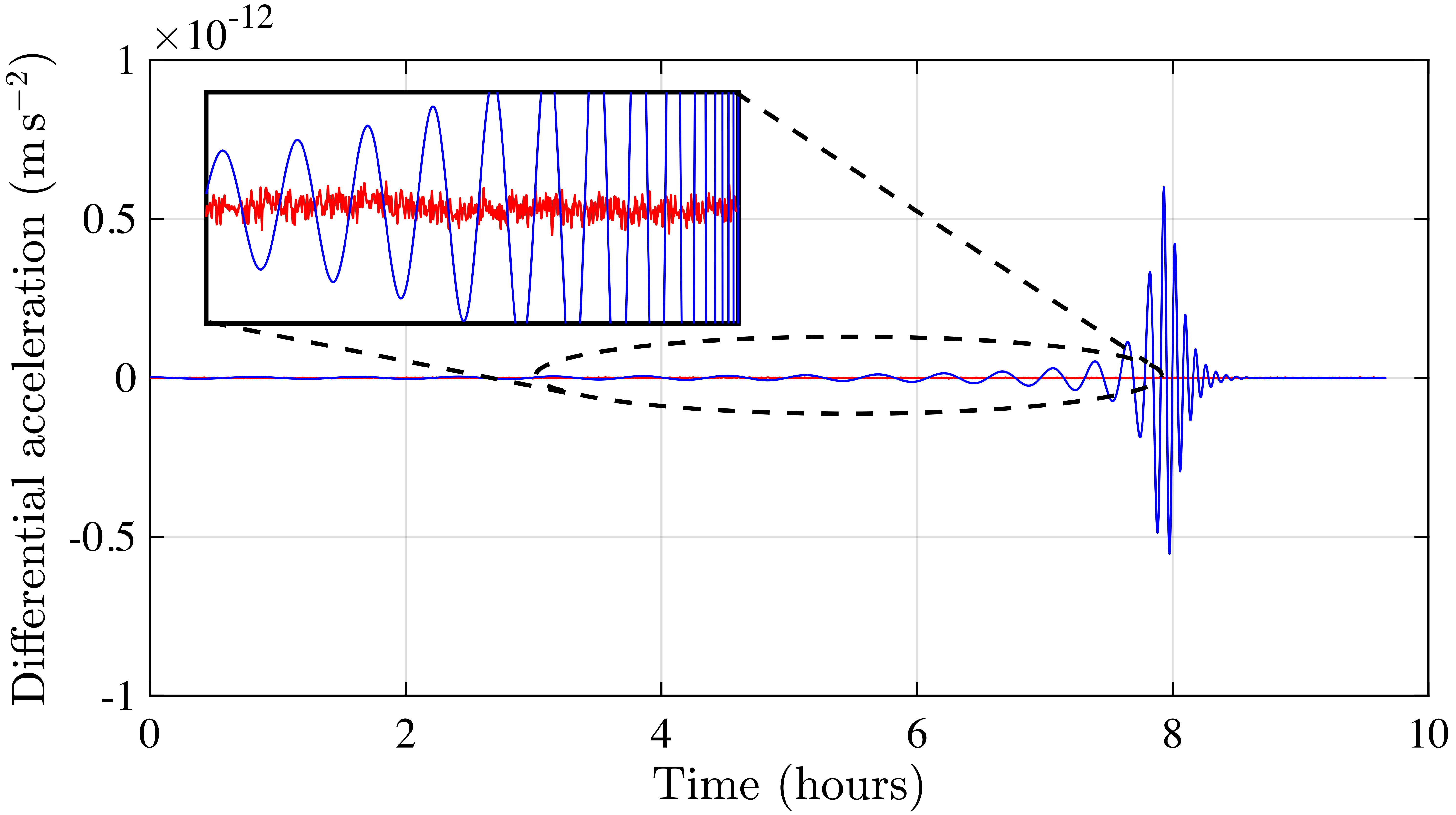

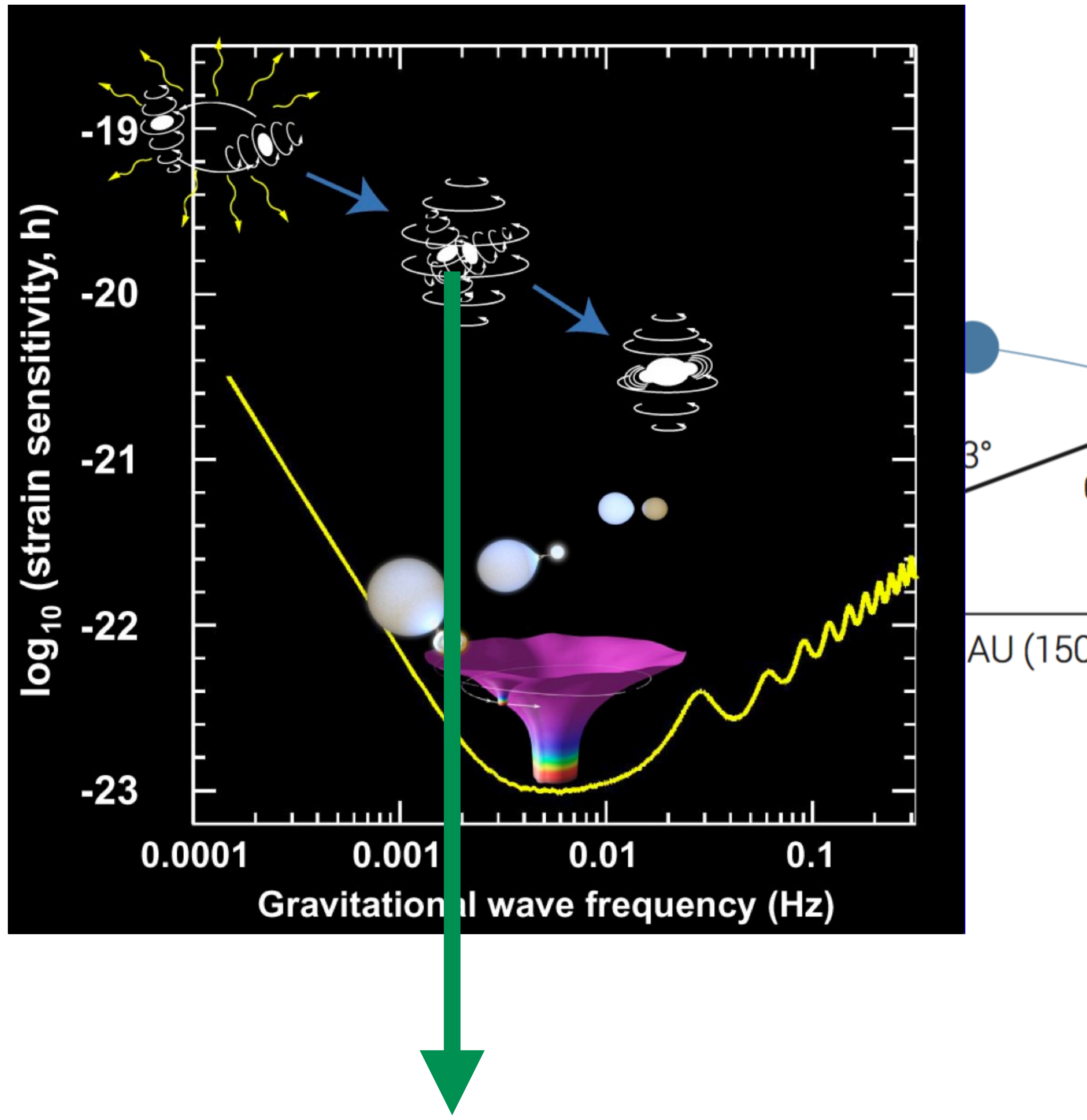

High Signal-to-Noise Ratio (SNR):

SNR ~ 50-5000

Black Holes

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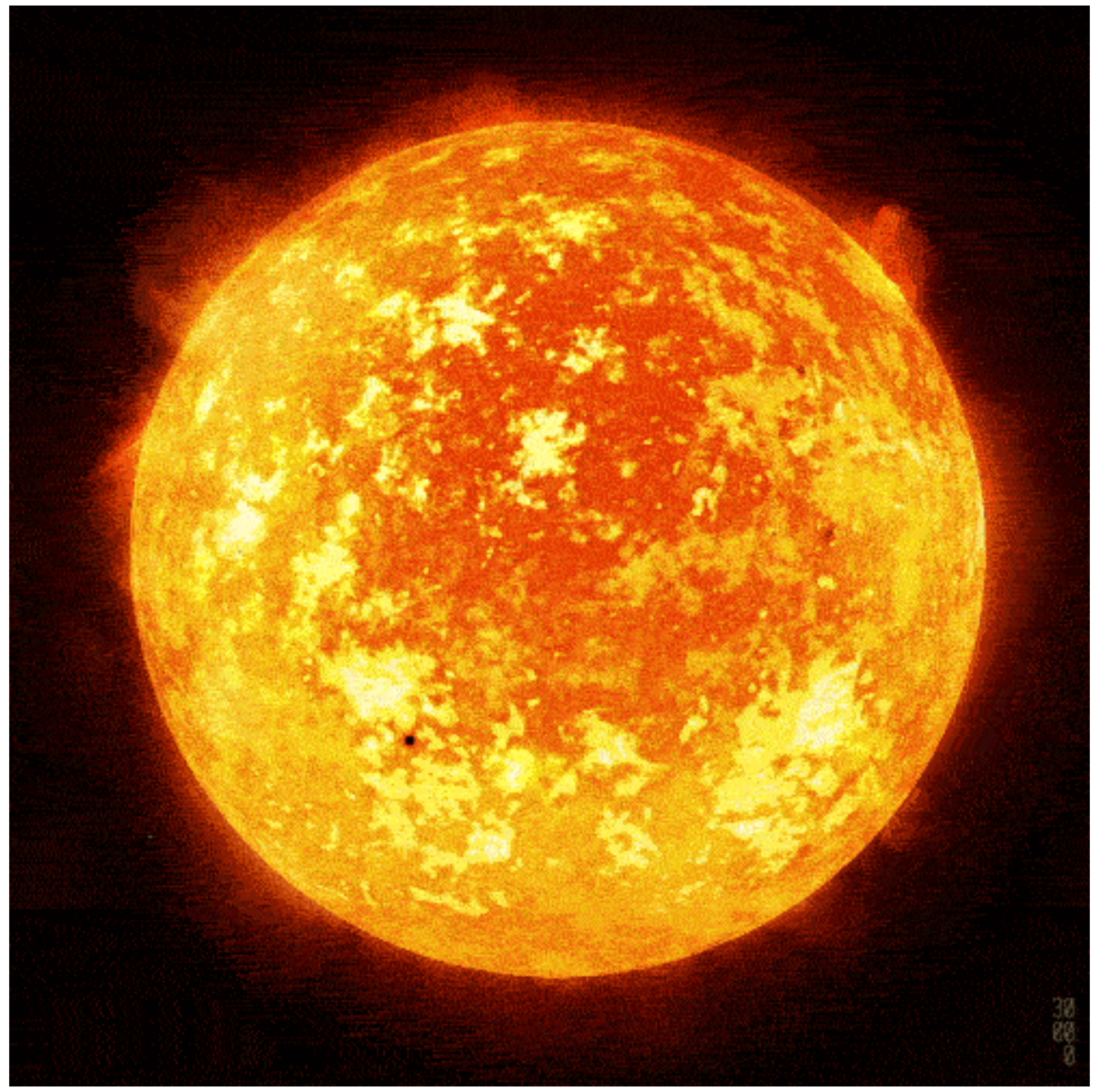
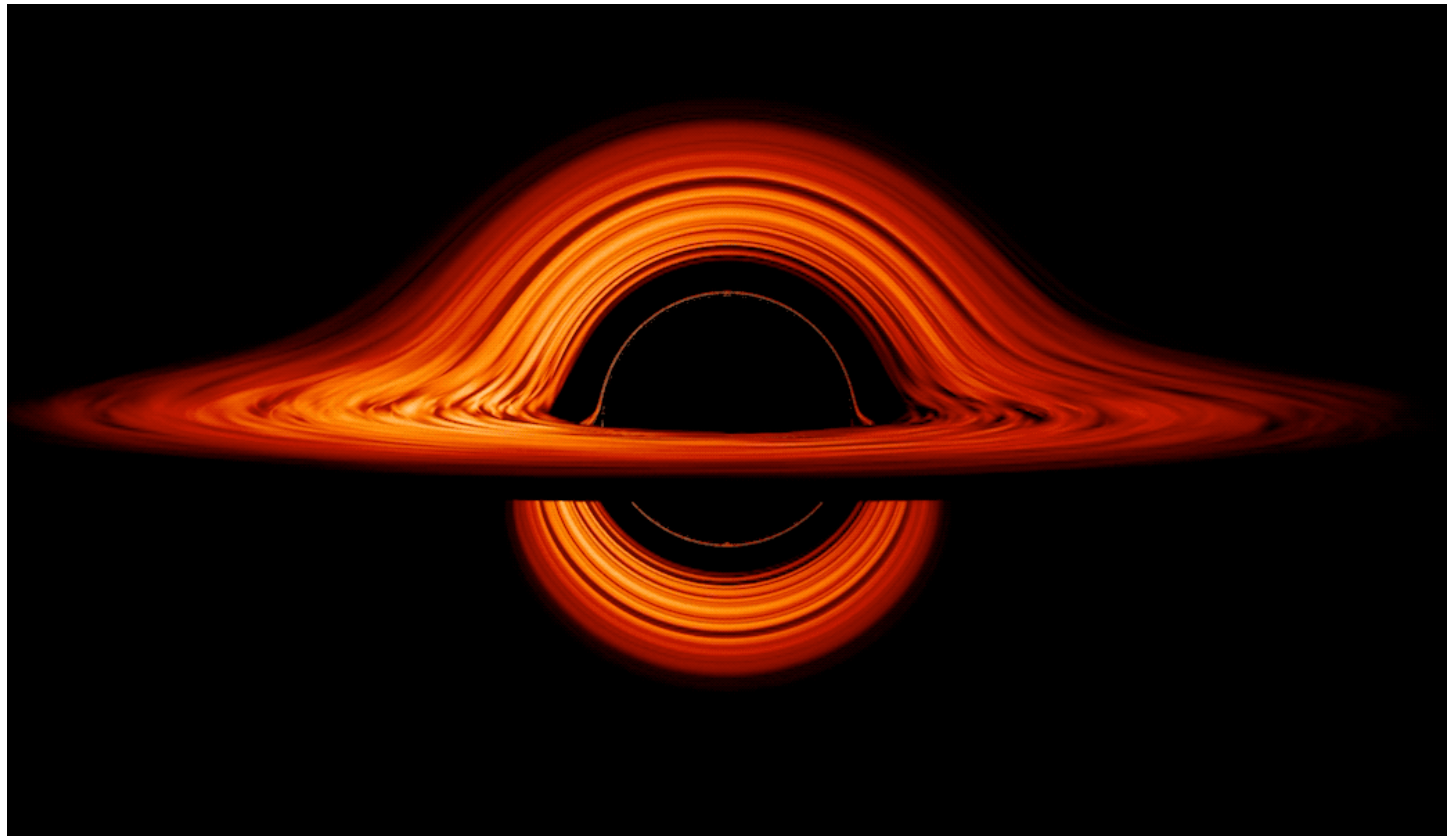
Historial Origin: A Black Hole is a <u>concept</u> that was established in the 1950-60's when people finally understood the physical implications of some of the solutions of the General Theory of Relativity (Schwarzschild solution).





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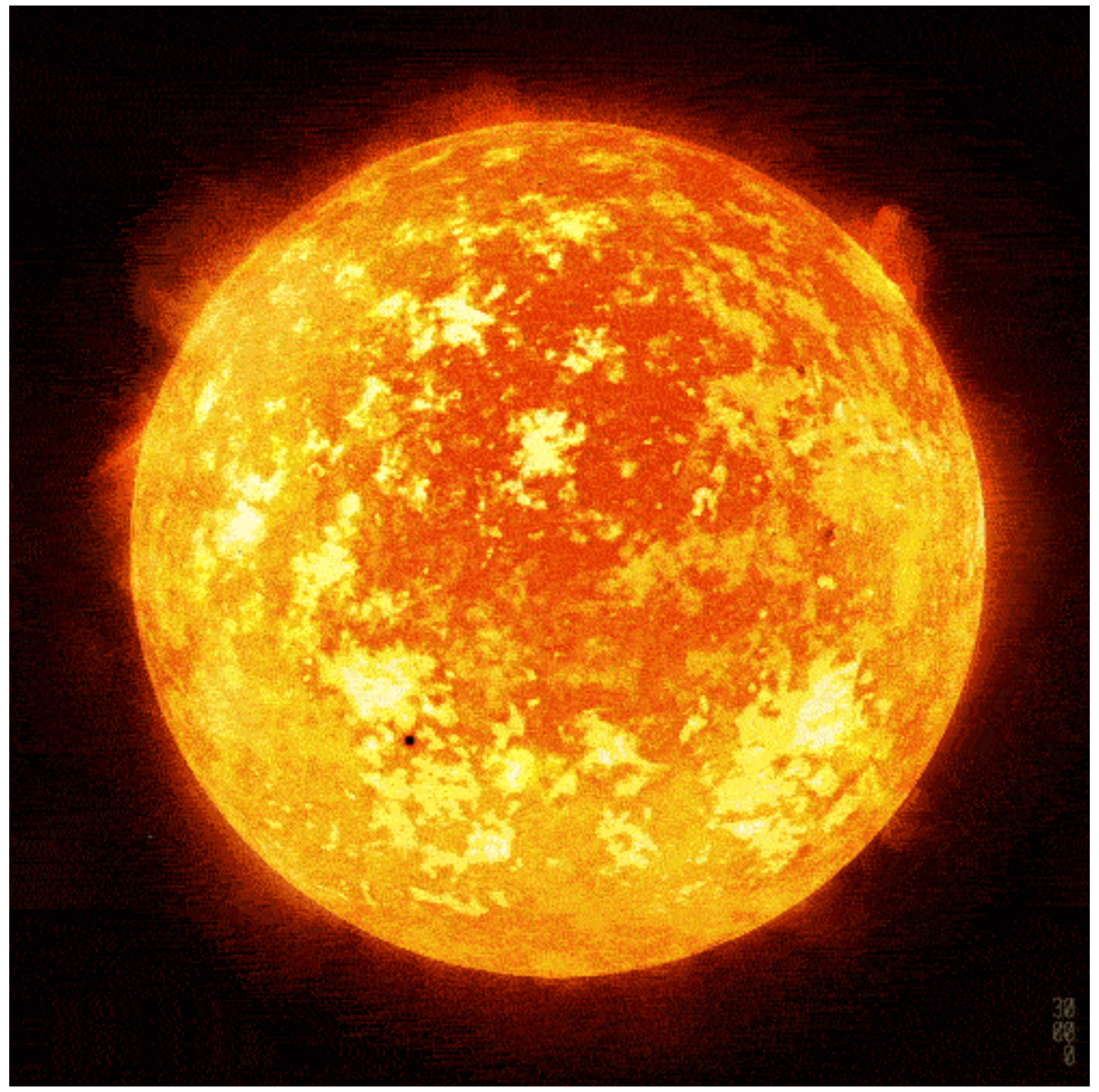

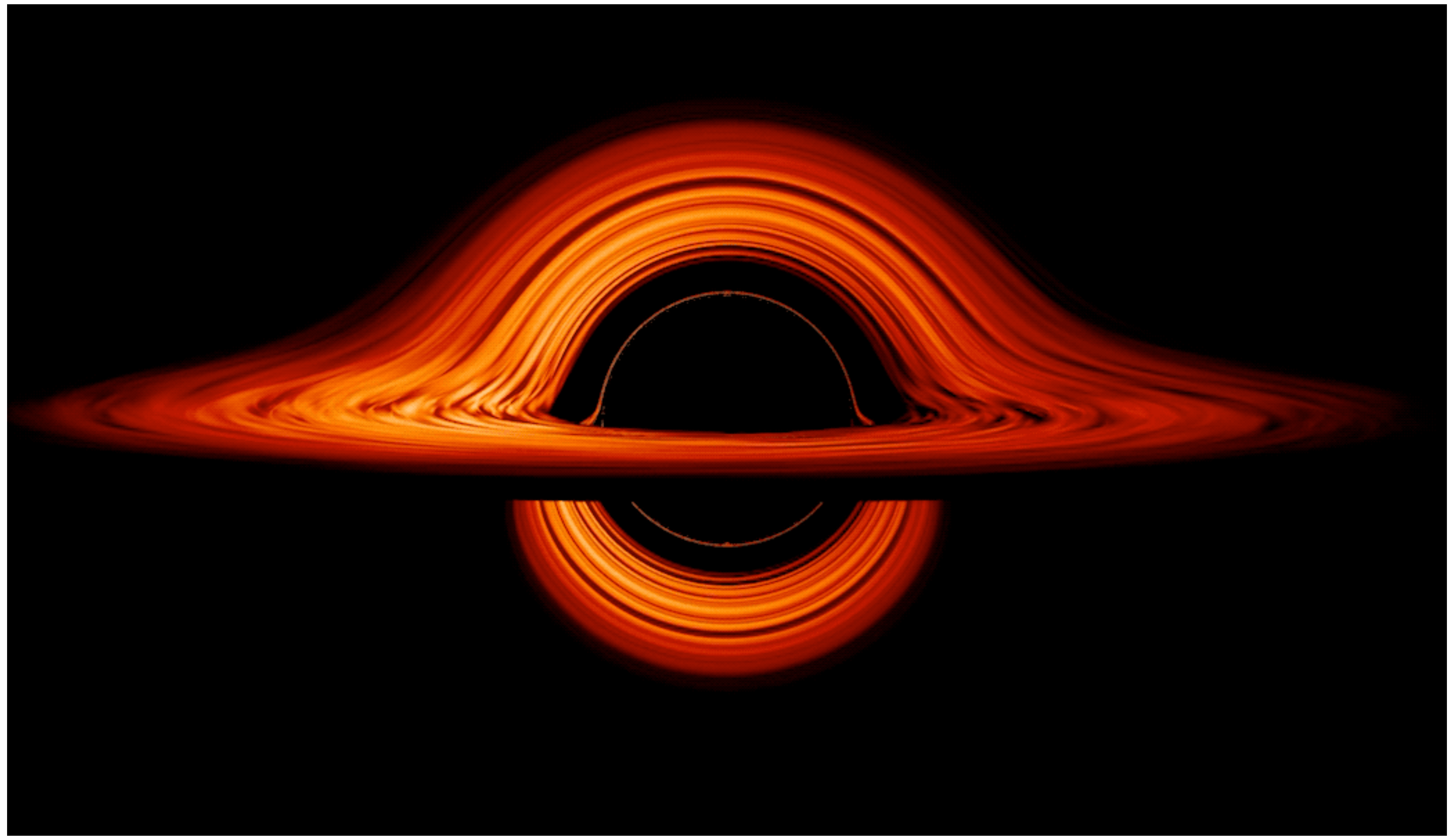

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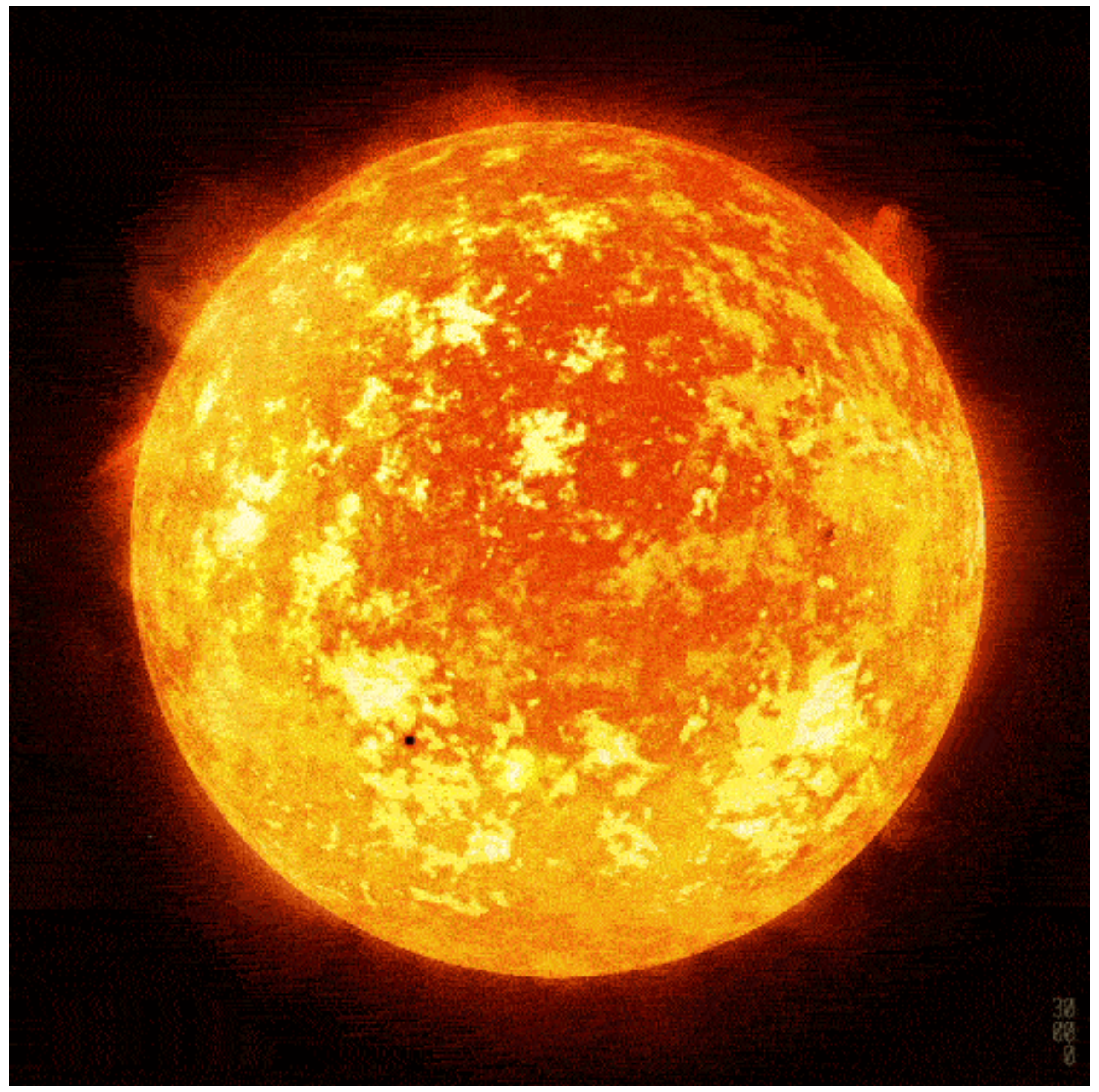
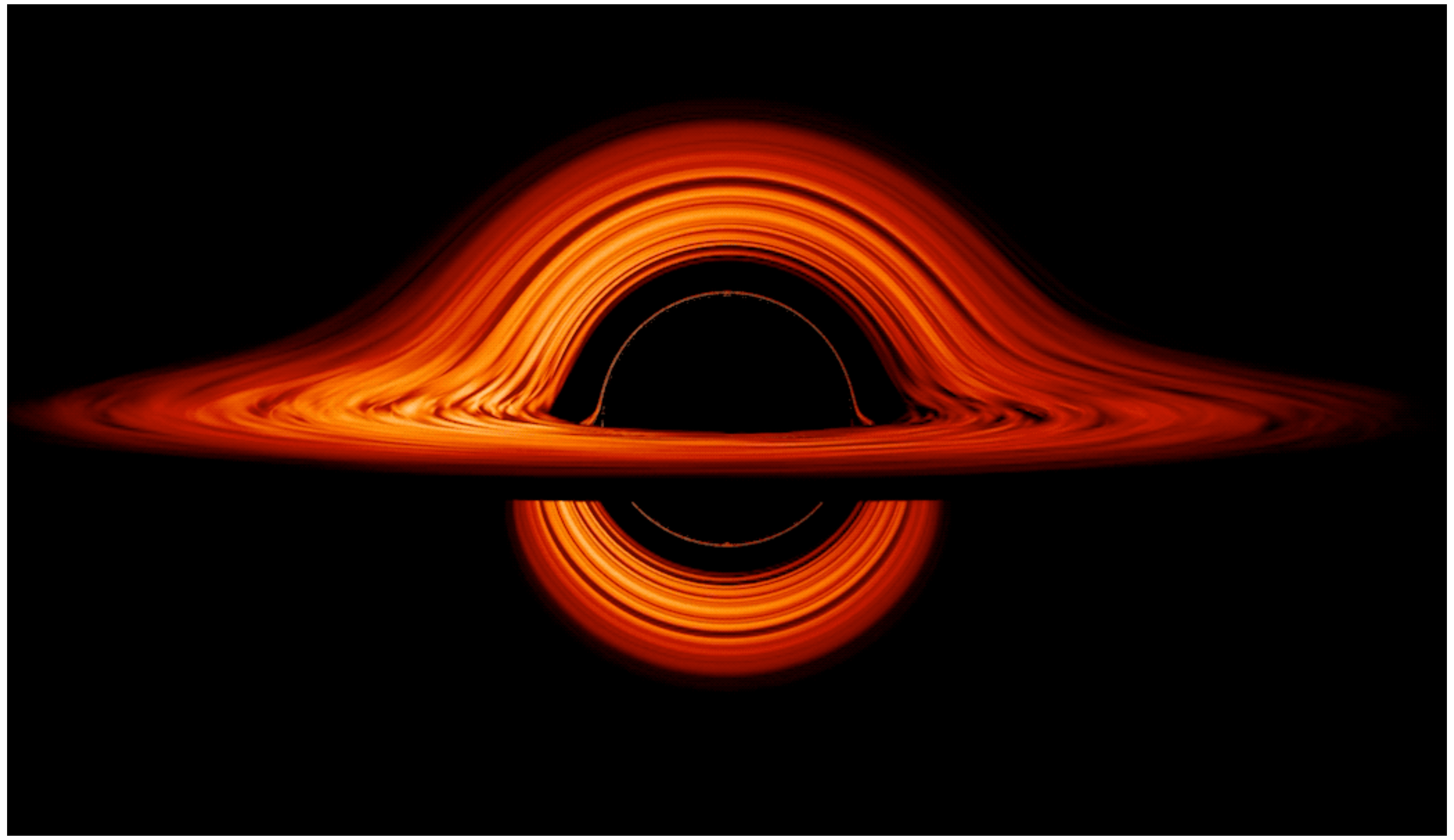


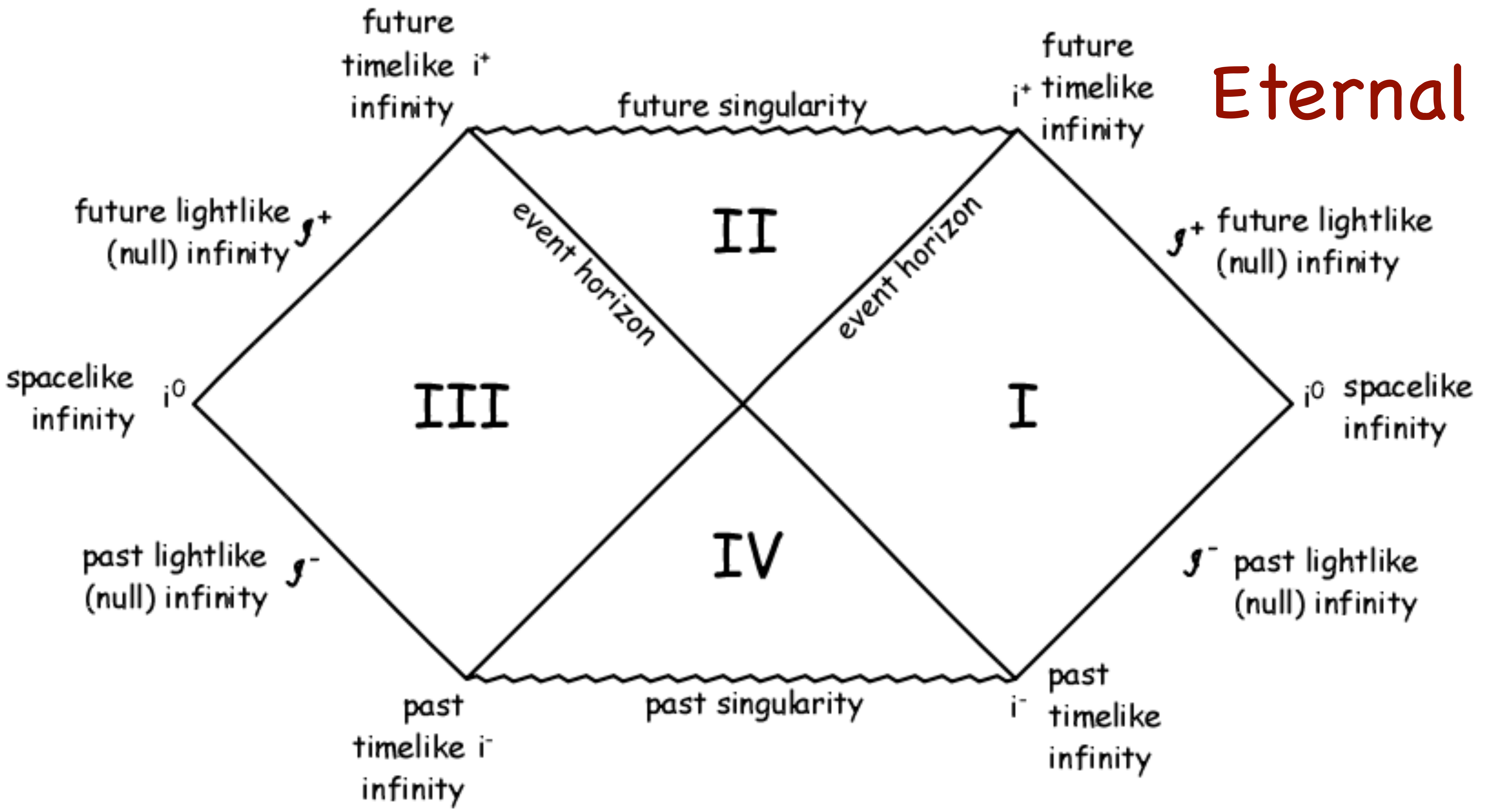


There is growing evidence (VLBI, X-ray, Gravitational Waves,…) of objects in the cosmos that present properties so close to this concept that makes us think that they must be Black Holes. However, it is impossible to observationally prove the existence of event horizons (the key element of BHs).

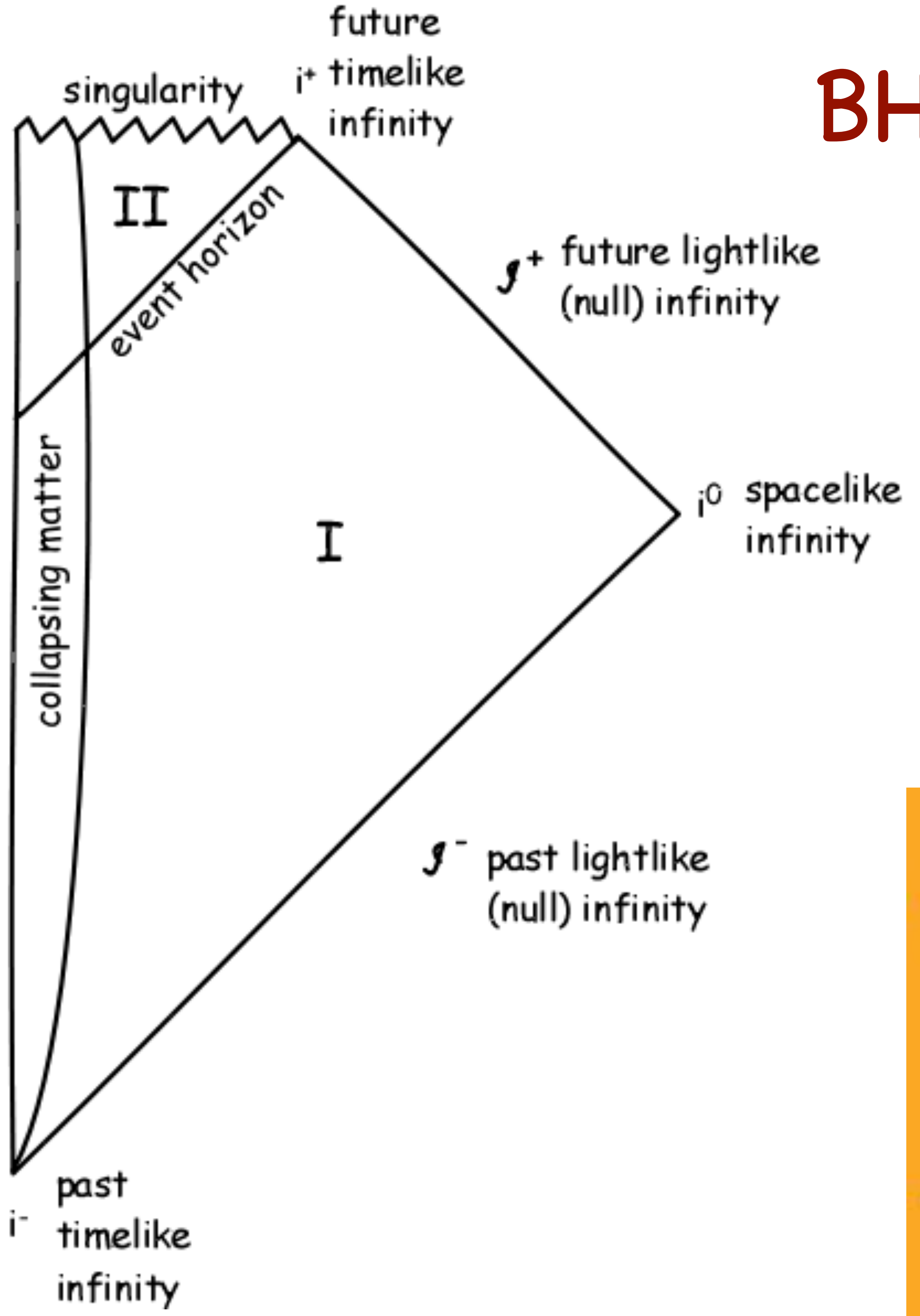
Black Holes

- A Black Hole is a region of the spacetime of no escape. From this region, no physical signals/ interactions can propagate outside. The BH horizon is the boundary of this region.

Eternal BH



BH Formation

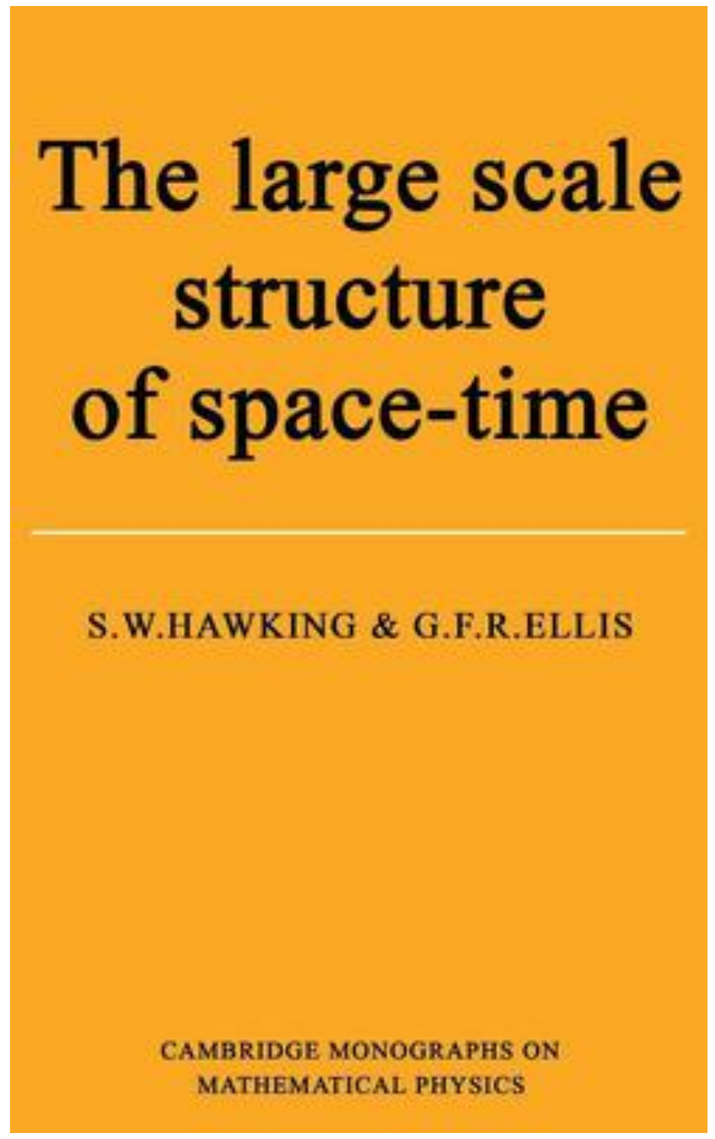


wrong. That this is so is shown rigorously by Penrose's theorem:

Theorem 1

Space–time $(\mathscr{M}, \mathbf{g})$ cannot be null geodesically complete if:

(1) $R_{ab}K^aK^b \geqslant 0$ for all null vectors K^a (cf. §4.3);
(2) there is a non-compact Cauchy surface $\mathscr{H}$ in $\mathscr{M}$;
(3) there is a closed trapped surface $\mathscr{T}$ in $\mathscr{M}$.



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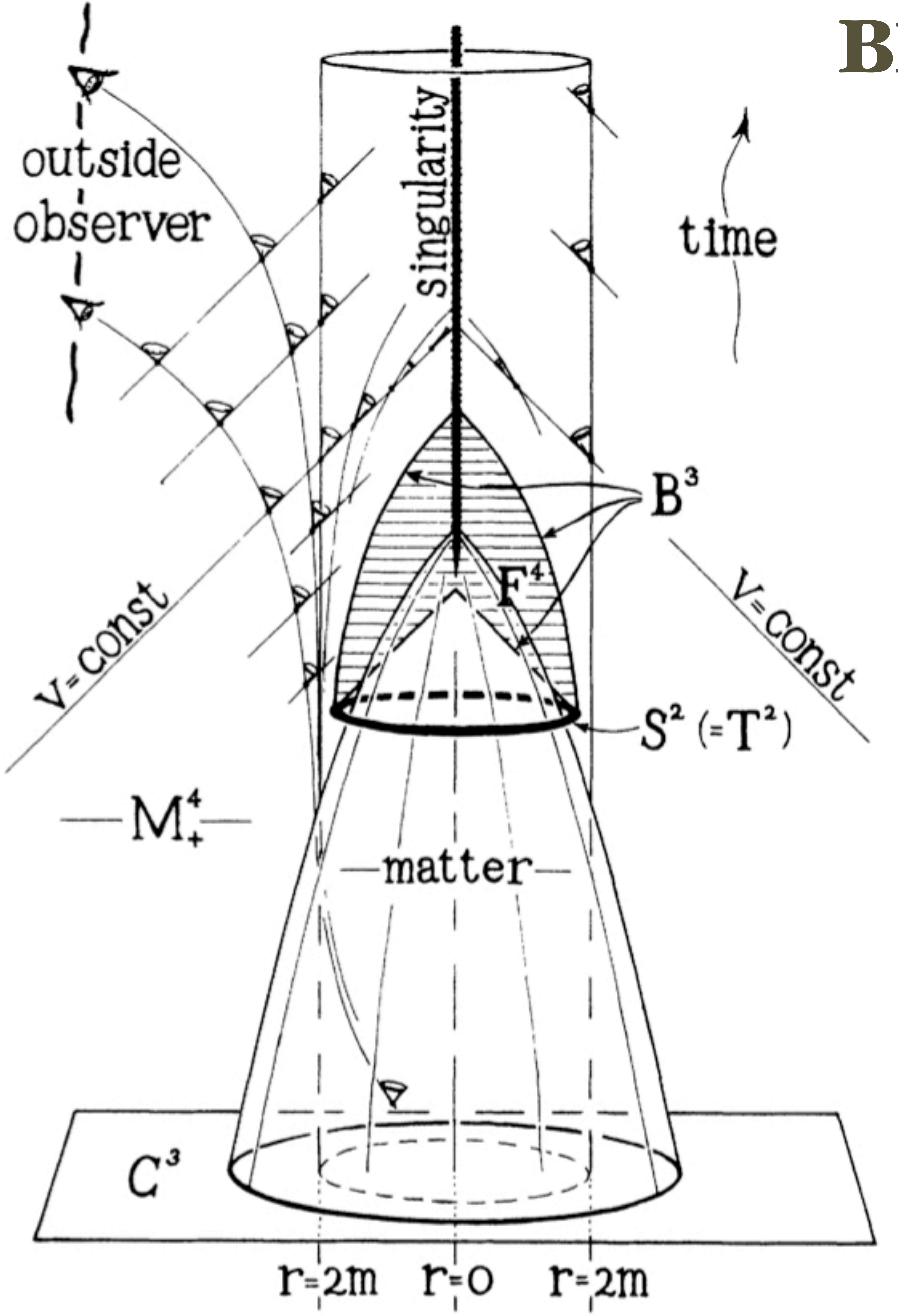


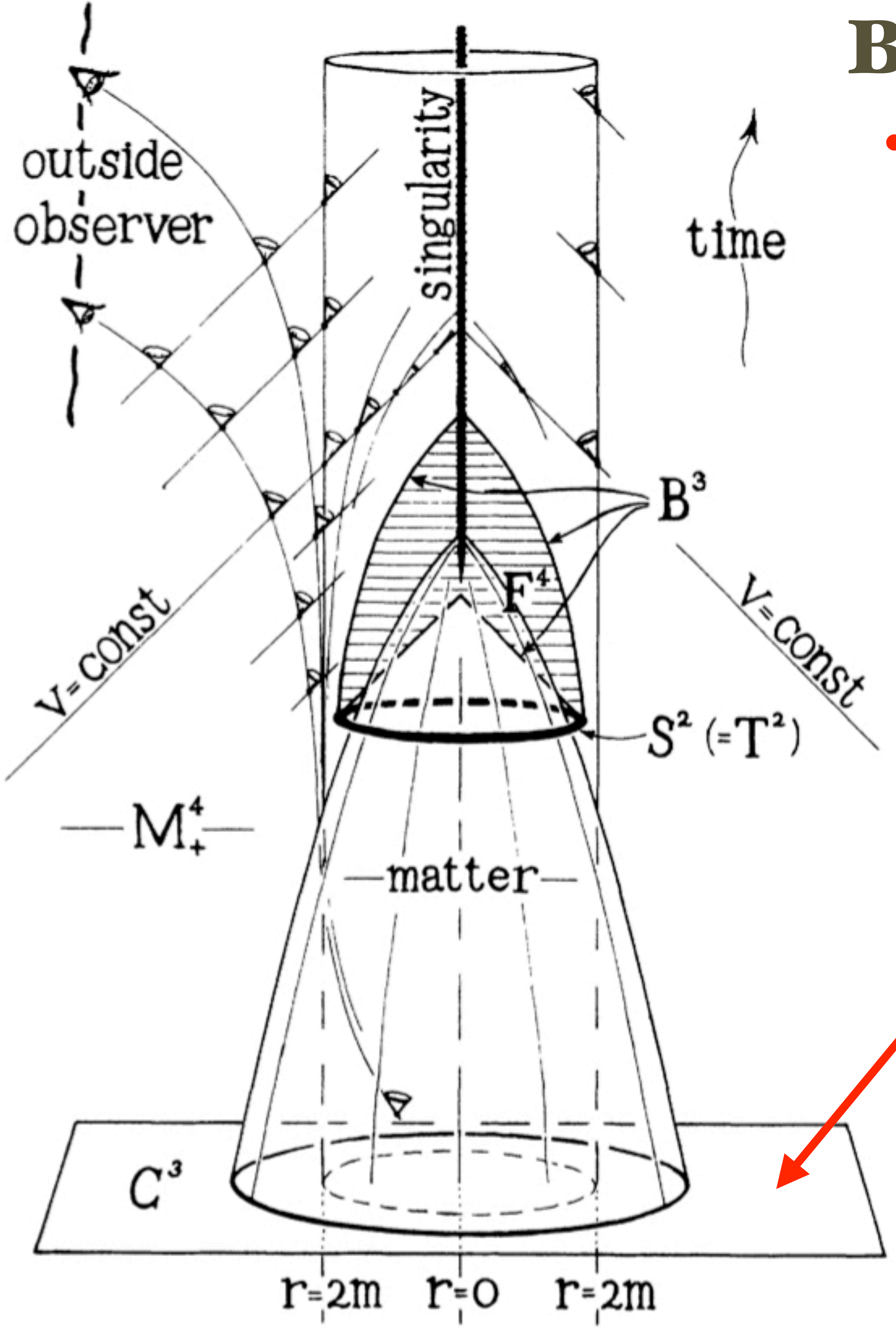
VOLUME 14, NUMBER 3 PHYSICAL REVIEW LETTERS 18 JANUARY 1965

GRAVITATIONAL COLLAPSE AND SPACE-TIME SINGULARITIES

Roger Penrose
Department of Mathematics, Birkbeck College, London, England
(Received 18 December 1964)

Black Hole Formation

- Consider an initial mass distribution (no symmetries assumed) that will be the source of the Gravitational Field according to Einstein's Equations.



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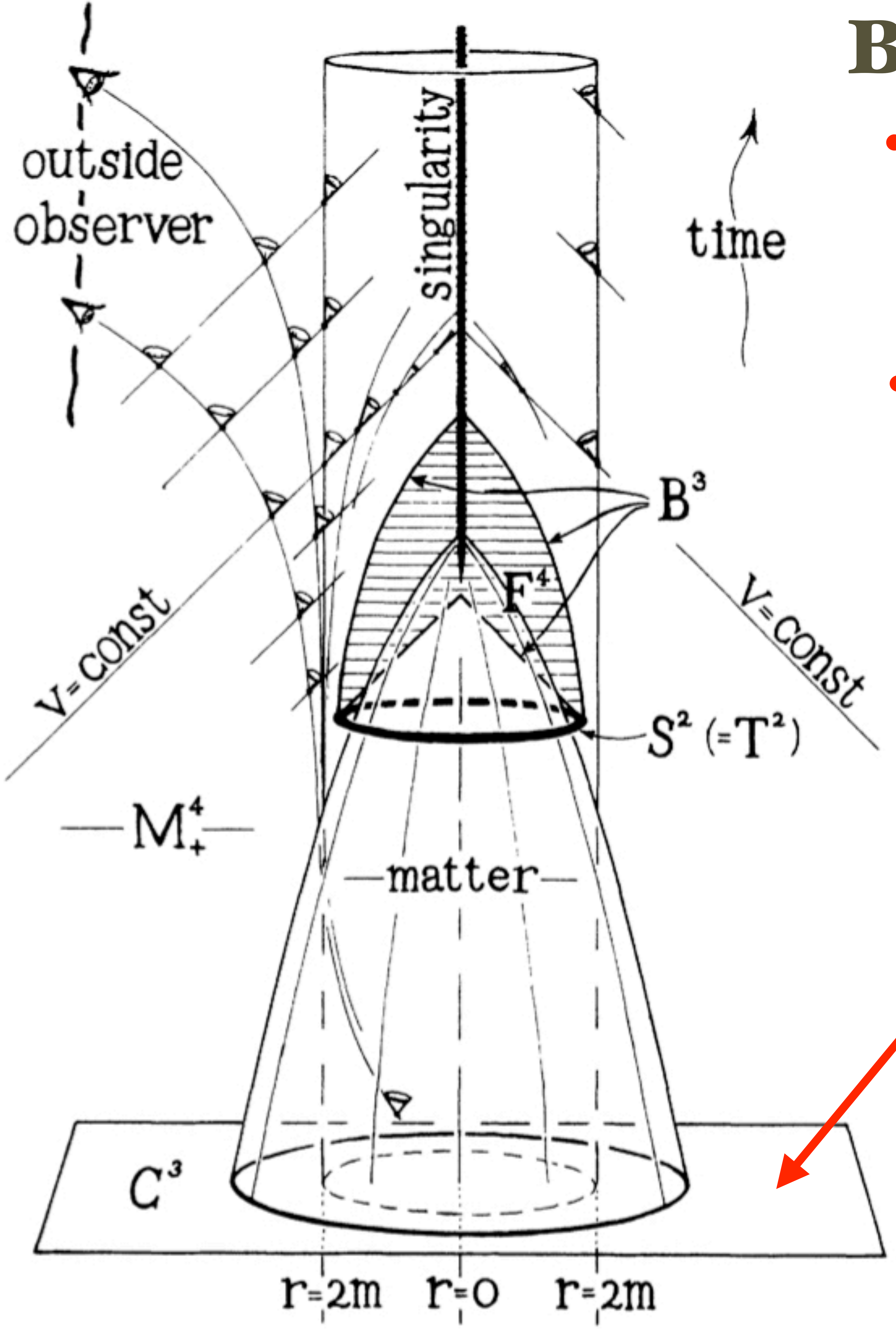

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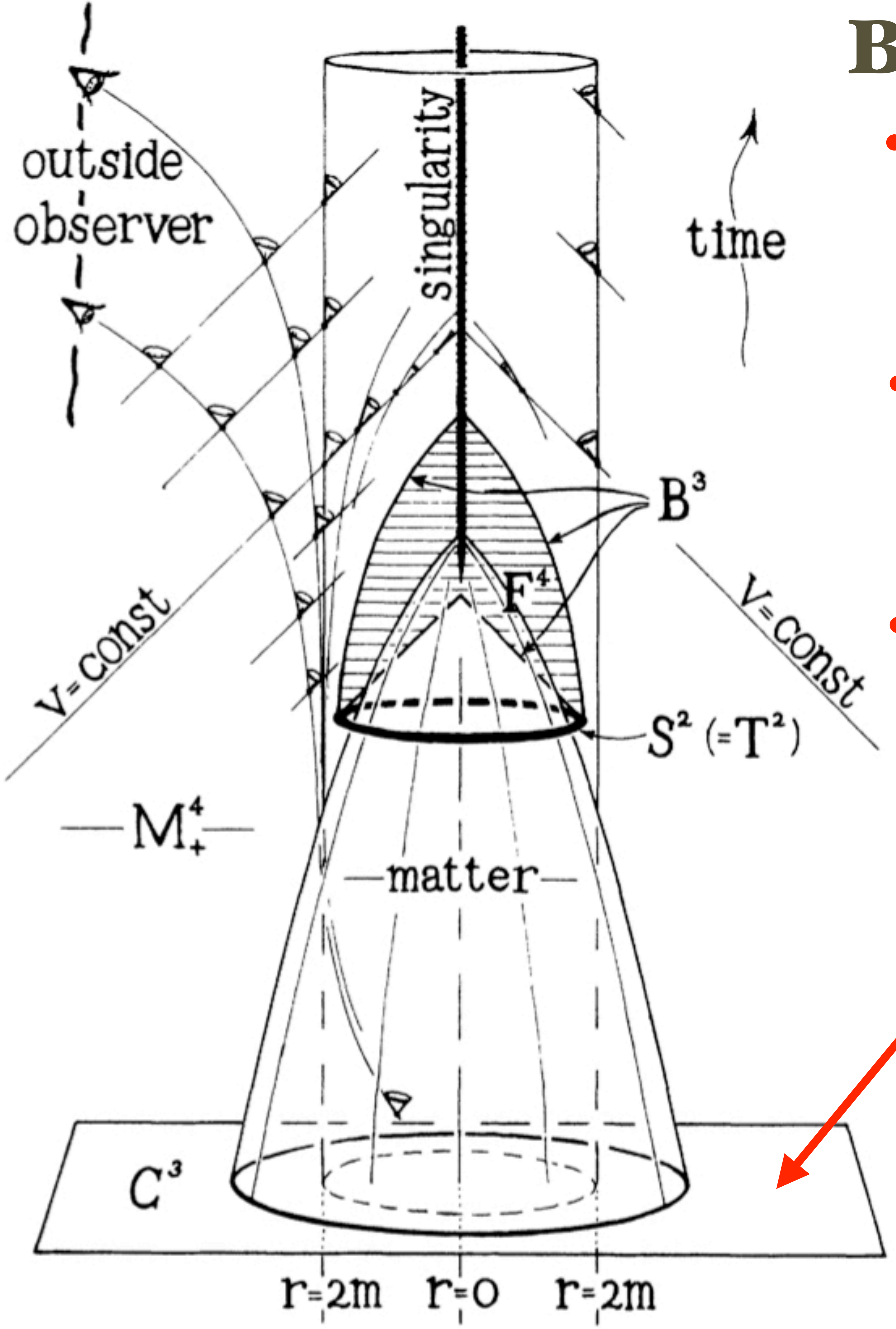

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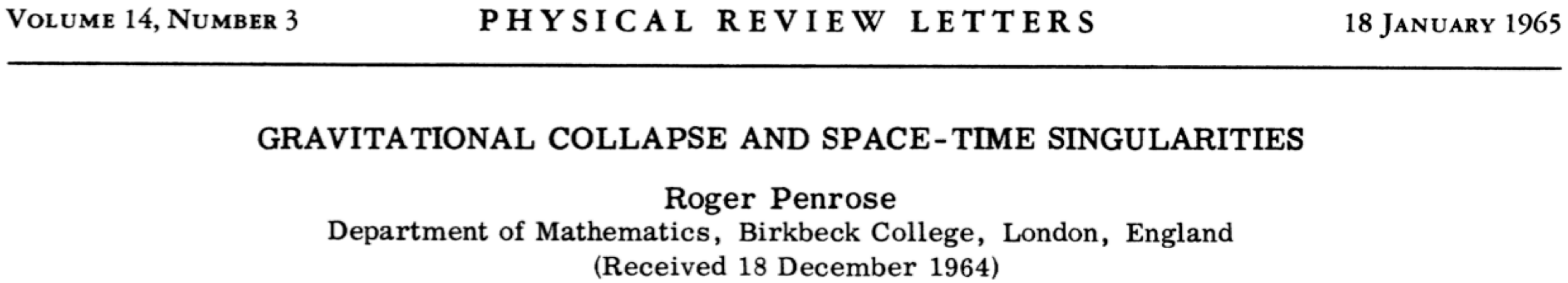
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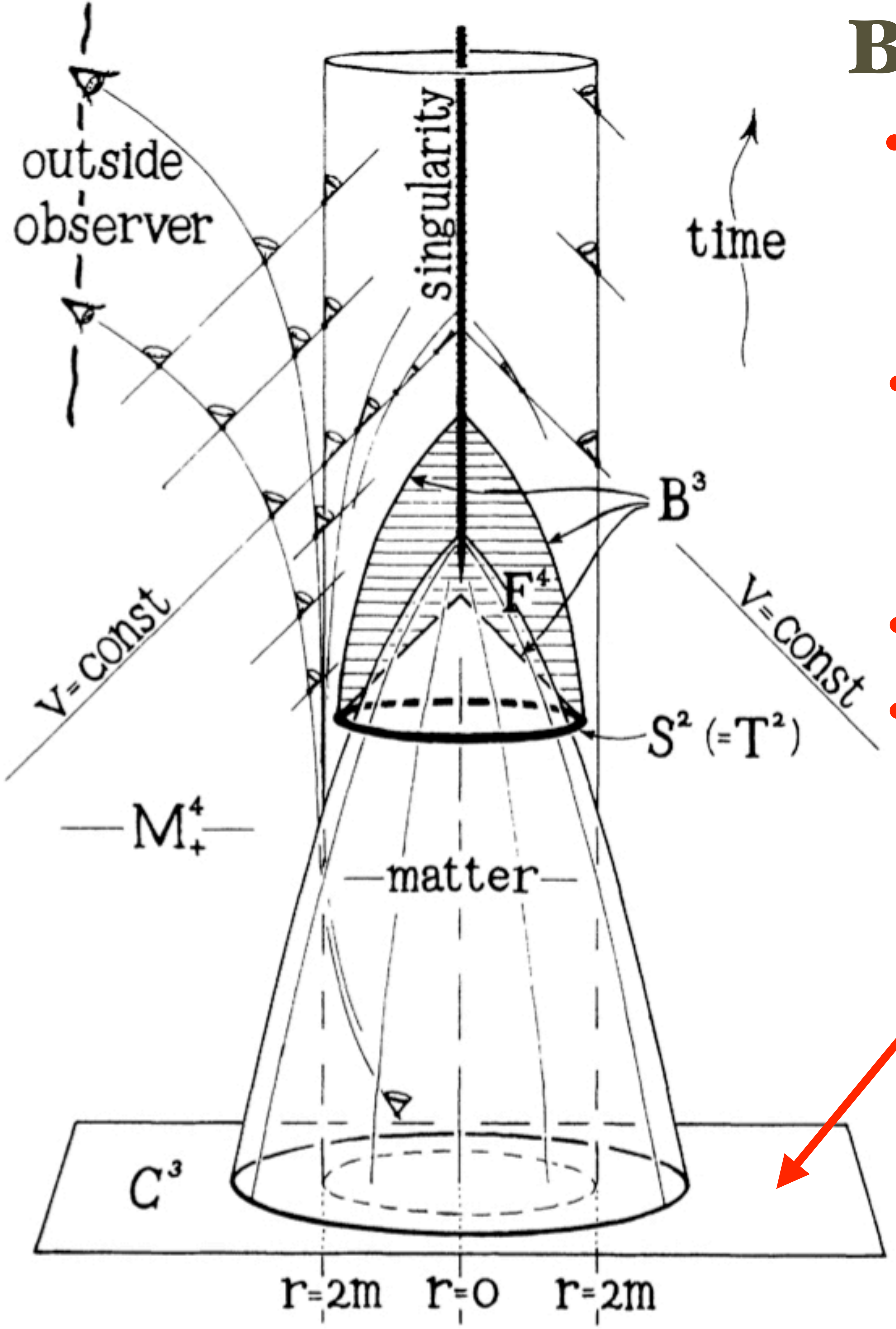

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- The matter distribution is assumed to satisfy certain energy-momentum conditions (like positivity of the mass-energy density).
- Existence of TRAPPED SURFACES.
- Under these assumptions, Penrose proved that deviations from spherical symmetry cannot prevent the appearance of space-time singularities: The spacetime manifold is incomplete or it losses its meaning due to unbounded curvature regions (quantum effects?).

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The Black Hole Spectrum

BH
Mass

The Black Hole Spectrum

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Uncharted territory:
Primordial BHs, Other?

Lensing, CMB polarization,...

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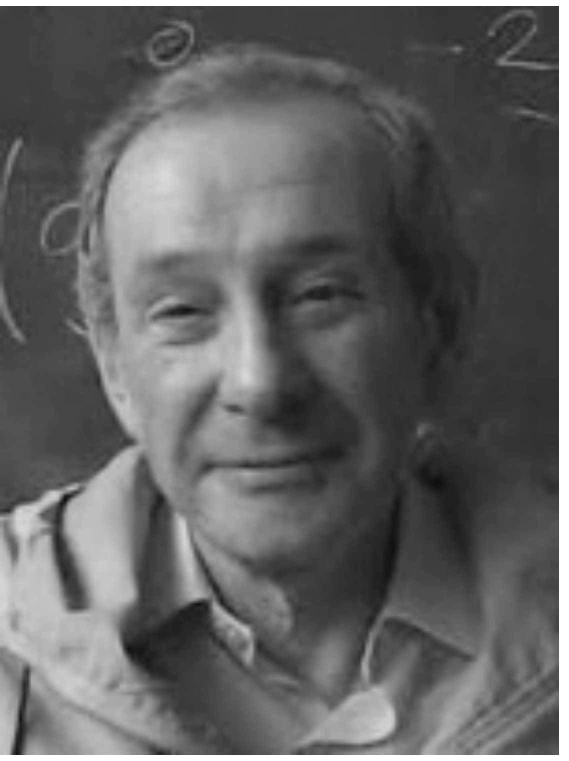
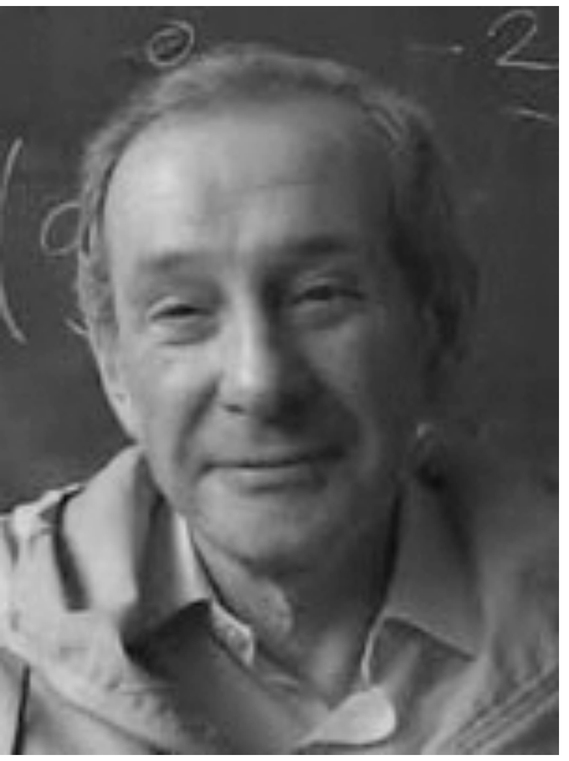
Black Holes in General Relativity



Werner Israel

1967: Werner Israel found that non-rotating Black Holes, when they have finally settle down to become stationary (static), would necessarily become completely spherically symmetrical.
Carter, Robinson and others generalized this to include rotating Black Holes, the implication being that the space-time geometry of the final state must be given by the Kerr family of Black Holes.

Black Holes in General Relativity



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Stephen Hawking

1972: The previous argument was, given by Hawking, that if there is any rotation present, then there must be complete axial symmetry. The very remarkable conclusion is that the Black Holes that we expect to find in Nature have to conform to the Kerr geometry!! (**No hair conjecture. Penrose** gave a formulation).
Hawking also proved that the topology of the Event Horizon of a Black Hole must be (homemorphic to) the 2-sphere.

Black Holes in General Relativity

➡ **Cosmic Censorship Conjecture [Penrose and others]:** When the outcome of gravitational collapse is a singularity, it must always be covered by a Horizon, and hence it must be a Black Hole.

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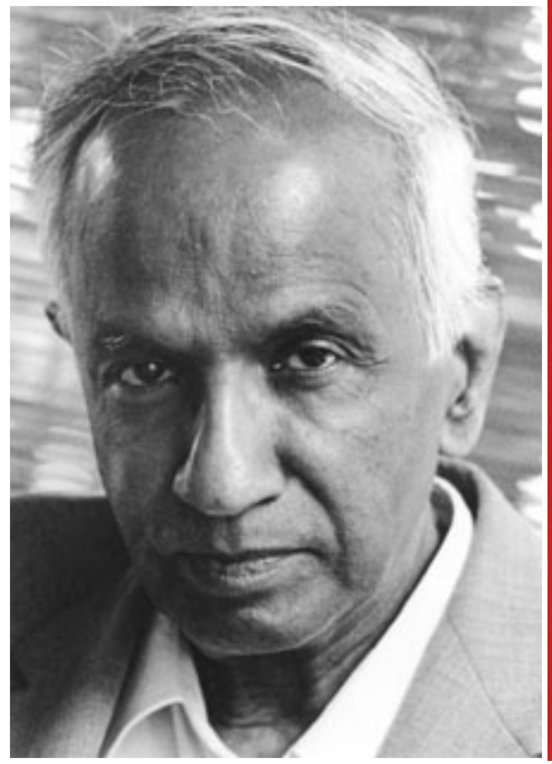
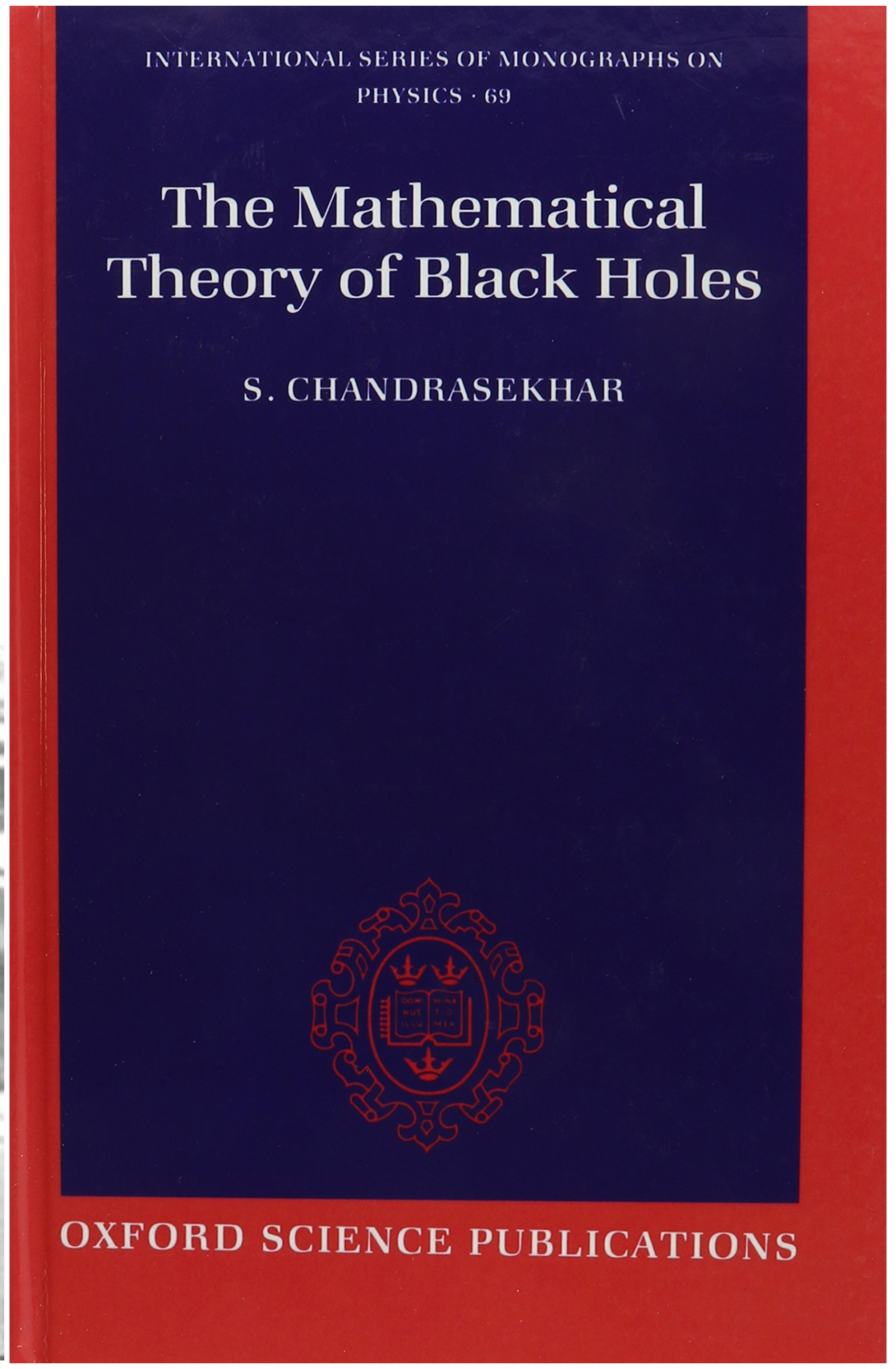
Subrahmanyan Chandrasekhar





"The Black Holes of Nature are the most perfect macroscopic objects there are in the Universe: The only elements in their construction are our concepts of space and time. And since the General Theory of Relativity provides only a single unique family of solutions for their description, they are the simplest objects as well".

Vacuum Einstein's equations: $R_{\mu\nu} = 0$

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Symmetry Reduction

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$\omega = -\,{}^{*}\xi_{(t)} \wedge \mathbf{d}\xi_{(t)}$. In vacuum: $\mathbf{d}\omega = 0 \;\Rightarrow\; \omega = \mathbf{d}\Omega$

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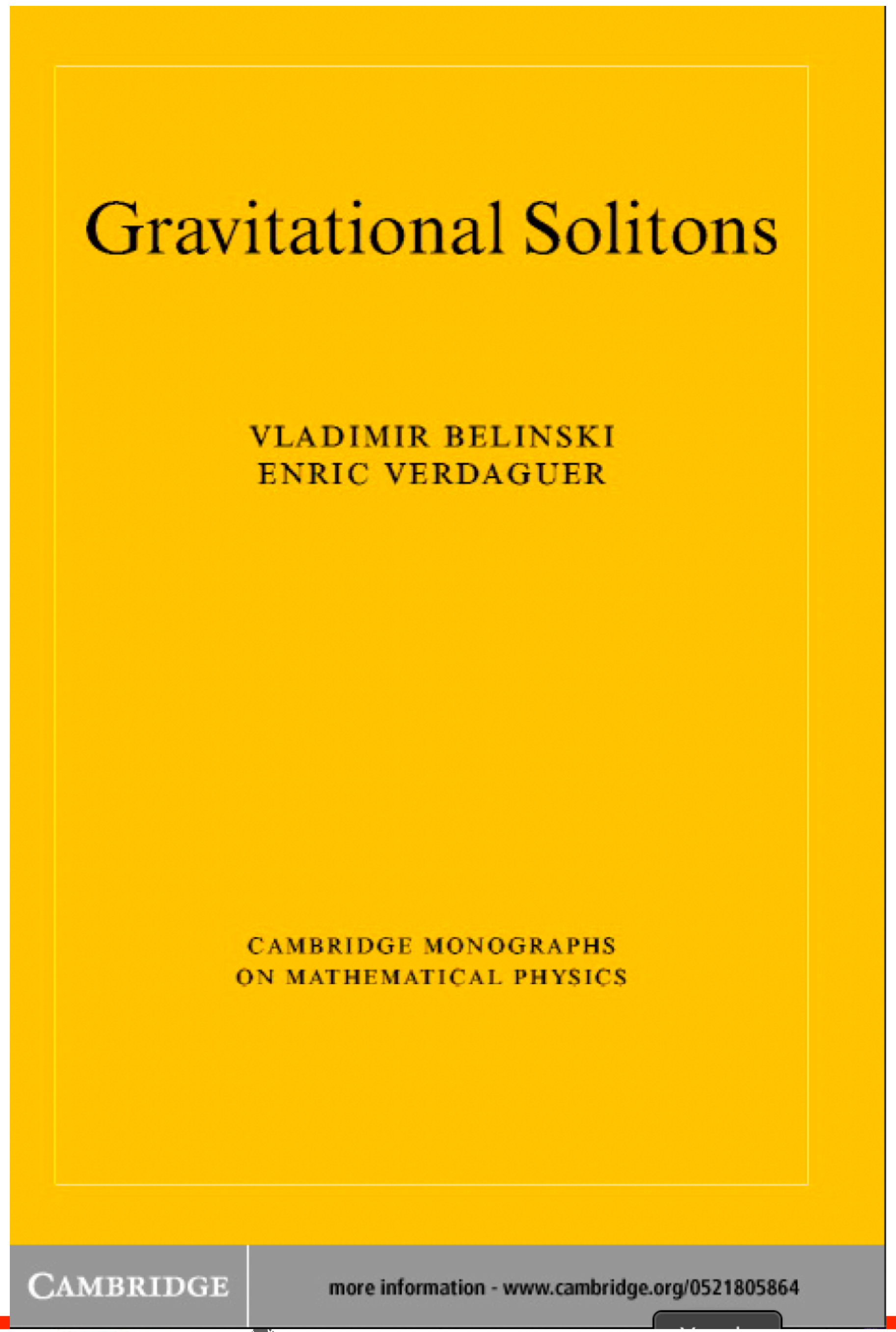
We can use the inverse scattering method



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Geometry Reduction

spacetime Weyl curvature tensor is Petrov type D

$$\tilde{C}_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma} + i * C_{\mu\nu\rho\sigma}$$

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Black Holes in General Relativity: Paths to the Kerr solution

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This is how the Kerr metric was first found!

VOLUME 11, NUMBER 5 PHYSICAL REVIEW LETTERS 1 SEPTEMBER 1963

GRAVITATIONAL FIELD OF A SPINNING MASS AS AN EXAMPLE OF ALGEBRAICALLY SPECIAL METRICS

Roy P. Kerr*

University of Texas, Austin, Texas and Aerospace Research Laboratories, Wright-Patterson Air Force Base, Ohio

(Received 26 July 1963)

The Kerr metric in Boyer-Lindquist Coordinates

$$ds^2 = -\,dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + (r^2 + a^2)\sin^2\theta\, d\varphi^2 + \frac{2Mr}{\rho^2}(dt - a\sin^2\theta\, d\varphi)^2$$

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Separability of the equations for tests fields and for gravitational perturbations

Connection between Killing symmetries and algebraic structure

Fayos & CFS, CQG, **18**, 353 (2001); **19**, 5489 (2002)

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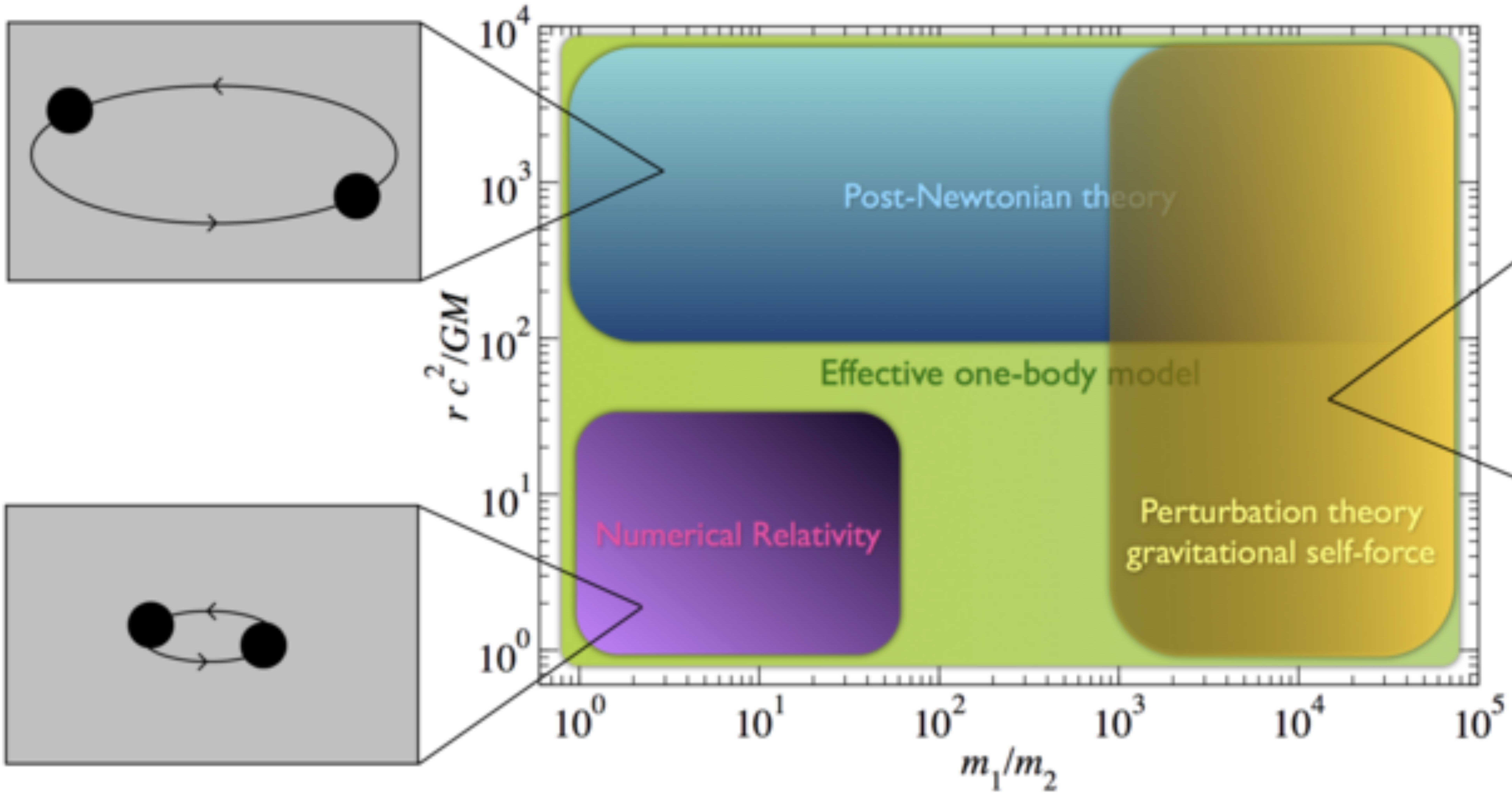
In Kerr, when $\xi = \xi_{(t)} = \partial/\partial t$, the Papapetrou field is the electromagnetic field of the Kerr-Newman solution.

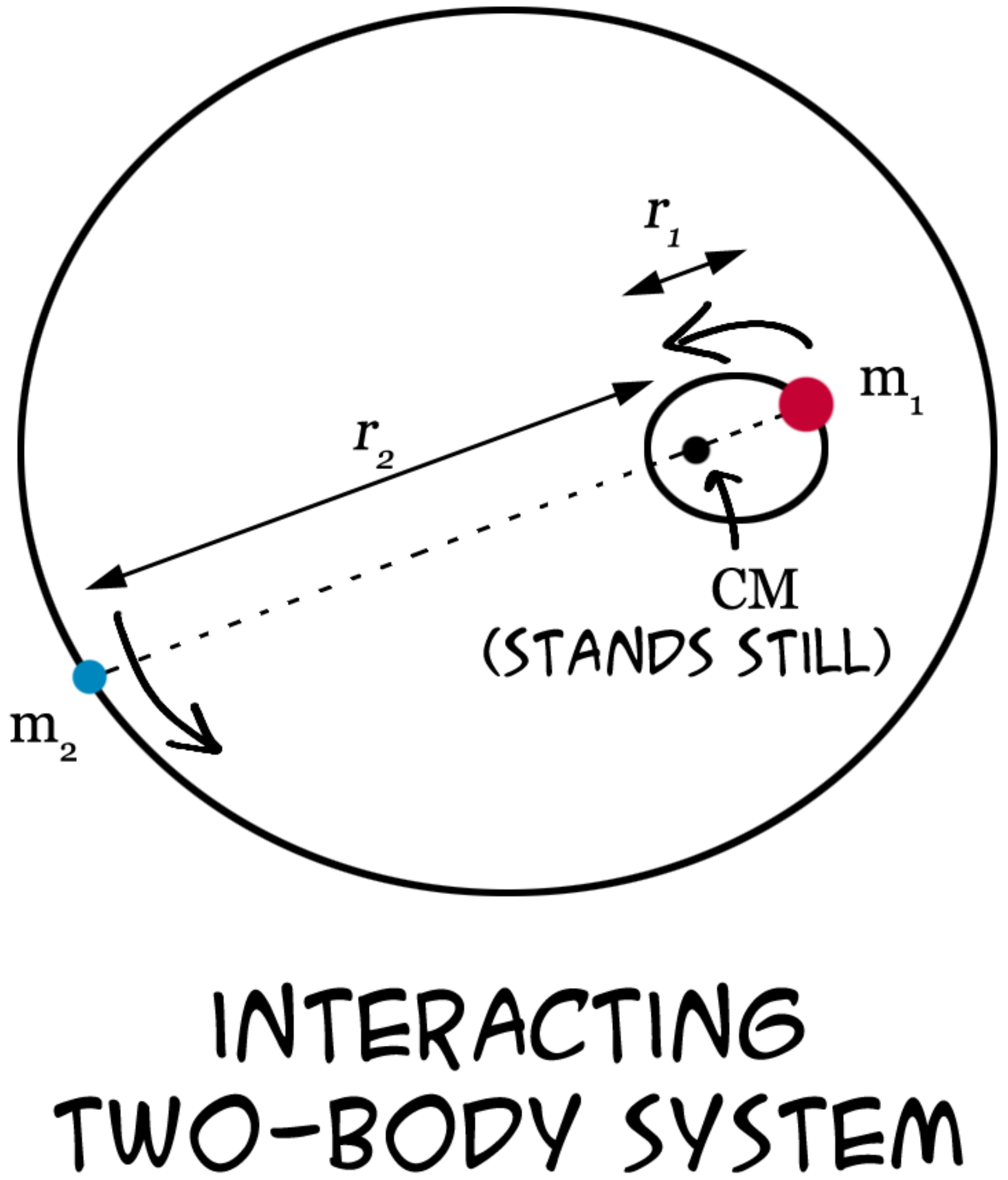
The BBH Problem
(the GR Two-Body Problem)

Mathematical approaches to the GR two-body dynamics



Credit: Buonanno & Sathyaprakash (2014)

The Newtonian Two-Body Problem



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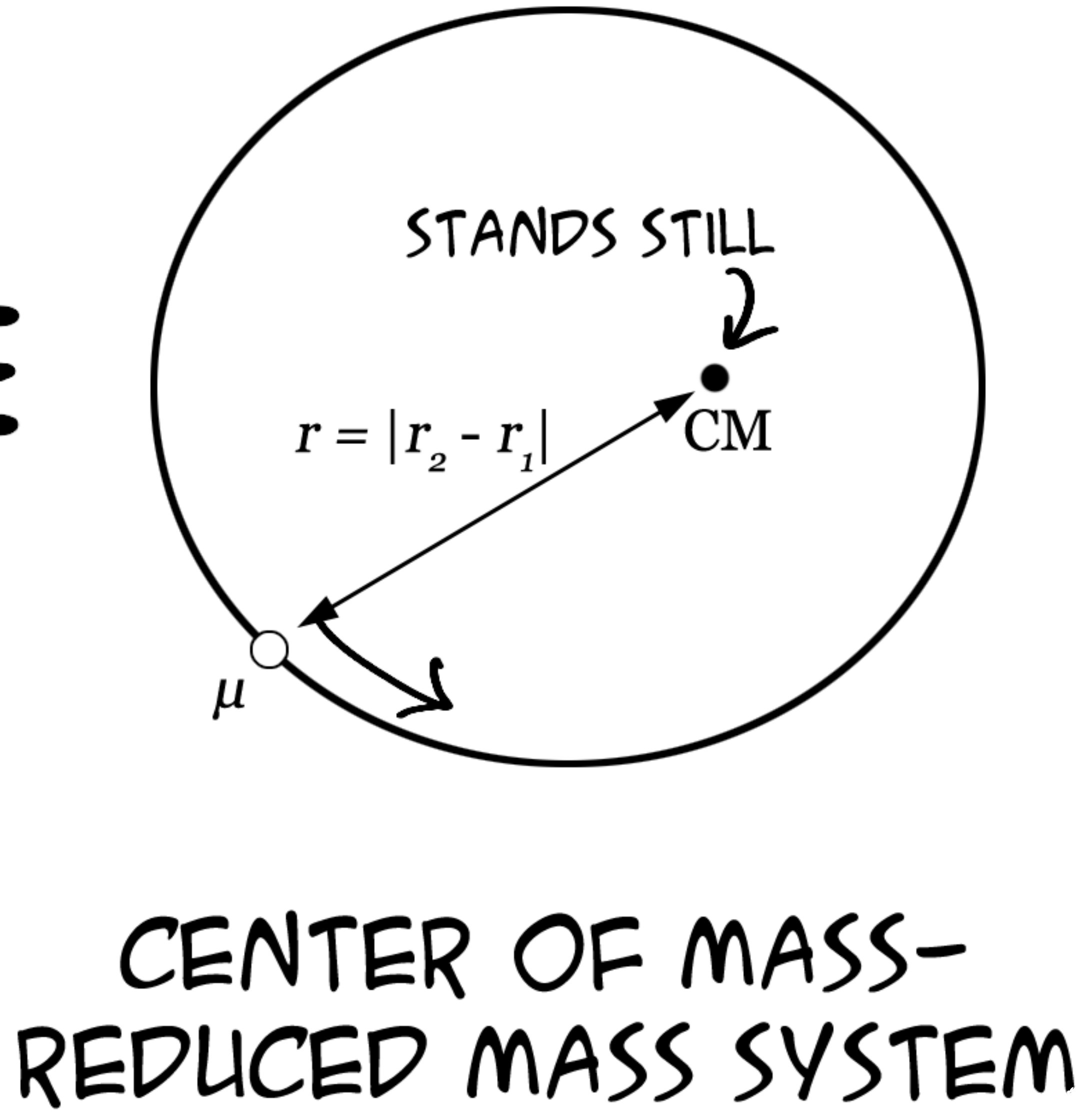

$M = m_1 + m_2$: Total Mass

$\mu = \dfrac{m_1 m_2}{m_1 + m_2}$: Reduced Mass

$q = \dfrac{m_1}{m_2}$: Mass Ratio

$\nu = \dfrac{\mu}{M}$: Symmetric Mass Ratio

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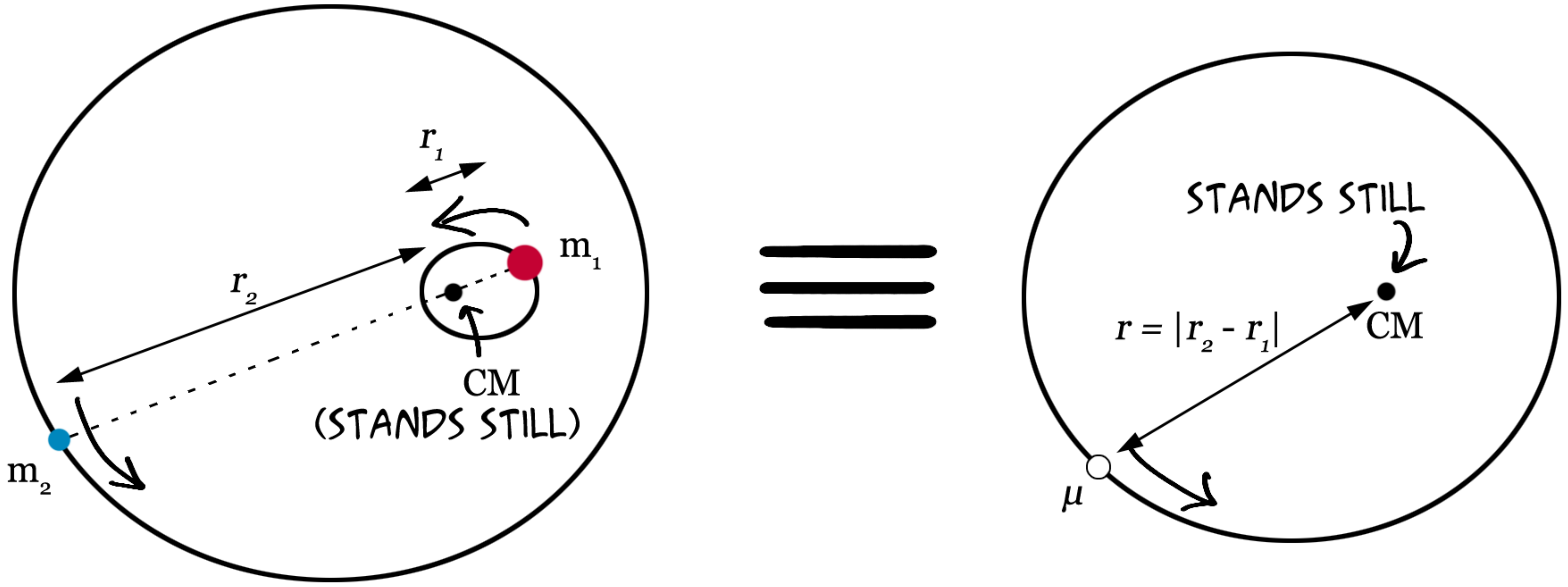

INTERACTING TWO-BODY SYSTEM ≡ CENTER OF MASS-REDUCED MASS SYSTEM

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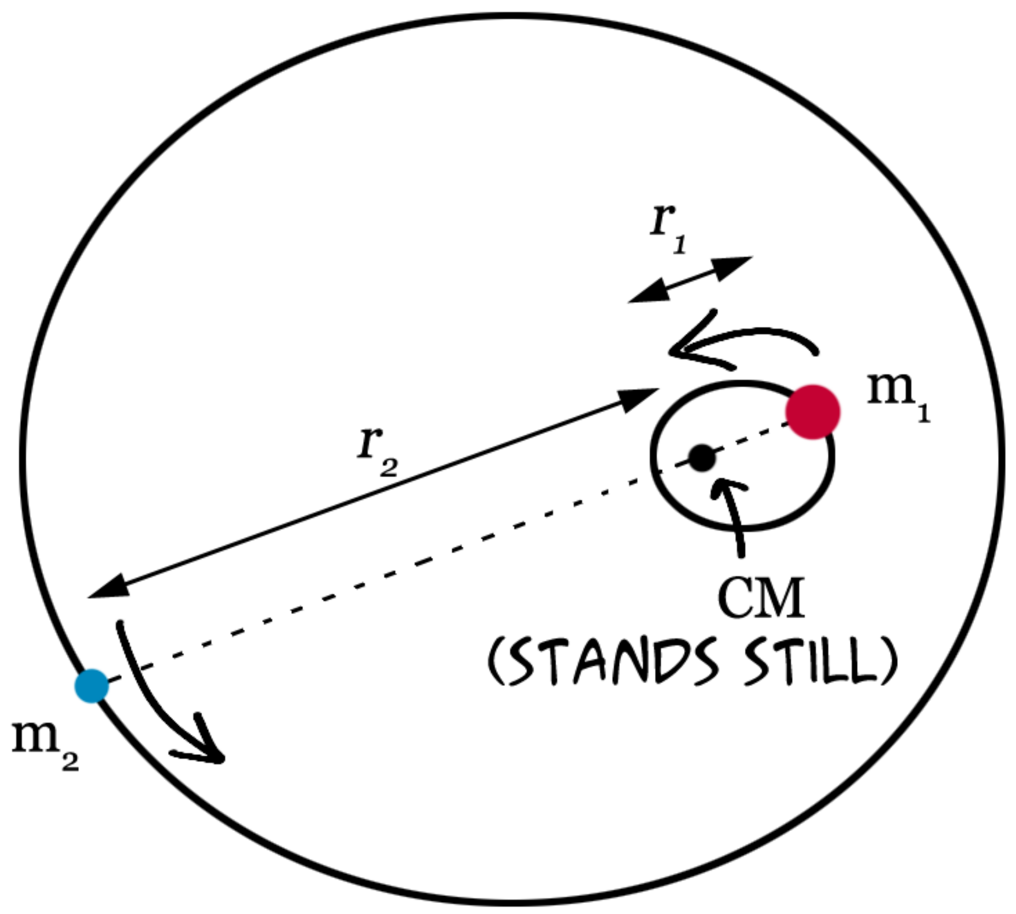

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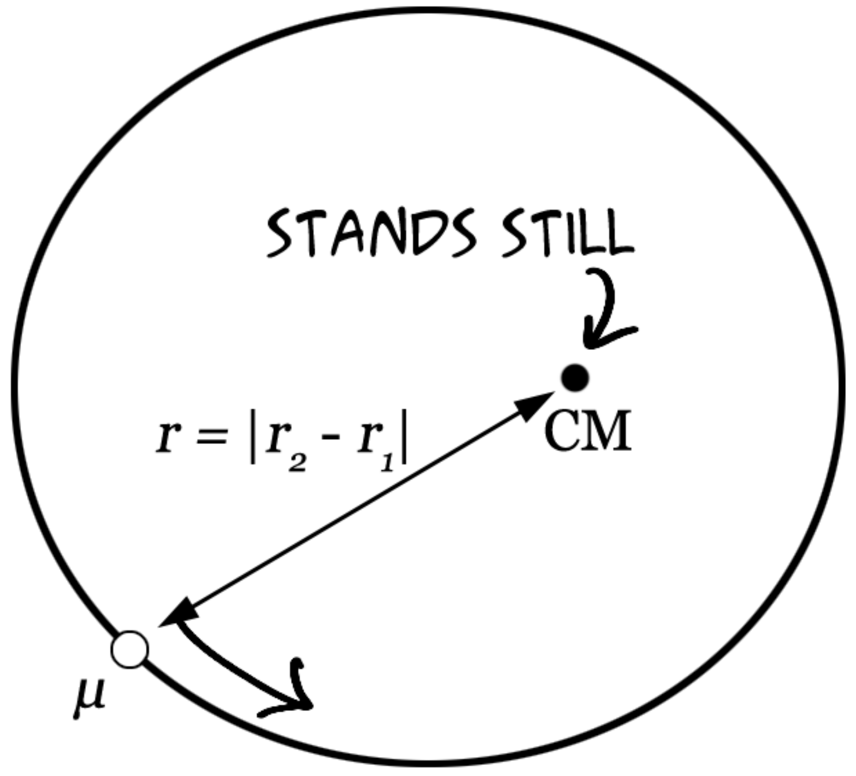

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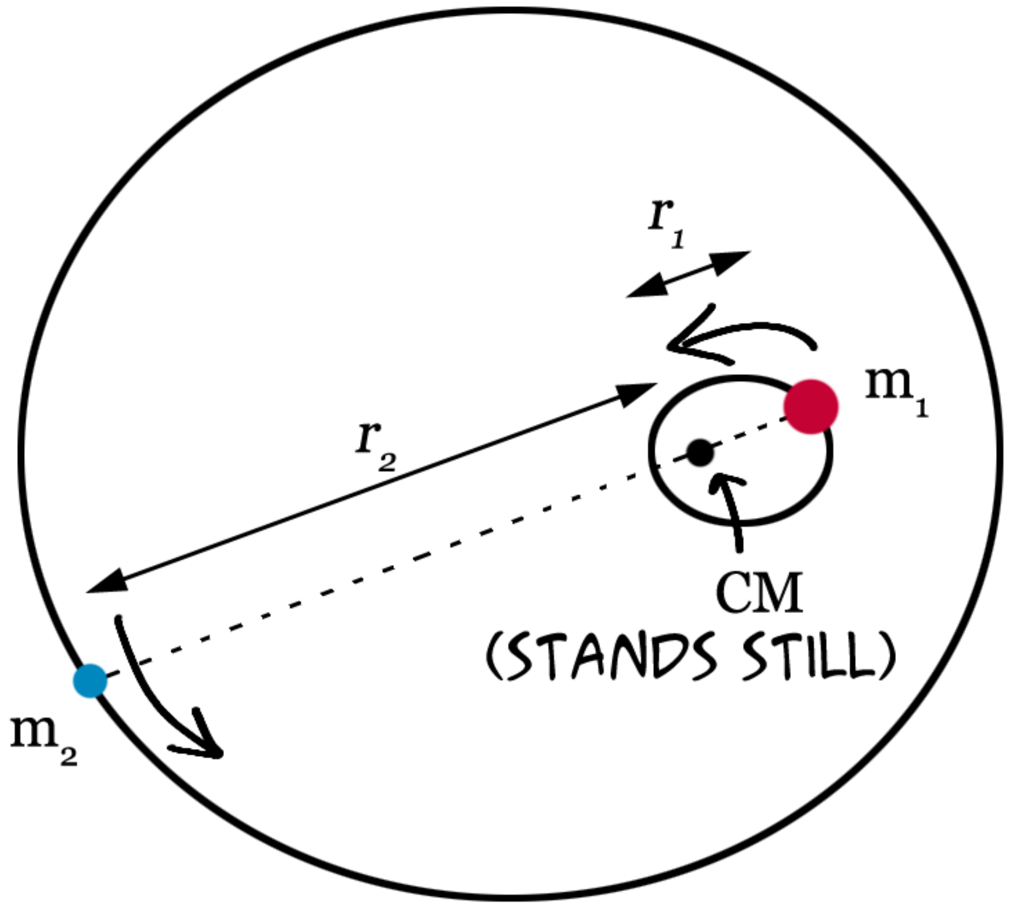

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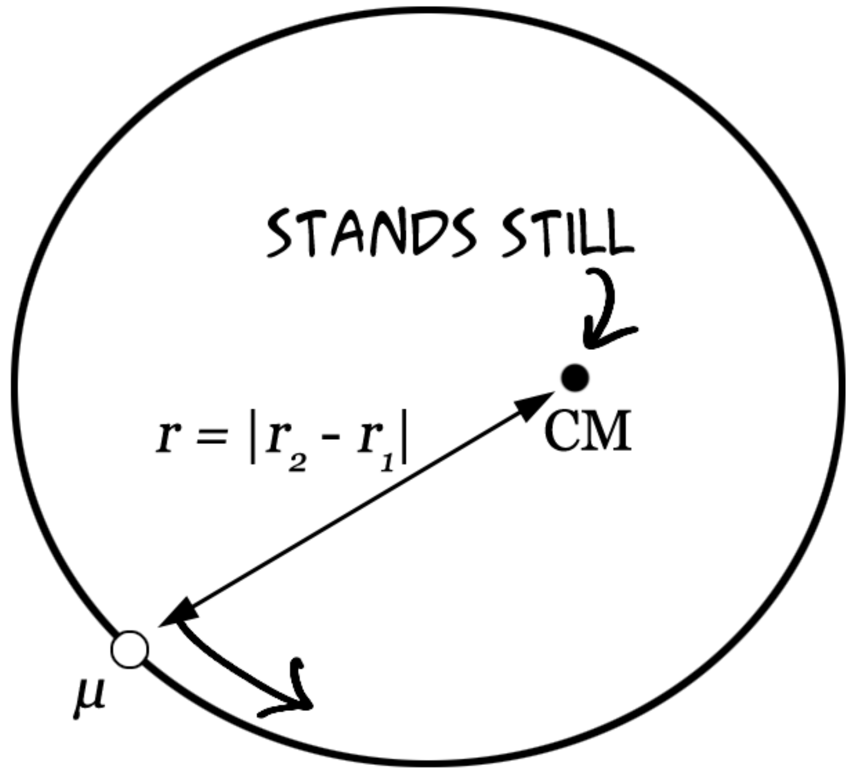

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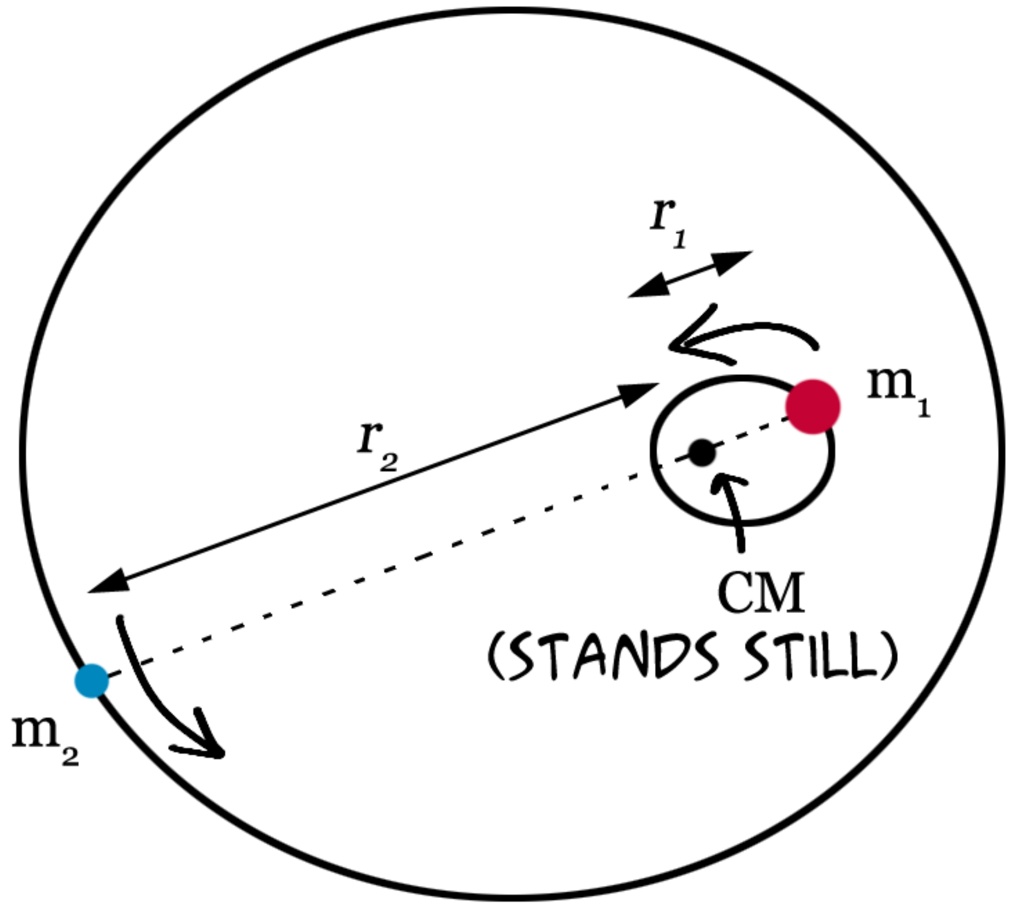
Conservation of Energy and Angular Momentum

The Newtonian Two-Body Problem



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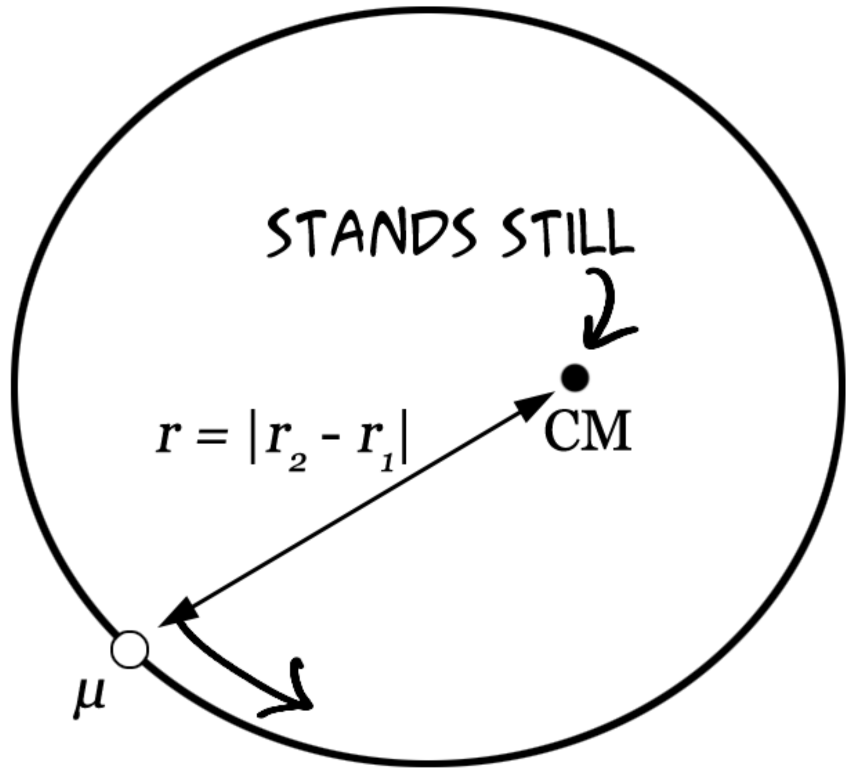

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Conservation of Energy and Angular Momentum

"Hidden" symmetry: Runge-Lenz vector

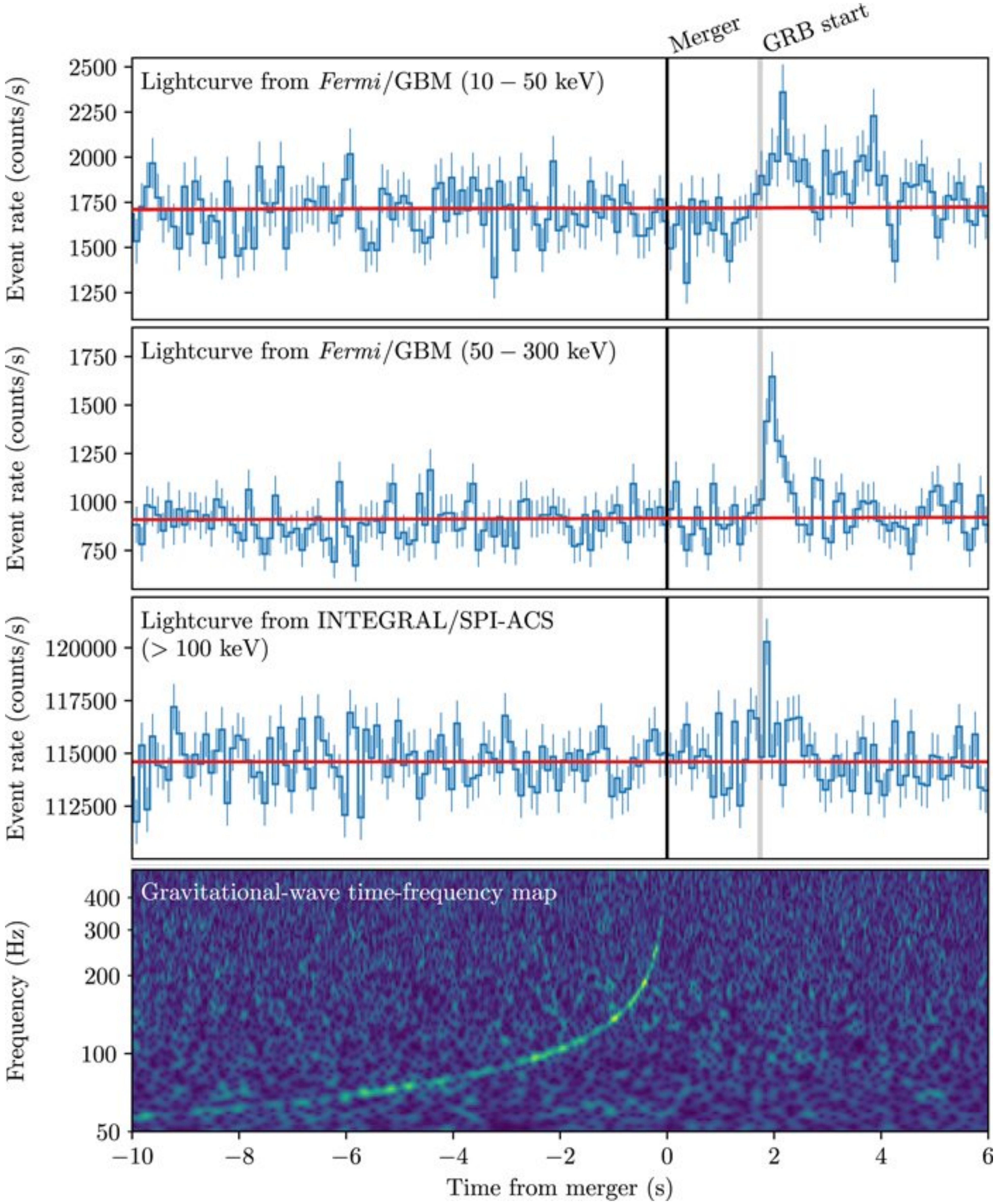
$$\mathbf{A} = \mu \dot{\mathbf{r}} \times \mathbf{L} - G\mu^2 M \hat{\mathbf{r}}$$

$$E = \frac{1}{2}\mu \dot{\mathbf{r}}^2 - \frac{G m_1 m_2}{r}$$

$$\mathbf{L} = \mu \mathbf{r} \times \dot{\mathbf{r}}$$

The “Newtonian” approximation to the Inspiral of Compact Binaries

Newtonian approximation: Newtonian orbital motion + Quadrupolar GW emission + Quadrupolar waveforms.



GW170817

Binary neutron star merger

A LIGO / Virgo gravitational wave detection with associated electromagnetic events observed by over 70 observatories.

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Mass quadrupole moment

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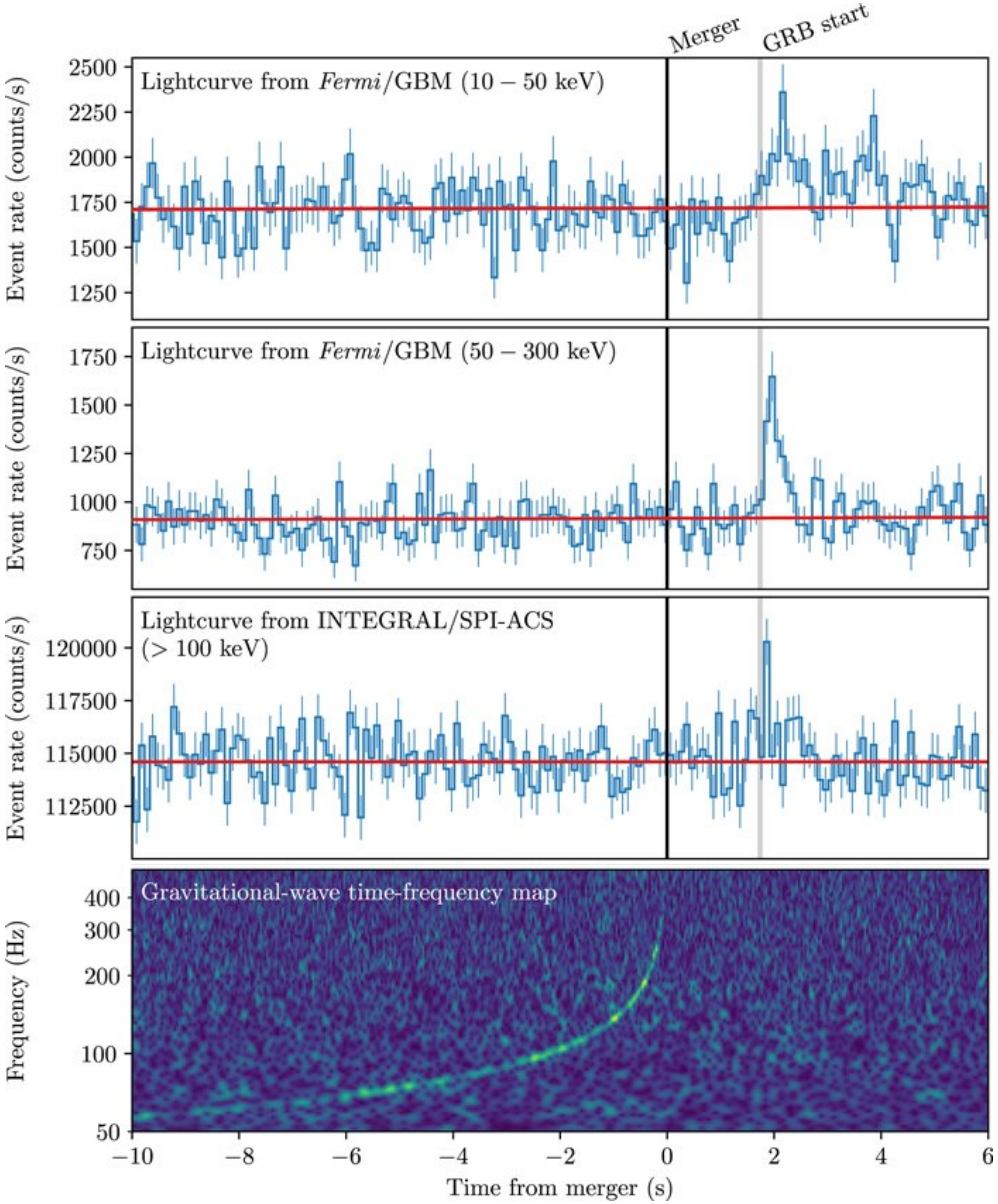

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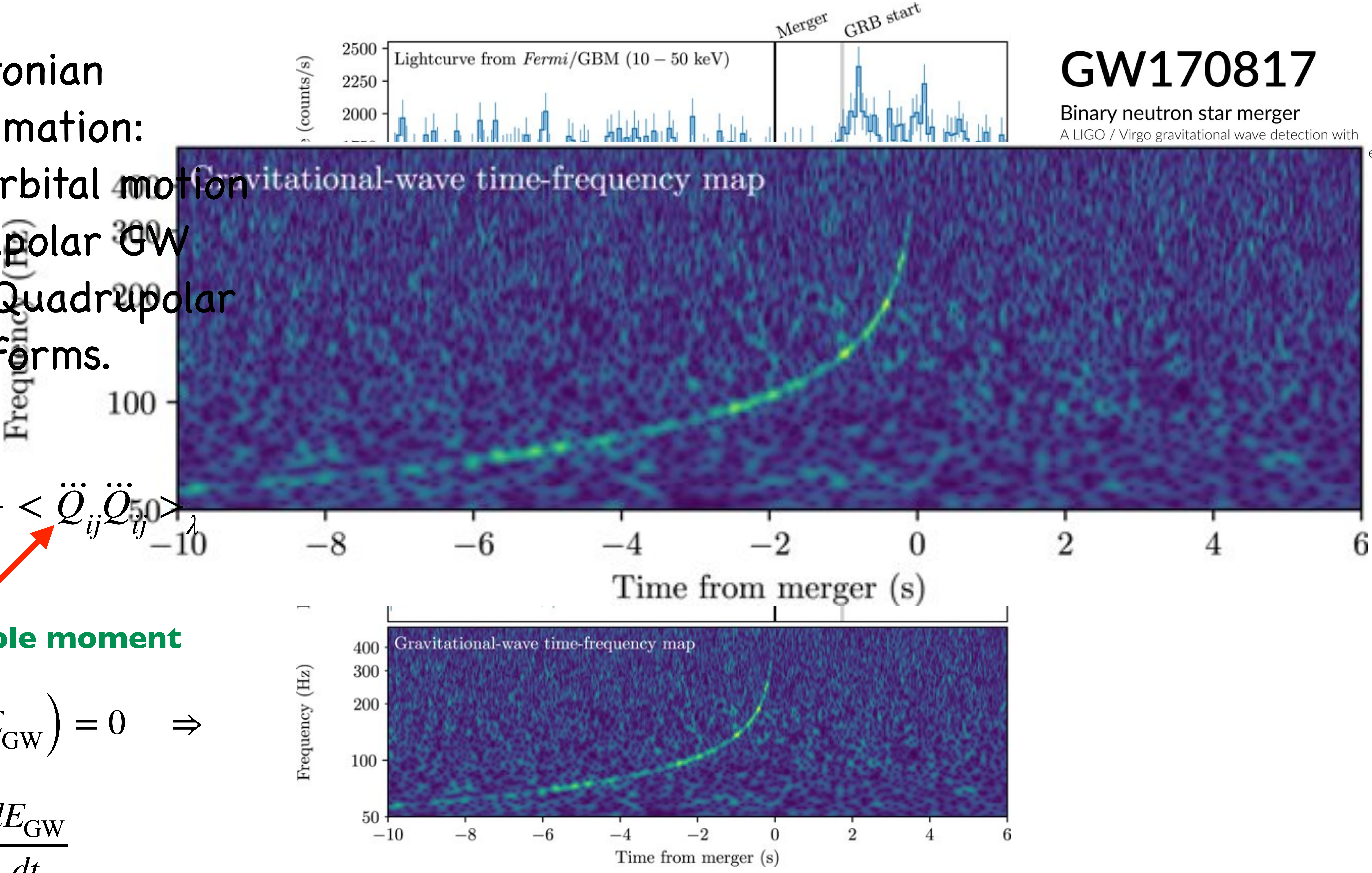

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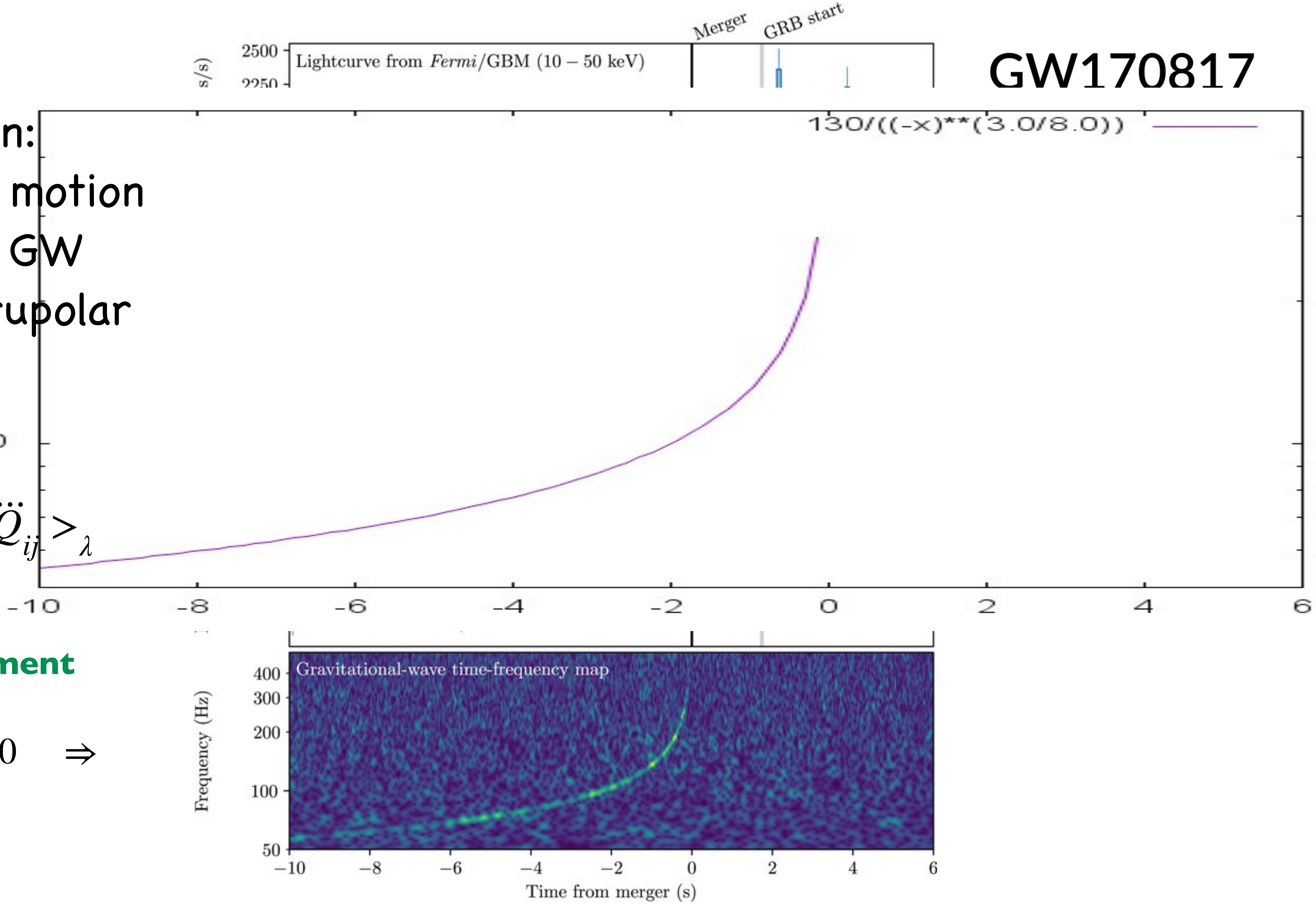

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Numerical Relativity

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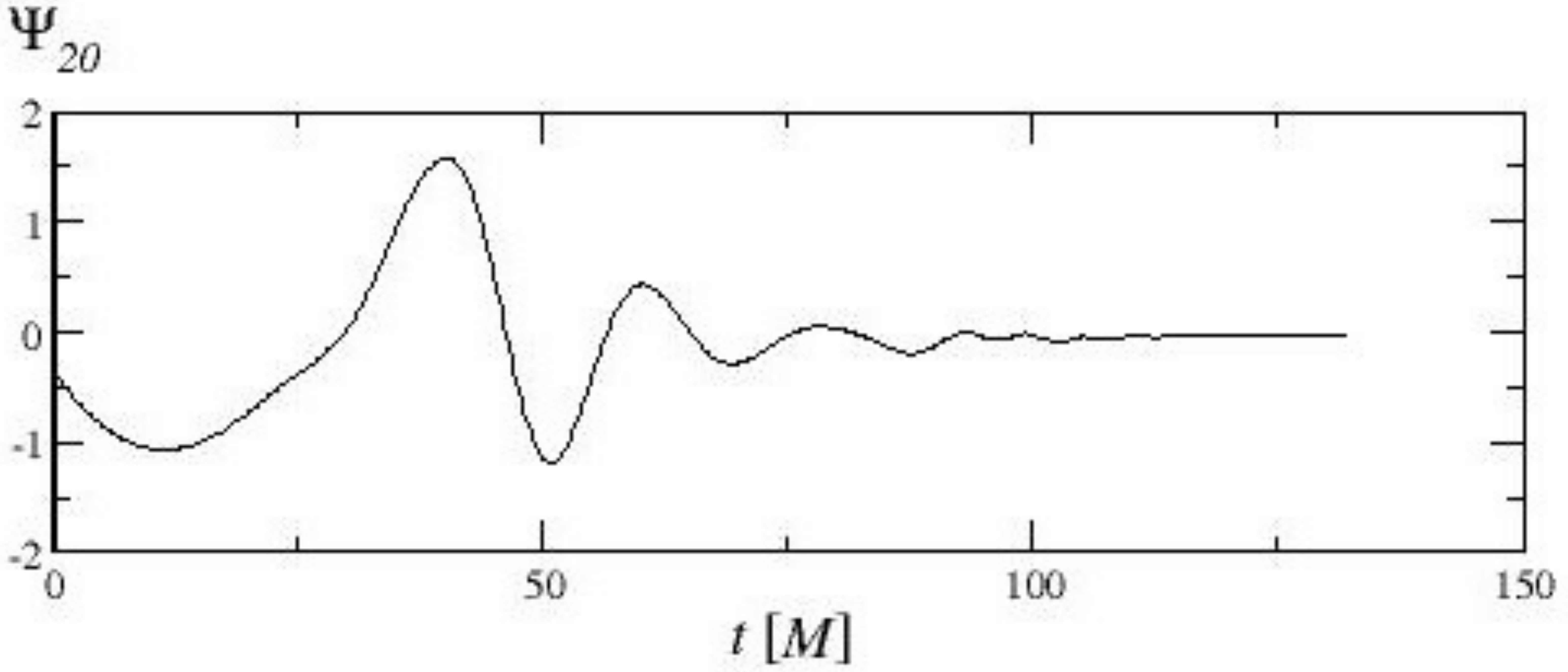
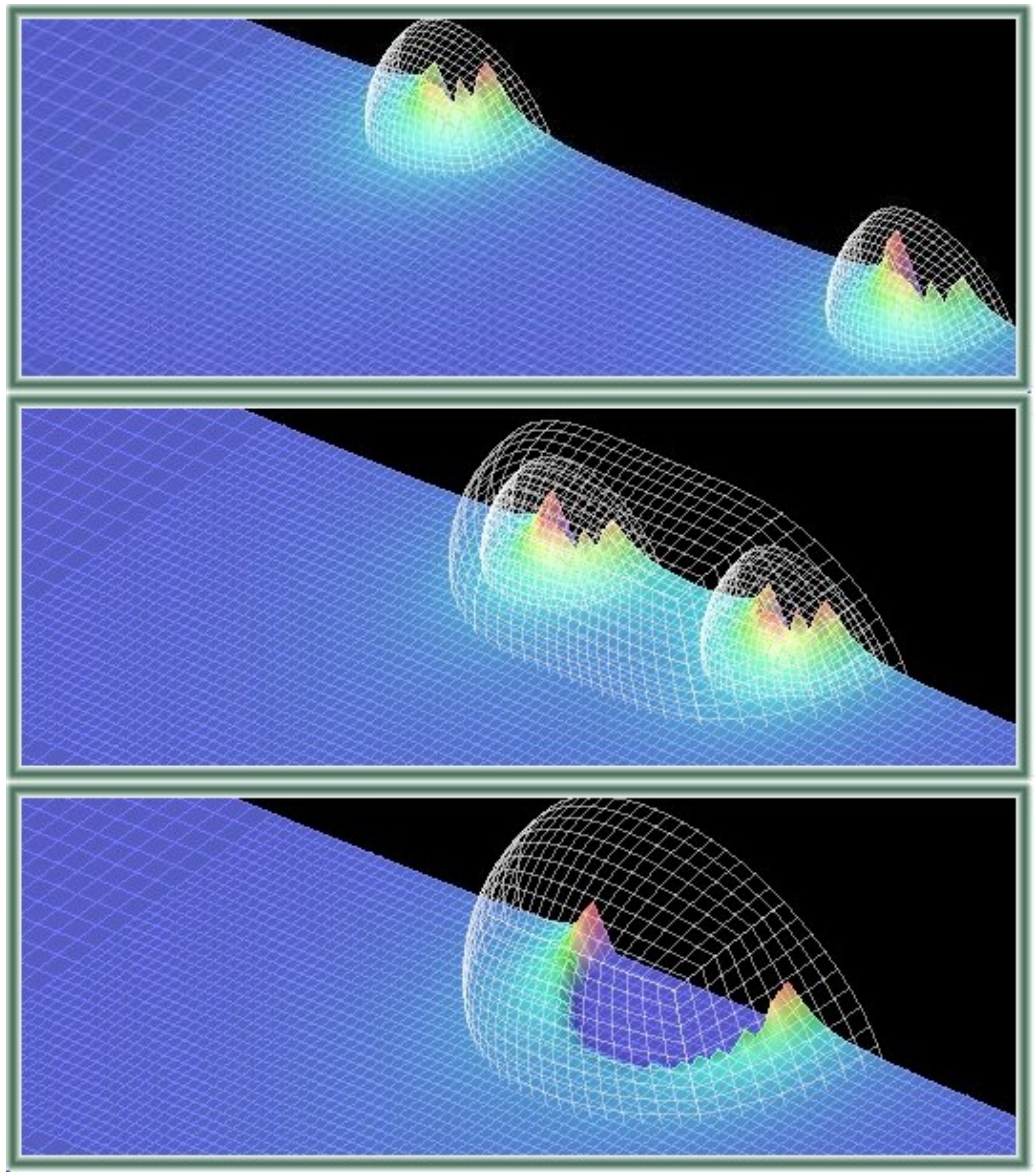
- First success: Head-on collisions of Black Holes:





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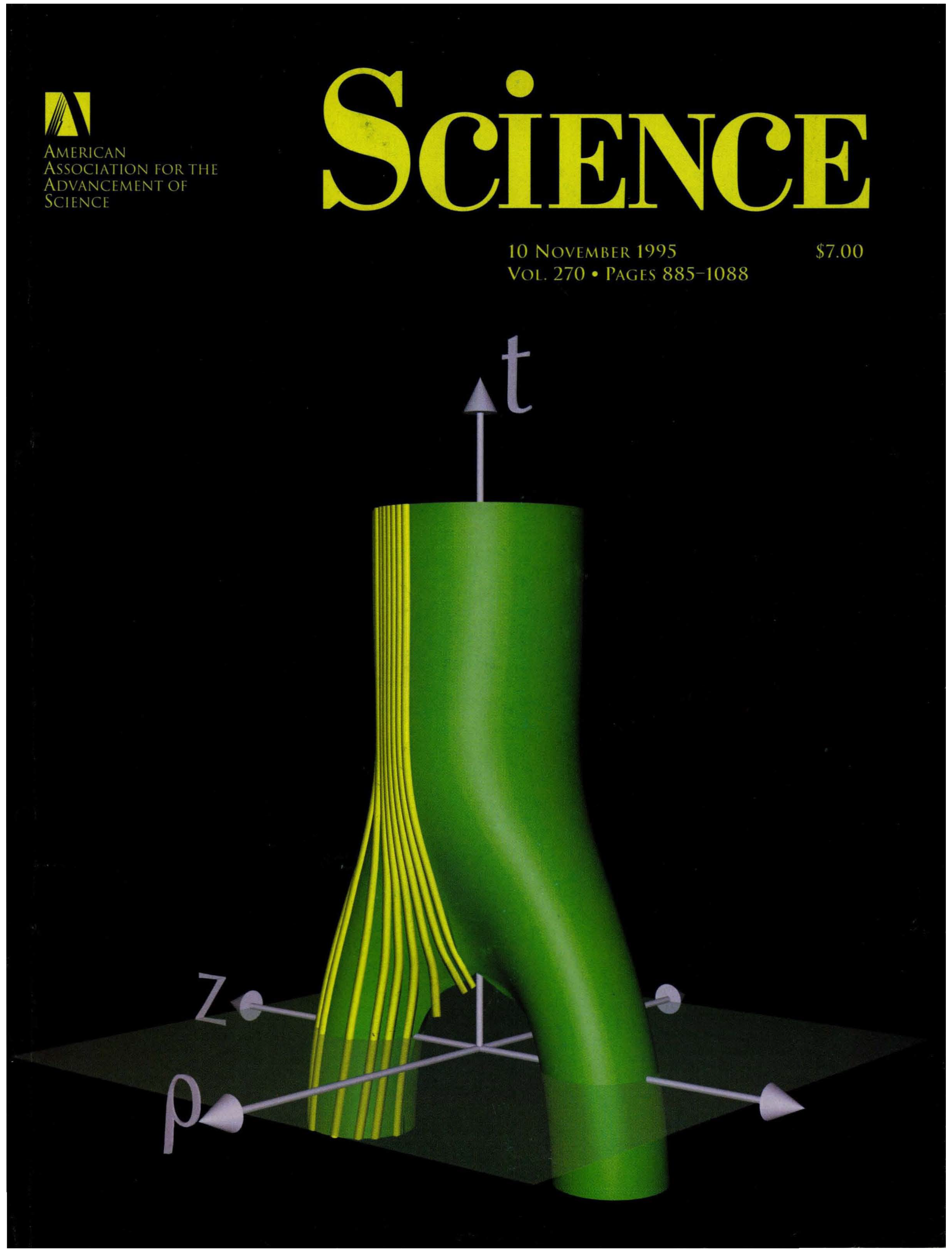
Geometry of a Black Hole Collision

Richard A. Matzner,* H. E. Seidel, Stuart L. Shapiro, L. Smarr, W.-M. Suen, Saul A. Teukolsky, J. Winicour

The Binary Black Hole Alliance was formed to study the collision of black holes and the resulting gravitational radiation by computationally solving Einstein's equations for general relativity. The location of the black hole surface in a head-on collision has been determined in detail and is described here. The geometrical features that emerge are presented along with an analysis and explanation in terms of the spacetime curvature inherent in the strongly gravitating black hole region. This curvature plays a direct, important, and analytically explicable role in the formation and evolution of the event horizon associated with the surfaces of the black holes.



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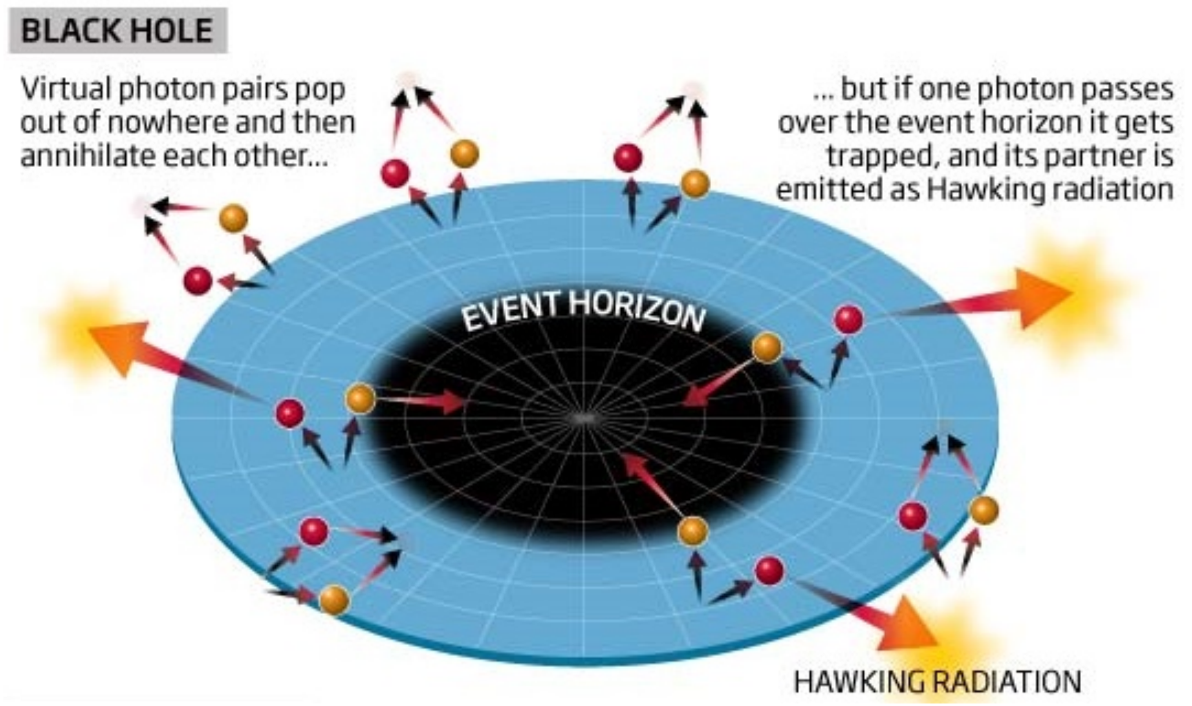
- The idea behind BH Perturbation Theory is that in a variety of physical processes around BHs, the spacetime metric can be well approximated by the BH metric (Kerr/Schwarzschild) plus perturbations.

$$g_{\mu\nu} = g_{\mu\nu}^{\mathrm{MBH}} + h_{\mu\nu}$$

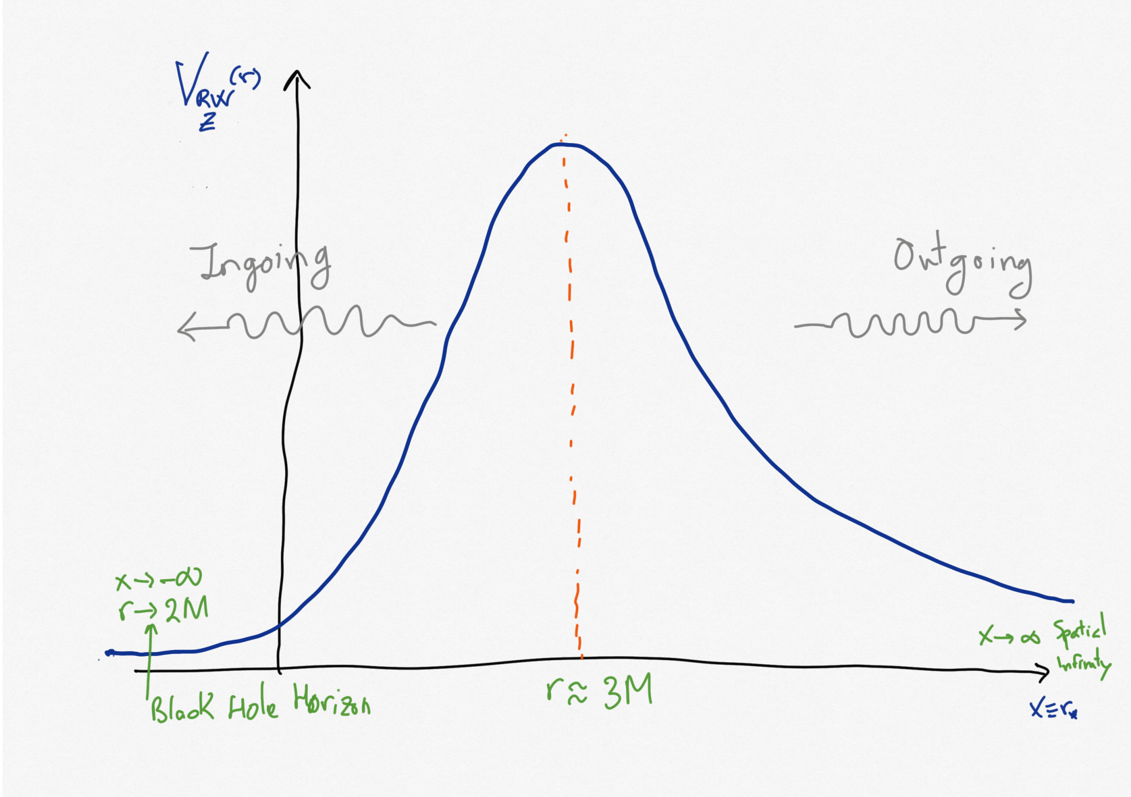
Scattering off a Black Hole

Incoming waves from spatial infinity. Penrose process for fields (spinning BHs: Blandford-Znajek process)



Outgoing waves from the horizon: Hawking radiation

Black Hole Quasinormal Oscillations



Damping Time

The quasinormal modes are solutions that satisfy the causal condition of purely ingoing waves crossing the event horizon, while at the same time behaving as purely outgoing waves reaching spatial infinity.

Frequency

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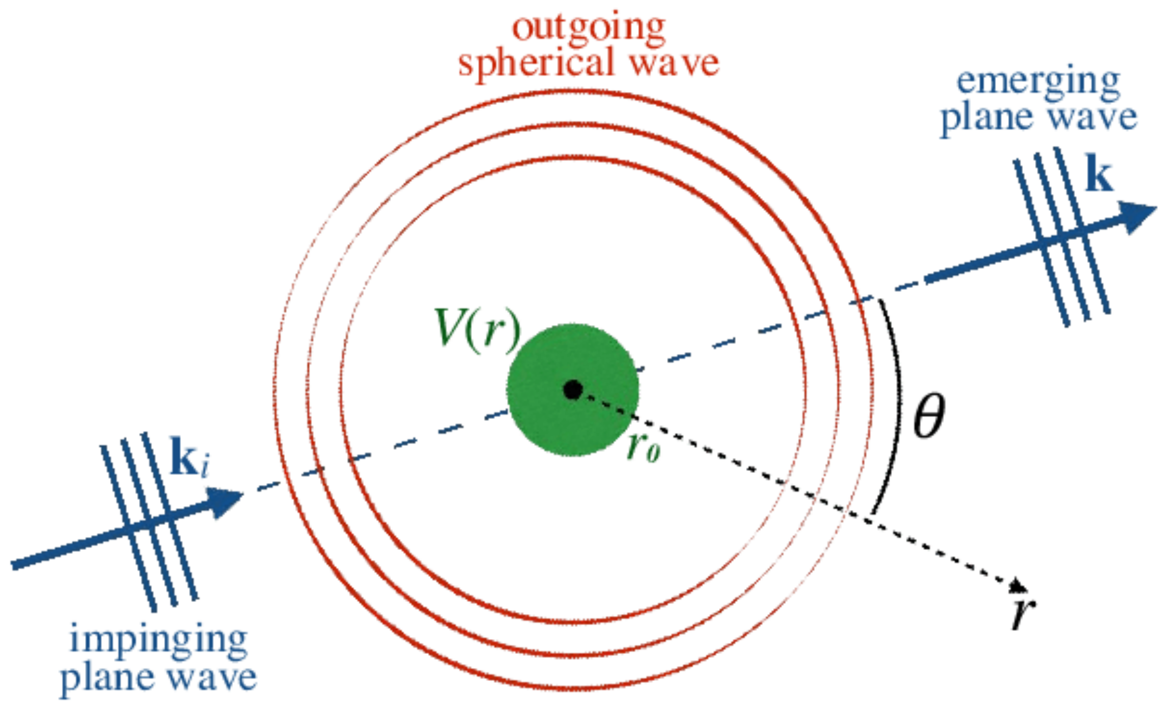

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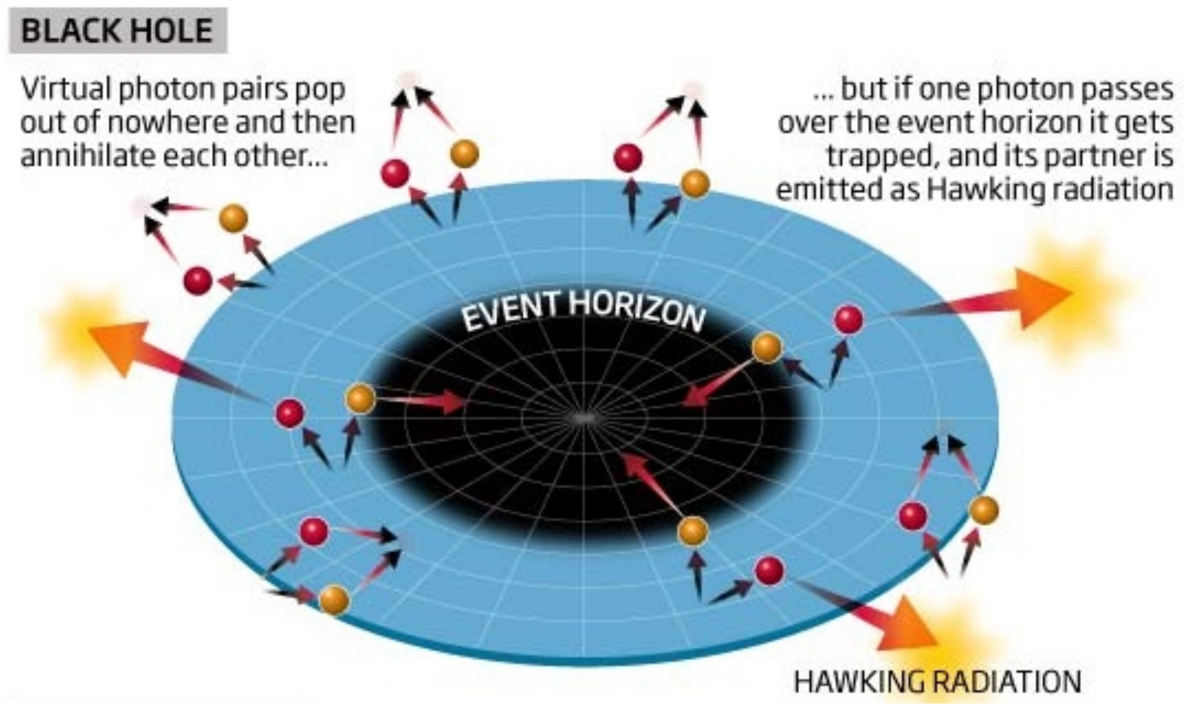

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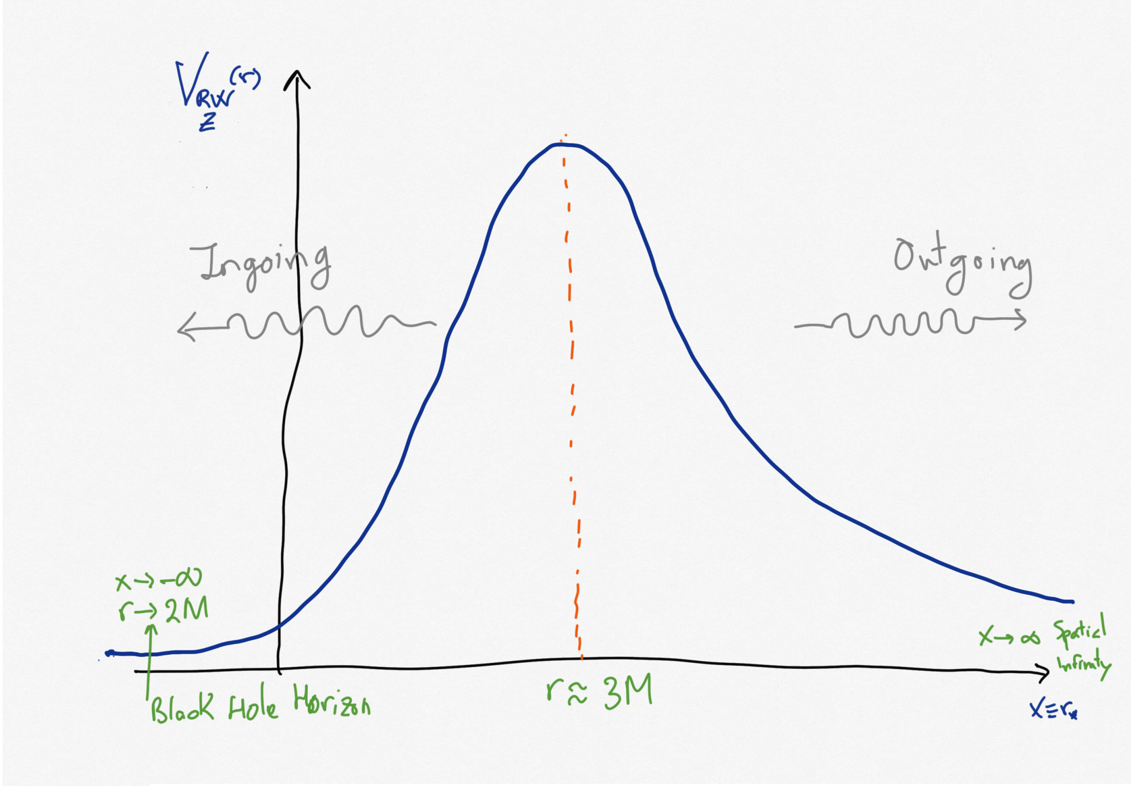

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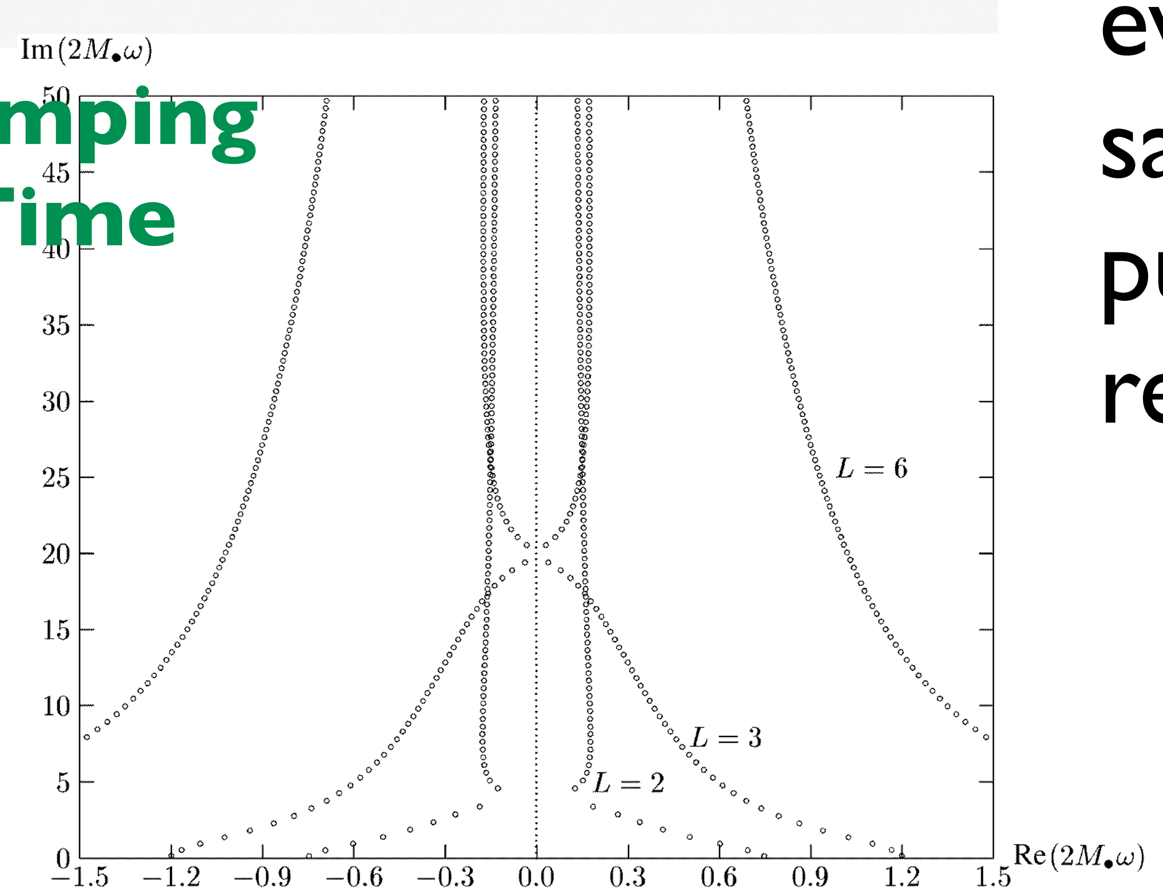

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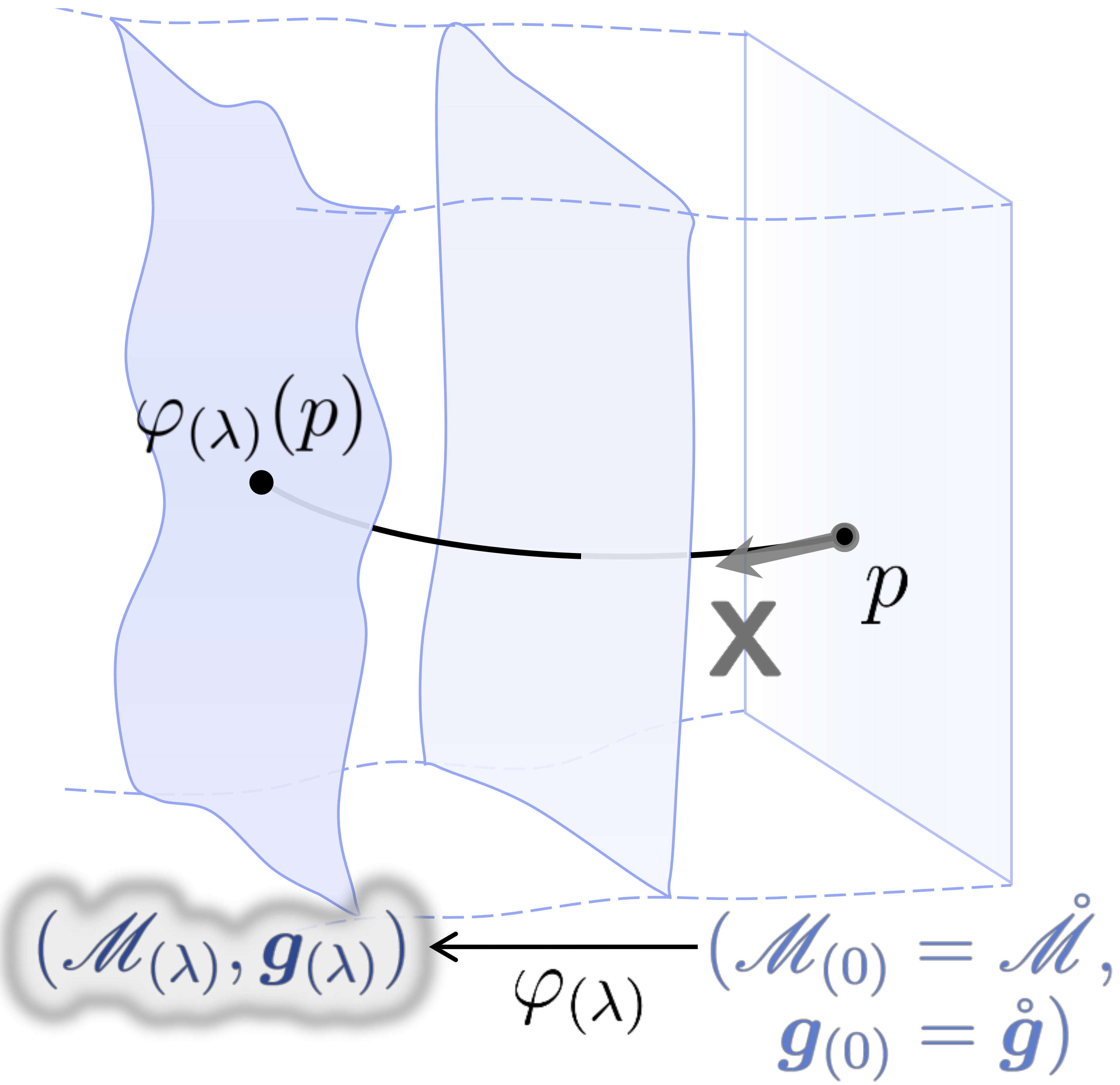
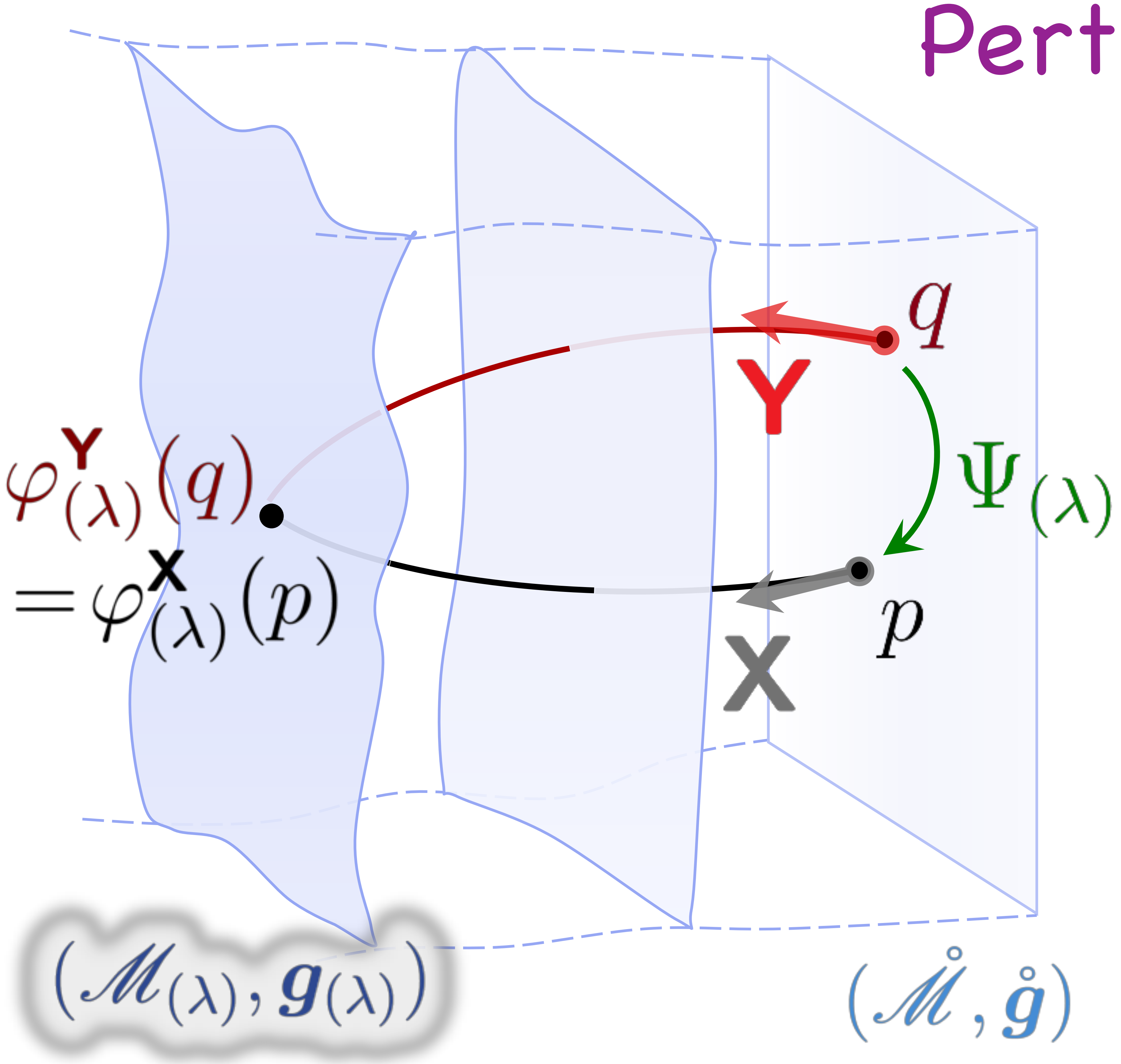


Perturbative Gauge



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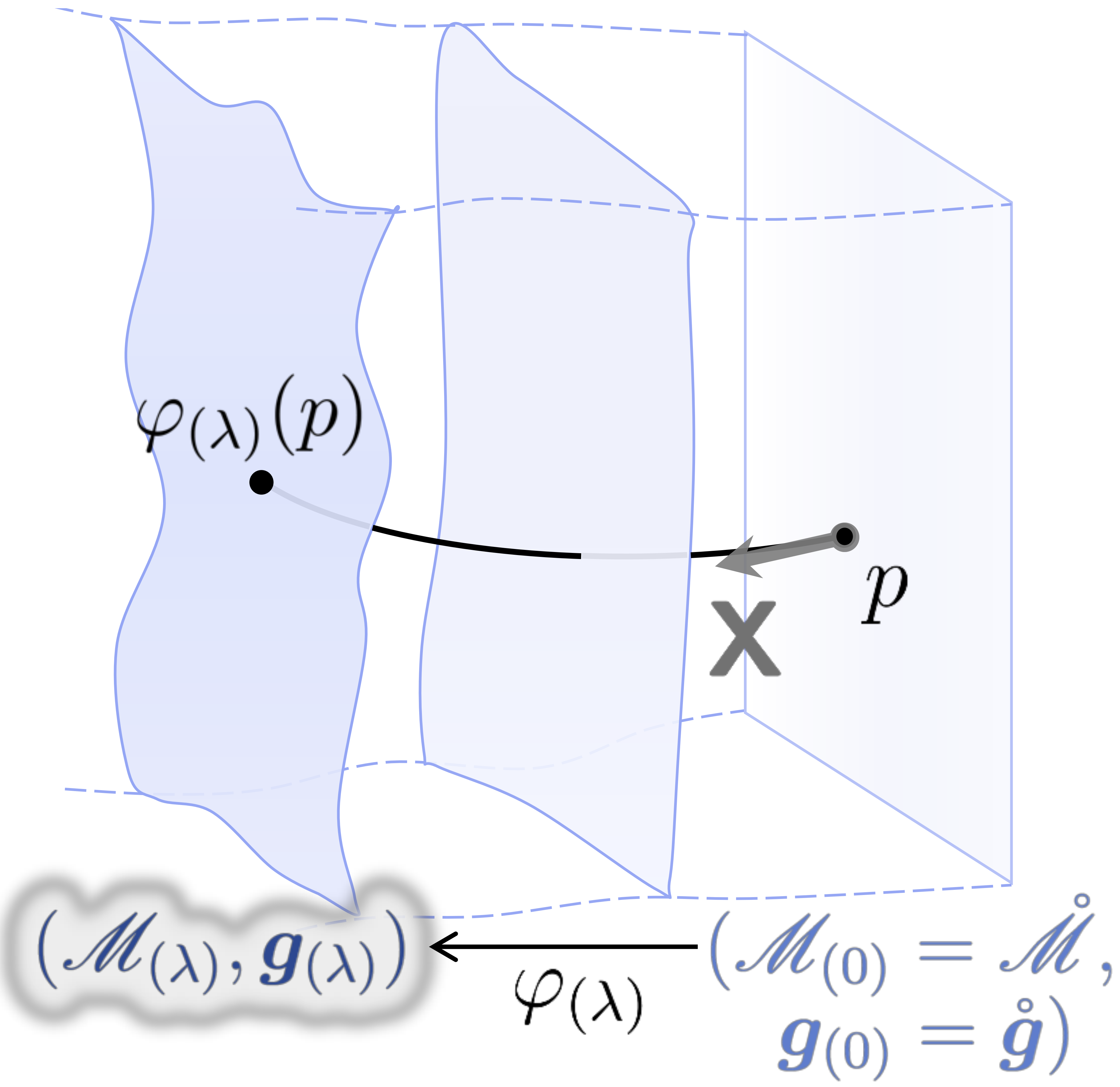

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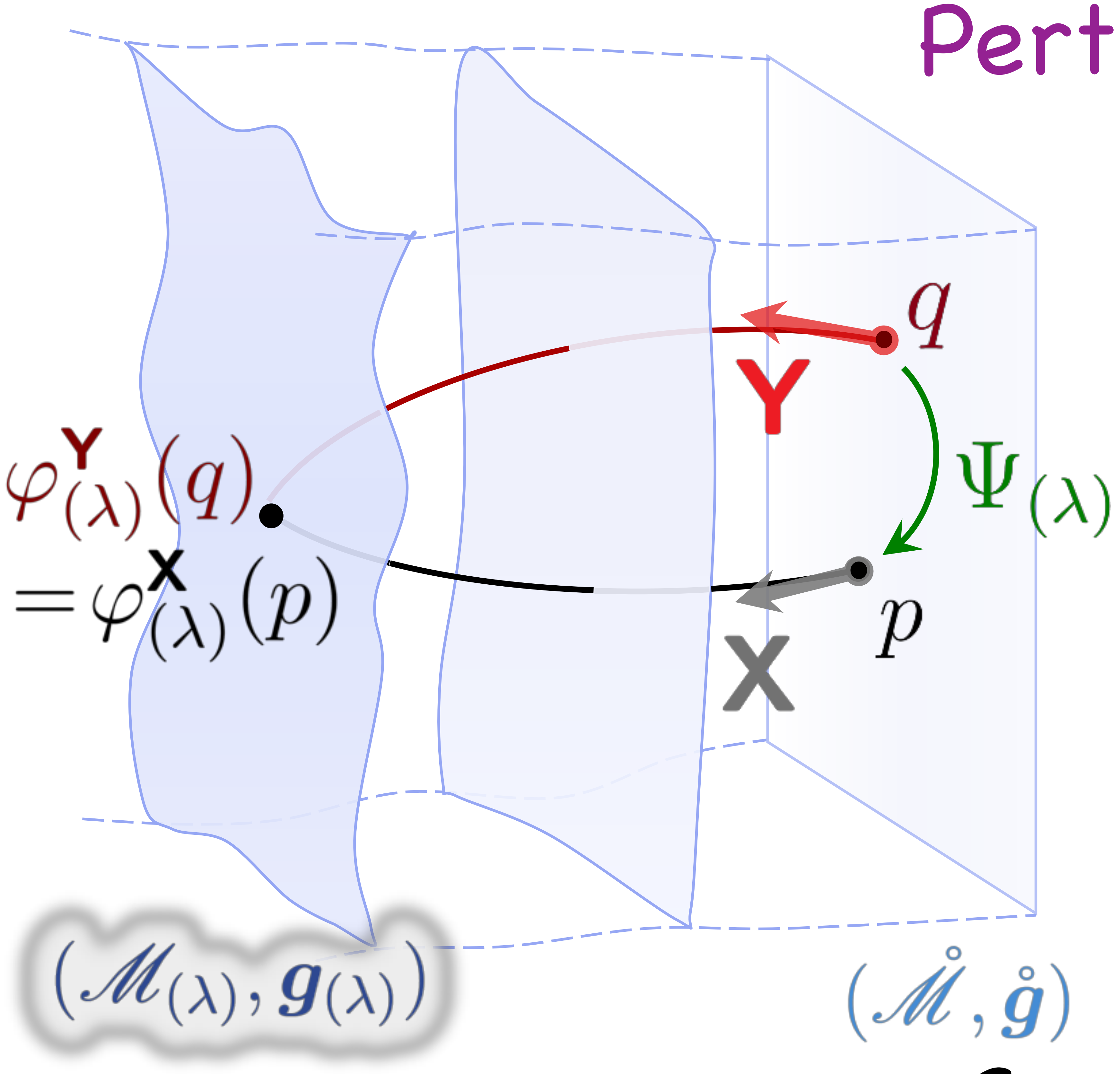

Gauge Transformation :

$$(\varphi^{Y})^{-1} \circ \varphi^{X}$$

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- The perturbed Einstein equations are:

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Zerilli Moncrief

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More details about the space of possible master functions and equations and hidden symmetries in M. Lenzi's talk.

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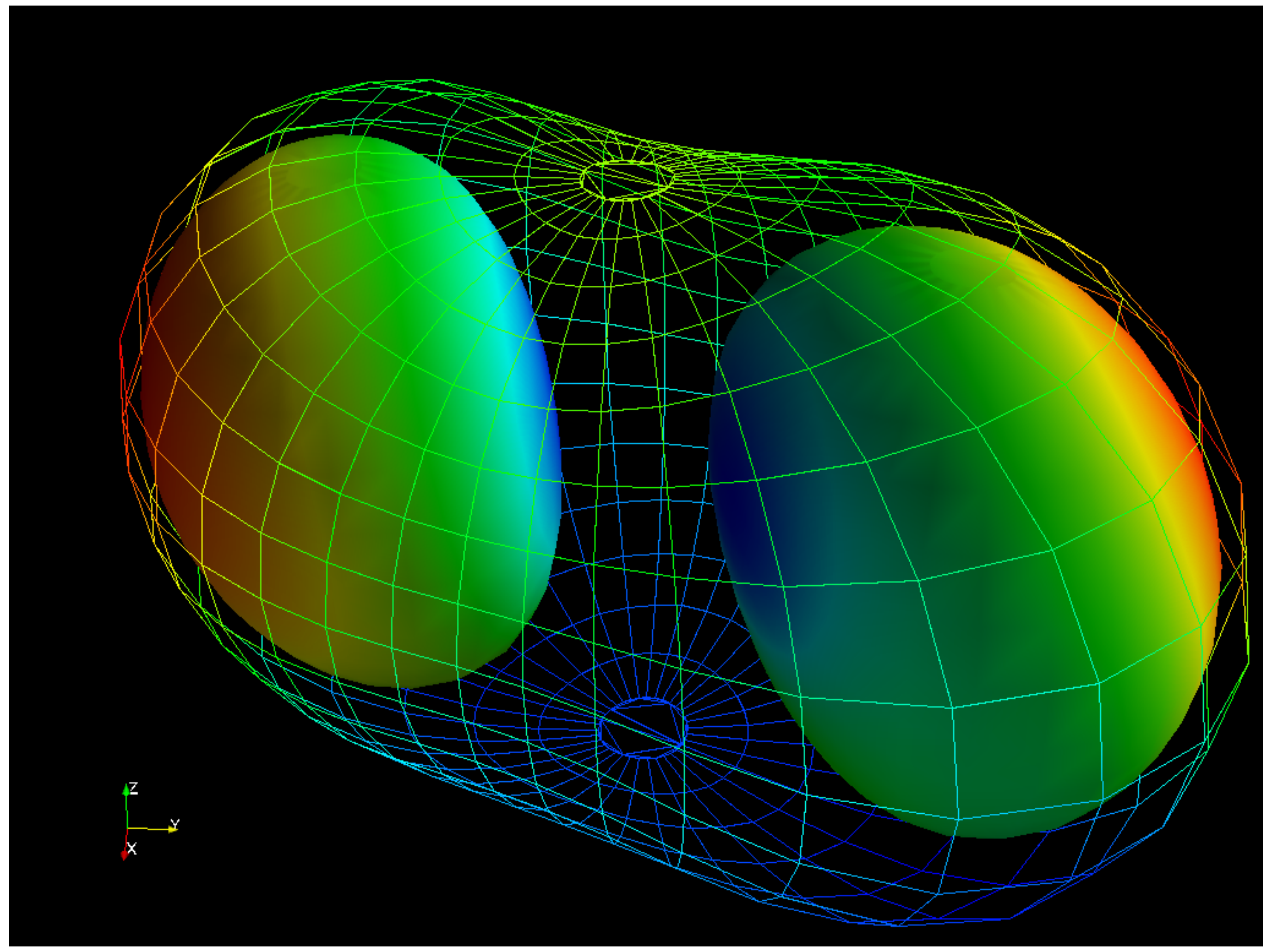
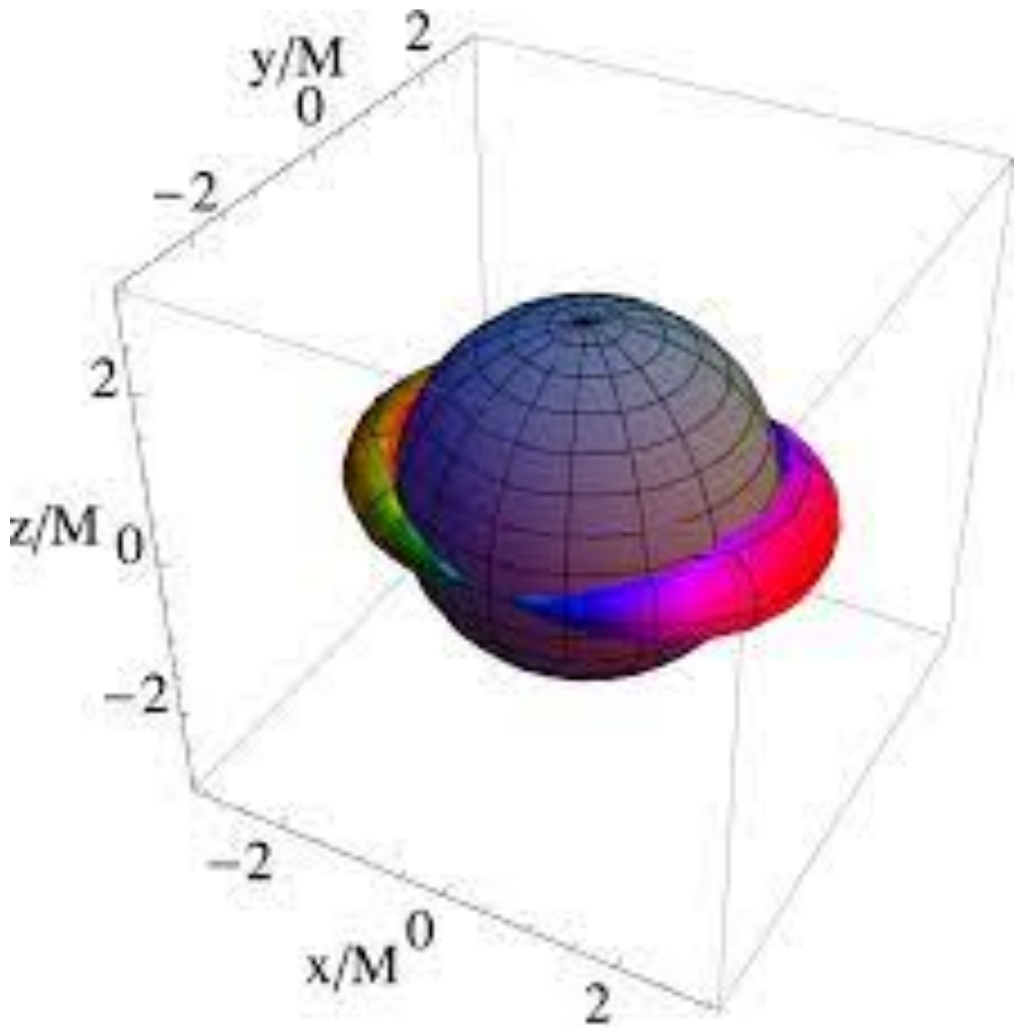
Black Hole Perturbation Theory

- The **Close Limit Approximation**:



Binary Black Hole evolution



Black Hole + Perturbations

At a given time t=to in the binary black hole evolution, a common **apparent horizon** forms. From the Numerical Relativity data at that slice t=to, we have all the information for the subsequent evolution: $(\alpha, \beta^i, h_{ij}, k_{ij})\,|_{t_o}$

From the Numerical Relativity data we construct:

$$\left\{ h^{\ell m}_{\mu\nu}[\alpha_o, \beta^i_o, q^o_{ij}, k^o_{ij}], \dot{h}^{\ell m}_{\mu\nu}[\alpha_o, \beta^i_o, q^o_{ij}, k^o_{ij}] \right\}$$

and from here:

$$\left\{ \Psi^{\ell m}[\alpha_o, \beta^i_o, q^o_{ij}, k^o_{ij}], \dot{\Psi}^{\ell m}[\alpha_o, \beta^i_o, q^o_{ij}, k^o_{ij}] \right\}$$

which can be evolved using the master equations of BH perturbation theory.

Black Hole Perturbation Theory

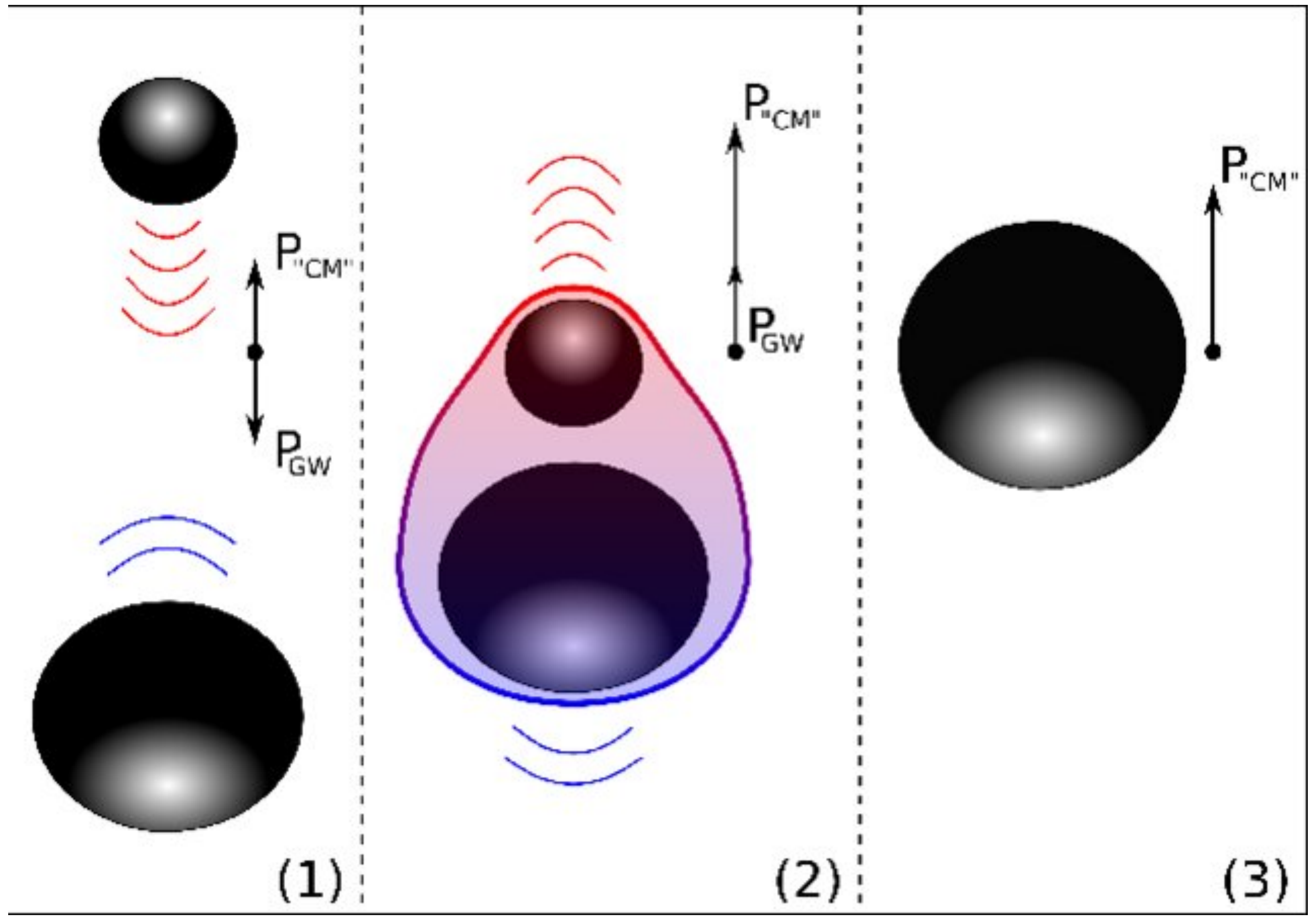
- The **Close Limit Approximation**:

PHYSICAL REVIEW LETTERS

VOLUME 77 | 25 NOVEMBER 1996 | NUMBER 22

Colliding Black Holes: How Far Can the Close Approximation Go?

Reinaldo J. Gleiser,[1] Carlos O. Nicasio,[1,2] Richard H. Price,[3] and Jorge Pullin[2]



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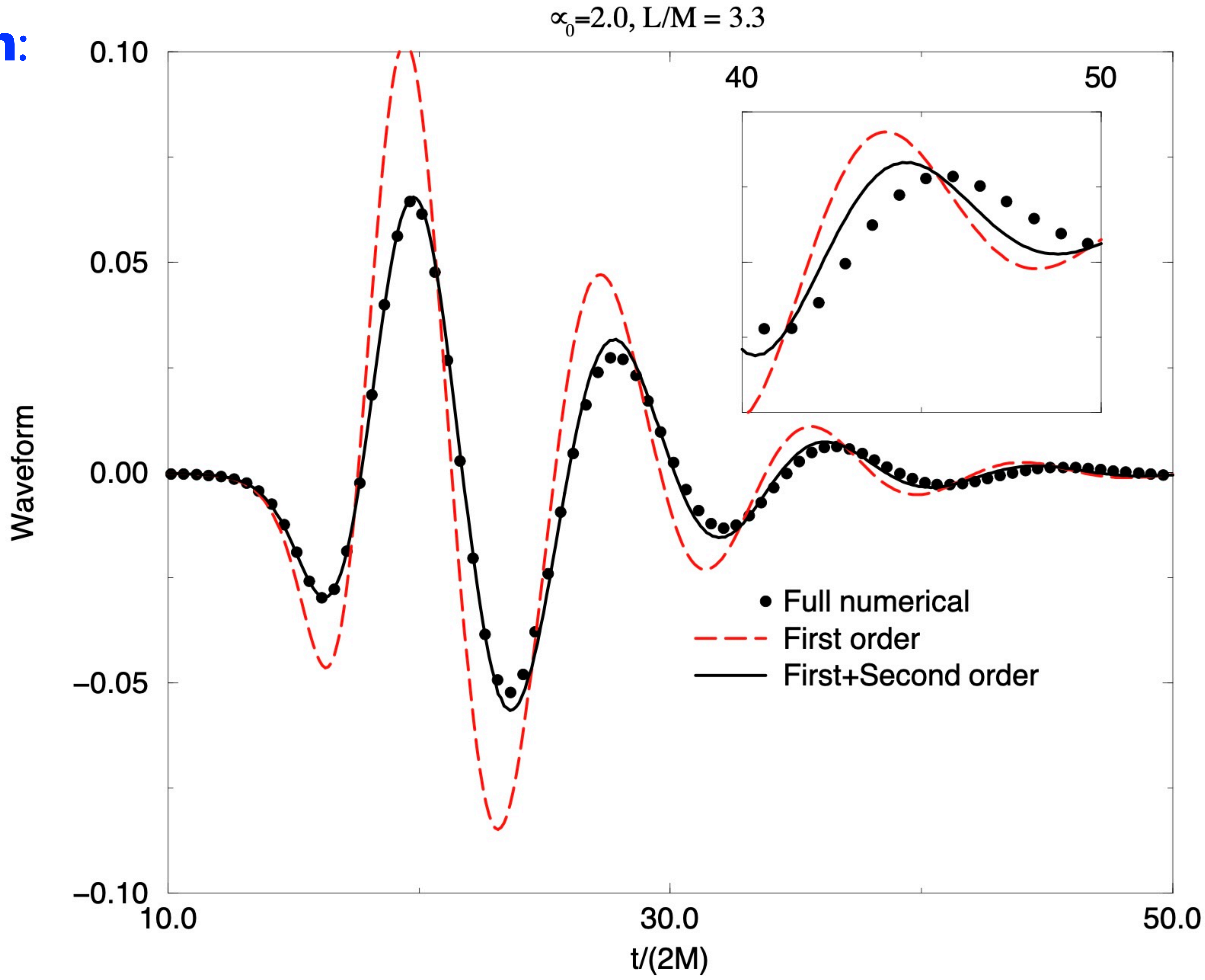

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PRL **95,** 121101 (2005) PHYSICAL REVIEW LETTERS week ending 16 SEPTEMBER 2005

Evolution of Binary Black-Hole Spacetimes

Frans Pretorius[1,2,*]

[1]*Theoretical Astrophysics, California Institute of Technology, Pasadena, California 91125, USA*

[2]*Department of Physics, University of Alberta, Edmonton, AB T6G 2J1 Canada*

(Received 6 July 2005; published 14 September 2005)

We describe early success in the evolution of binary black-hole spacetimes with a numerical code based on a generalization of harmonic coordinates. Indications are that with sufficient resolution this scheme is capable of evolving binary systems for enough time to extract information about the orbit, merger, and gravitational waves emitted during the event. As an example we show results from the evolution of a binary composed of two equal mass, nonspinning black holes, through a single plunge orbit, merger, and ringdown. The resultant black hole is estimated to be a Kerr black hole with angular momentum parameter $a \approx 0.70$. At present, lack of resolution far from the binary prevents an accurate estimate of the energy emitted, though a rough calculation suggests on the order of 5% of the initial rest mass of the system is radiated as gravitational waves during the final orbit and ringdown.

DOI: 10.1103/PhysRevLett.95.121101 PACS numbers: 04.25.Dm, 04.30.Db, 04.70.Bw

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Numerical Relativity Breakthrough!

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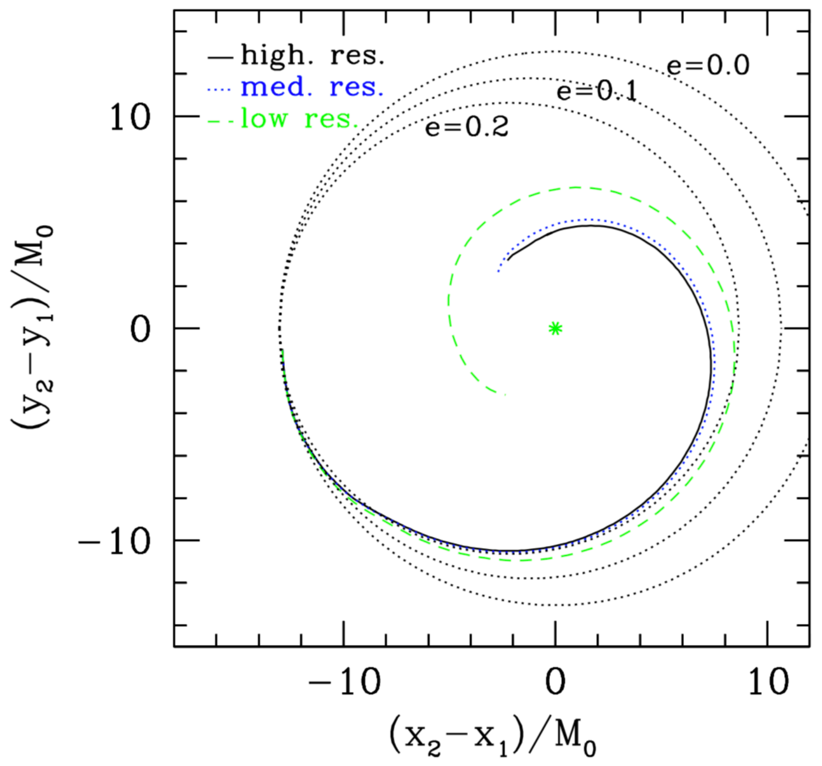
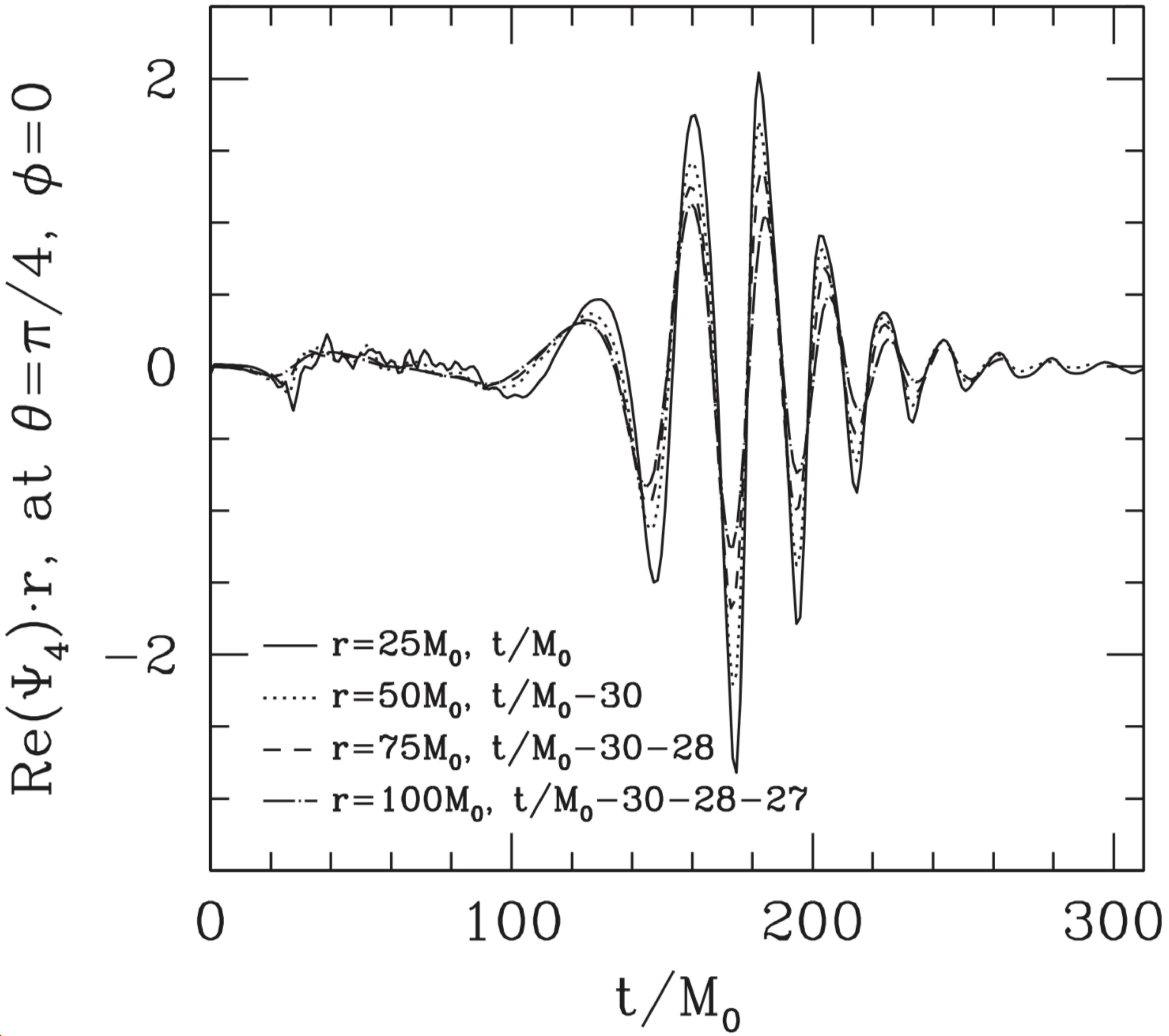
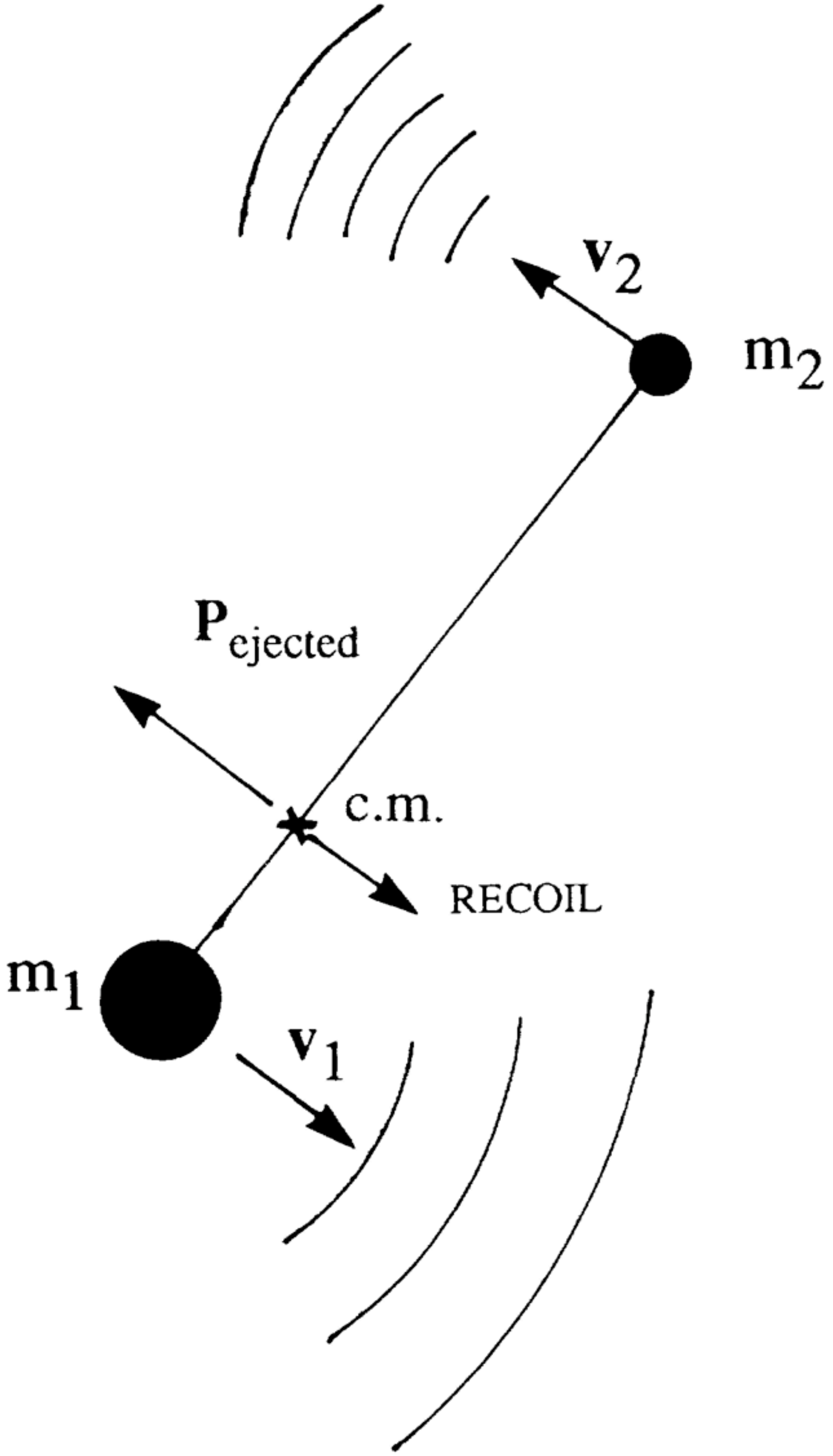




Gravitational Recoil from Binary Black Hole Mergers



- For unequal-mass Mergers, apart from energy and angular momentum, GWs carry linear momentum. Then, momentum conservation implies that the resulting Black Hole will experience a recoil.

- Depending on its magnitude, this recoil can have important astrophysical and cosmological consequences.

- We studied this question in 2005-2006 using the **Close Limit Approximation**.

$m_1 > m_2$

$|\mathbf{v}_1| < |\mathbf{v}_2|$

From: Wiseman, PRD **46** 1517 (1992)

Gravitational Recoil from Binary Black Hole Mergers

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$$\delta\dot{P}^x_{GW} = -\frac{i}{64\pi}\sum_{\ell\geq 2,m}(\ell+2)(\ell-1)\left\{\sqrt{(\ell-m)(\ell+m+1)}\left(\dot{\Psi}^{\ell m}_{ZM}\dot{\bar{\Psi}}^{\ell,m+1}_{CPM} - \dot{\Psi}^{\ell m}_{CPM}\dot{\bar{\Psi}}^{\ell,m+1}_{ZM}\right)\right.$$

$$\left. + \sqrt{(\ell+m)(\ell-m+1)}\left(\dot{\Psi}^{\ell m}_{ZM}\dot{\bar{\Psi}}^{\ell,m-1}_{CPM} - \dot{\Psi}^{\ell m}_{CPM}\dot{\bar{\Psi}}^{\ell,m-1}_{ZM}\right)\right\}$$

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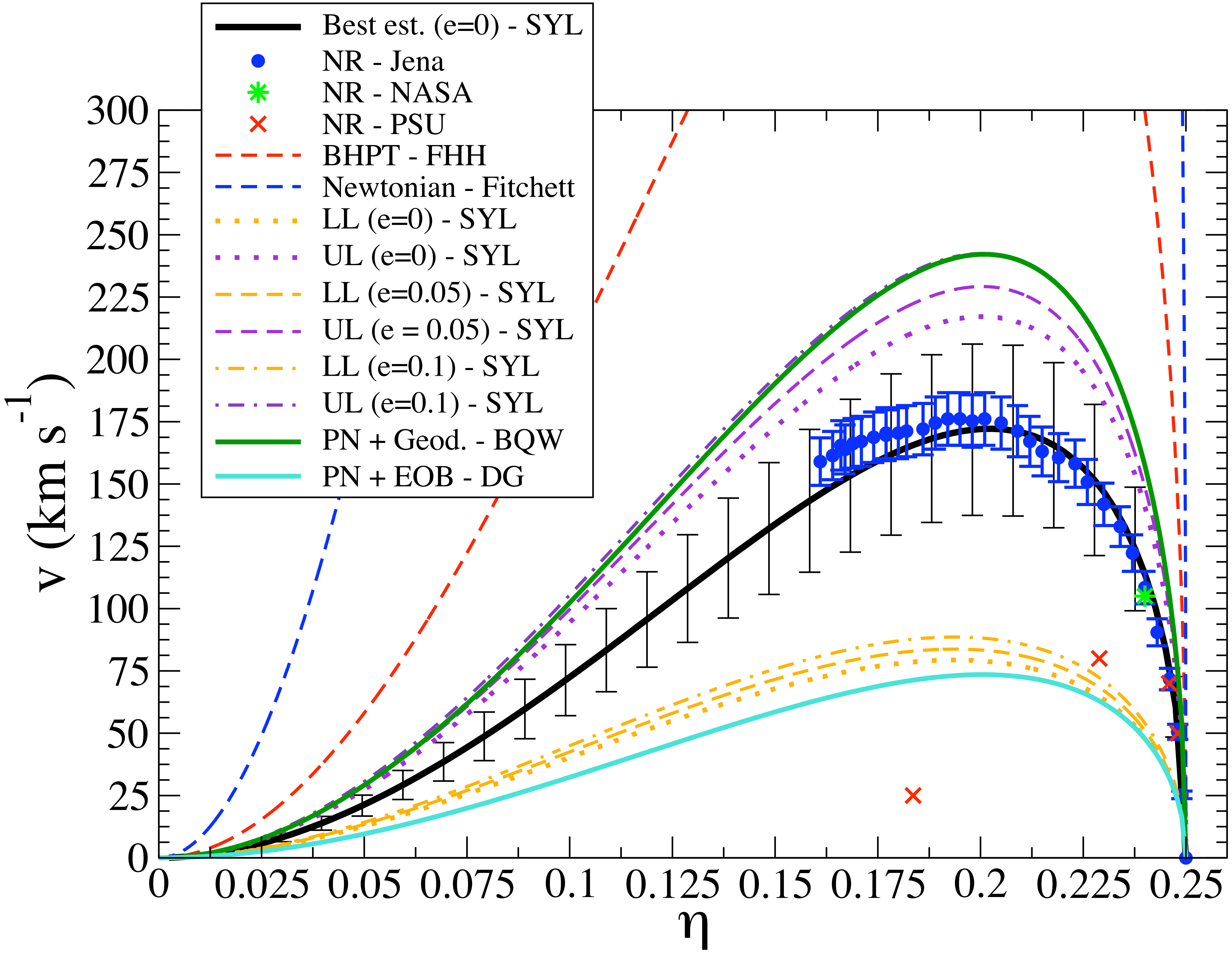
$$\left. - \sqrt{(\ell+m)(\ell-m+1)}\left(\dot{\Psi}^{\ell m}_{ZM}\dot{\bar{\Psi}}^{\ell,m-1}_{CPM} - \dot{\Psi}^{\ell m}_{CPM}\dot{\bar{\Psi}}^{\ell,m-1}_{ZM}\right)\right\}$$

$$\delta\dot{P}^z_{GW} = -\frac{i}{32\pi}\sum_{\ell\geq 2,m} m\,(\ell+2)(\ell-1)\left(\dot{\Psi}^{\ell m}_{ZM}\dot{\bar{\Psi}}^{\ell,m}_{CPM} - \dot{\Psi}^{\ell m}_{CPM}\dot{\bar{\Psi}}^{\ell,m}_{ZM}\right)$$

Gravitational Recoil from Binary Black Hole Mergers

$$\eta = \frac{m_1\, m_2}{(m_1 + m_2)^2} = \frac{m_1\, m_2}{M^2}$$



From: CFS, Yunes & Laguna, PRD **74** 124010 (2006)

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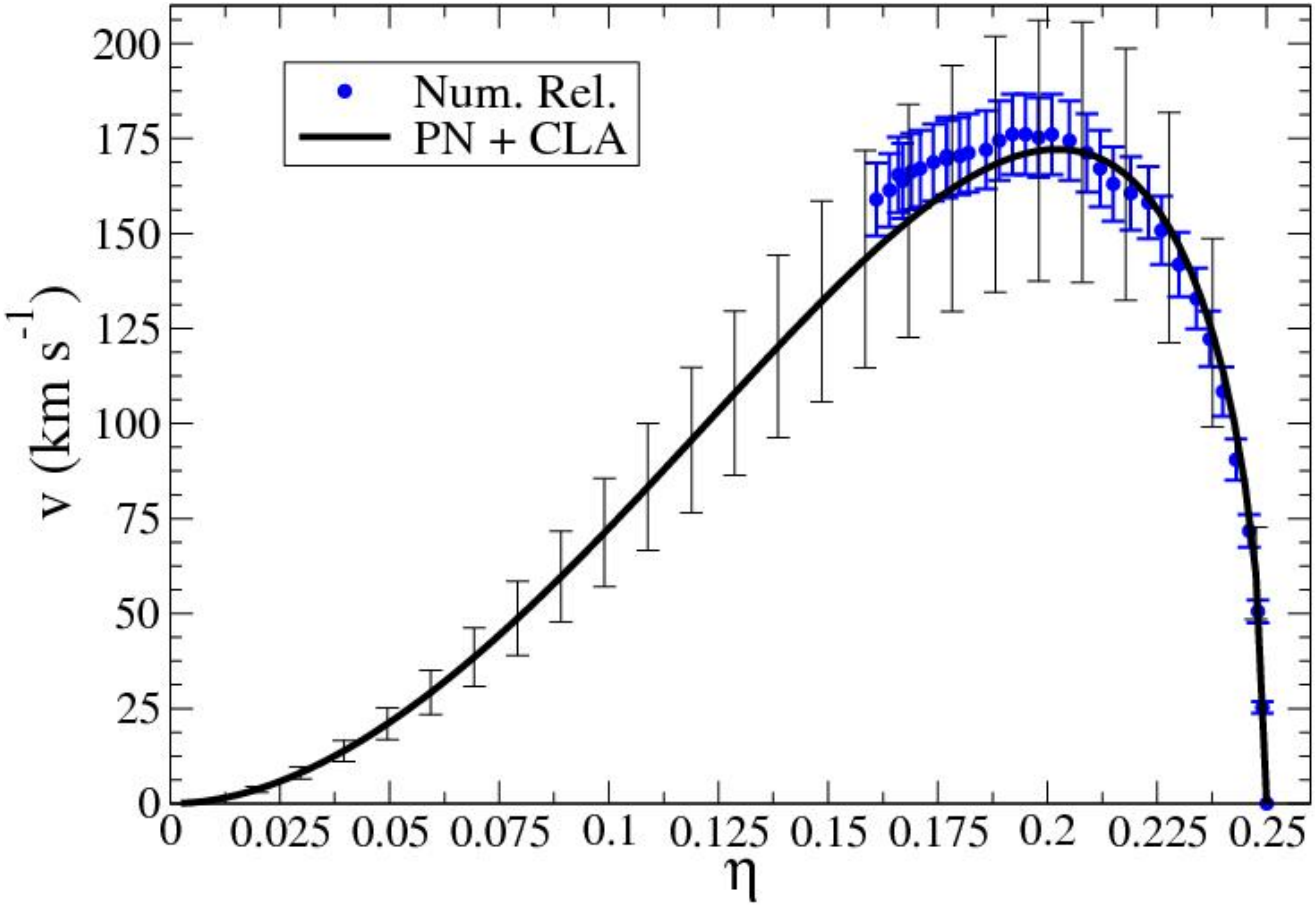

Comparison with: Gonzalez et al, PRL **98** 091101 (2007)

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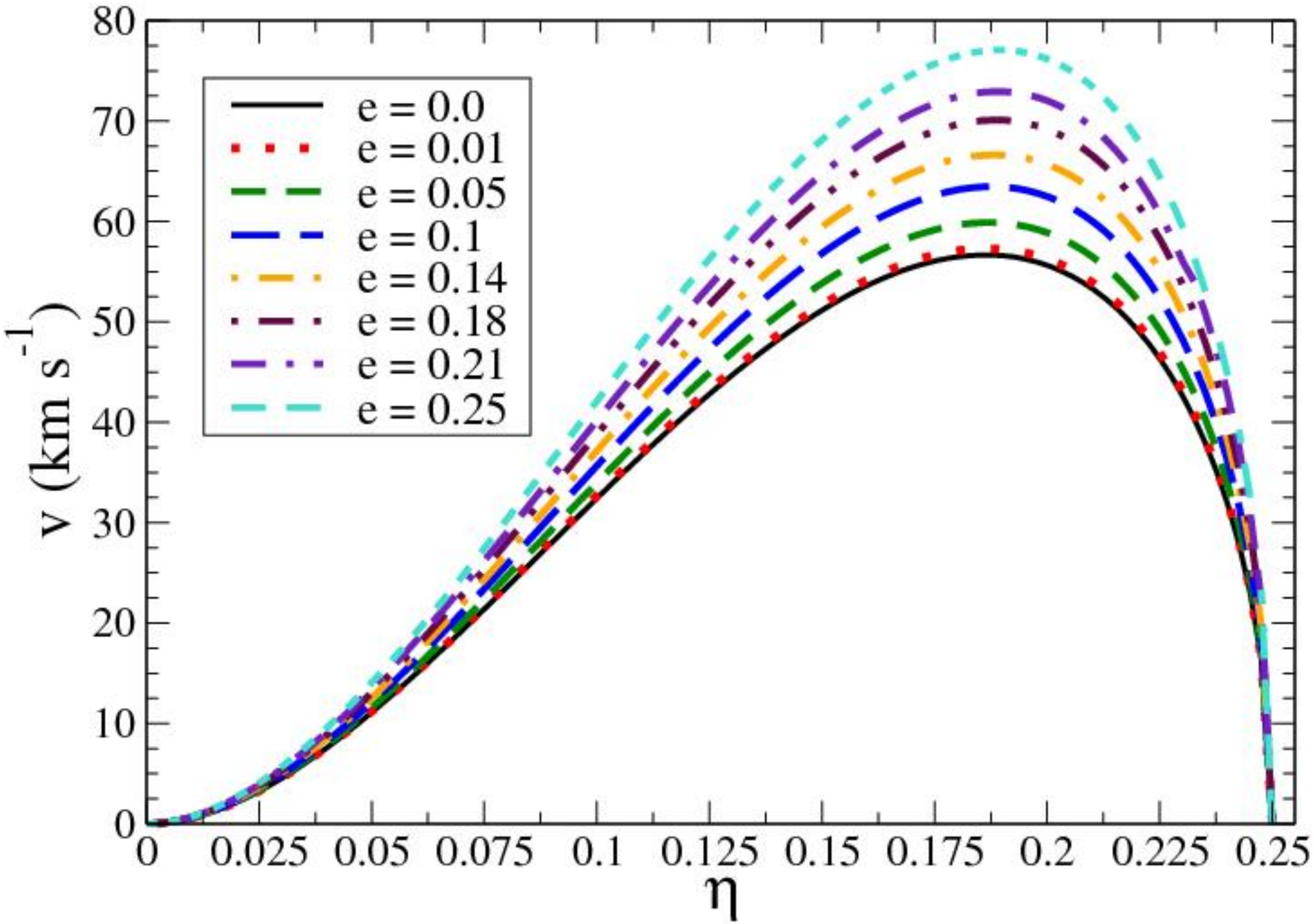

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From: CFS, Yunes & Laguna, ApJ **656** L9 (2007)

More about Black Hole Perturbation Theory

- Perturbation Theory of the **Kerr Black Hole**:
- The description of the perturbations of a Kerr BH uses a **geometric reduction** of the Einstein's perturbative equations instead of a **symmetry reduction** as in the case of a Schwarzschild BH. Teukolsky (1972) used the Newman-Penrose formalism to decouple the equations. Due to the symmetries of the Kerr metric, the **Teukolsky equation is separable**:

$$
-\left[\frac{\left(r^2+a^2\right)^2}{\Delta}-a^2\sin^2\theta\right]\partial_{tt}\Psi-\frac{4Mar}{\Delta}\partial_{t\varphi}\Psi-2s\left[r-\frac{M\left(r^2-a^2\right)}{\Delta}-ia\cos\theta\right]\partial_t\Psi+\Delta^{-s}\partial_r\left(\Delta^{s+1}\partial_r\Psi\right)
$$

$$
+\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta\Psi\right)+\left[\frac{1}{\sin^2\theta}-\frac{a^2}{\Delta}\right]\partial_{\varphi\varphi}\Psi+2s\left[\frac{a\left(r-M\right)}{\Delta}+\frac{i\cos\theta}{\sin^2\theta}\right]\partial_\varphi\Psi-(s^2\cot^2\theta-s)\Psi=0\,,
$$

Field Type	Field Quantity	Spin Weights
Scalar	ϕ	0
Spinorial	χ_0	1/2
Idem	$\rho^{-1}\chi_1$	$-1/2$
Electromagnetic	Φ_0	1
Idem	$\rho^{-2}\Phi_2$	-1
Gravitational	Ψ_0	2
Idem	$\rho^{-4}\Psi_4$	-2

More about Black Hole Perturbation Theory

- The Self-Force Program: The SCO is treated as a point-like object and deviations from geodesic motion are described by the action of a local force, the self-force. The equation of motion for the SCO is the so-called the MiSaTaQuWa equation [Mino, Sasaki & Tanaka (1997); Quinn & Wald (1997)]:

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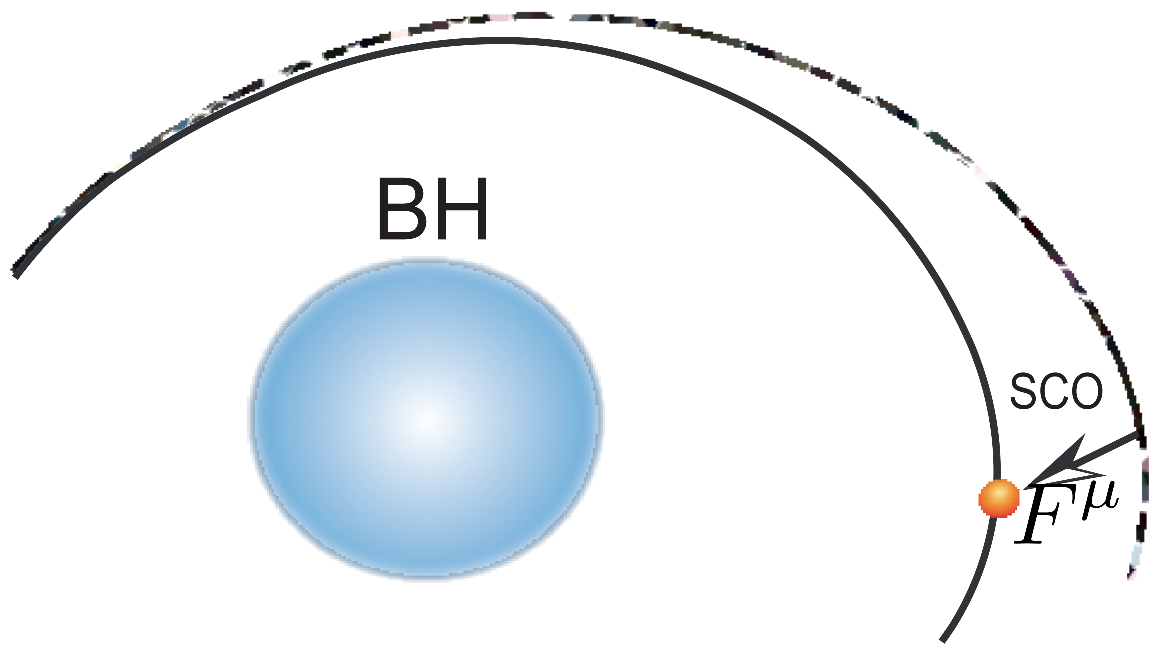

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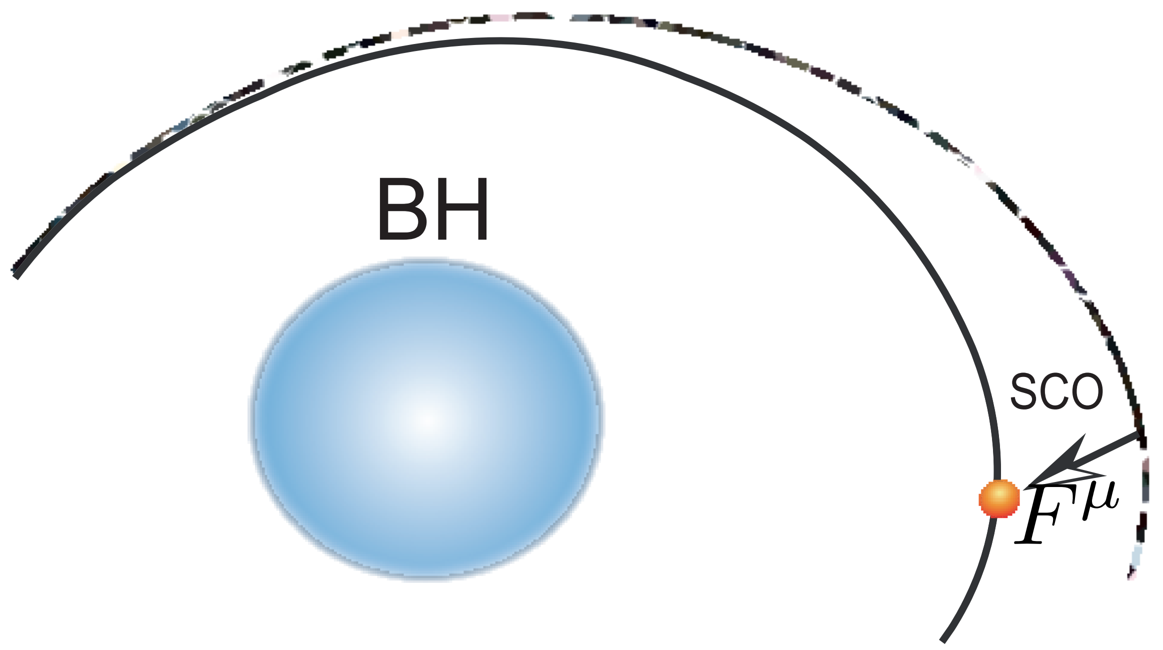

$$\frac{D^2 z^\mu}{D\tau^2} = -\frac{1}{2}\left(g^{\mu\nu} + \frac{dz^\mu}{d\tau}\frac{dz^\nu}{d\tau}\right)\left(2\nabla_\rho h^{\rm R}_{\nu\sigma} - \nabla_\nu h^{\rm R}_{\rho\sigma}\right)\Big|_{z(\tau)}\frac{dz^\rho}{d\tau}\frac{dz^\sigma}{d\tau}$$

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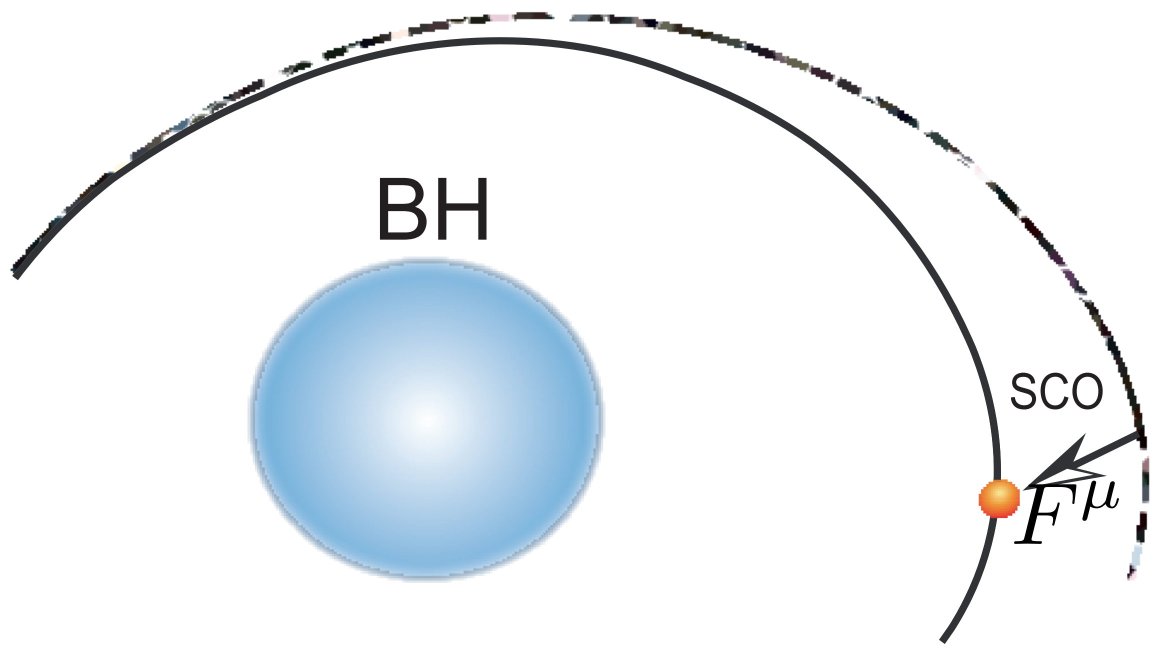

The spin of the SCO has been neglected!

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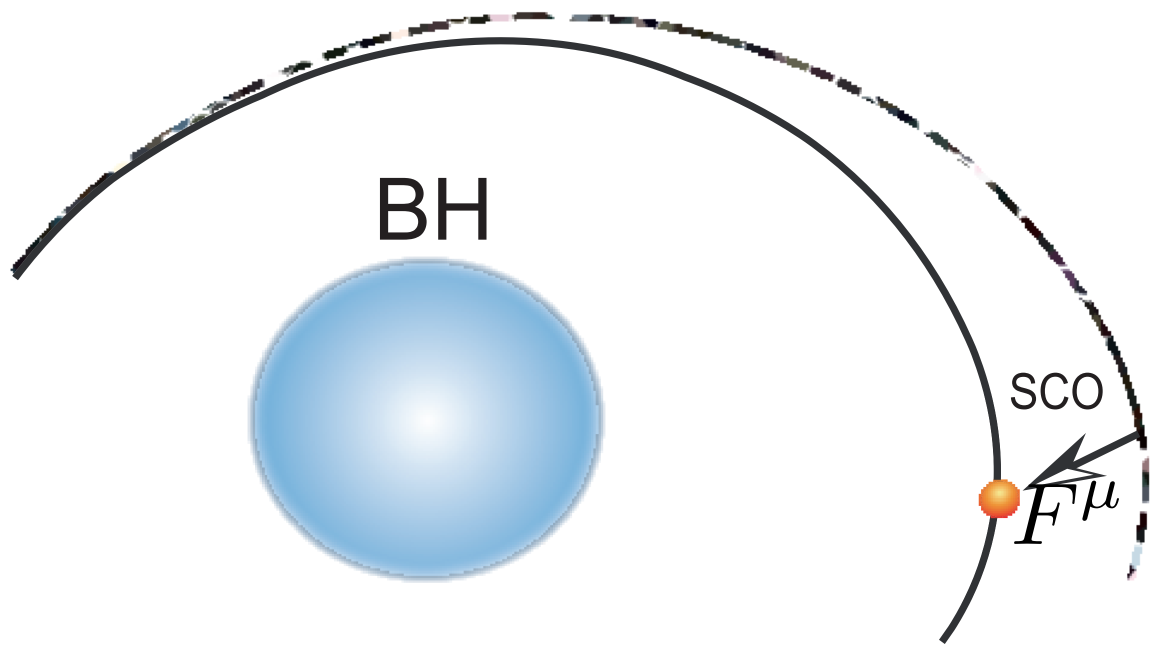

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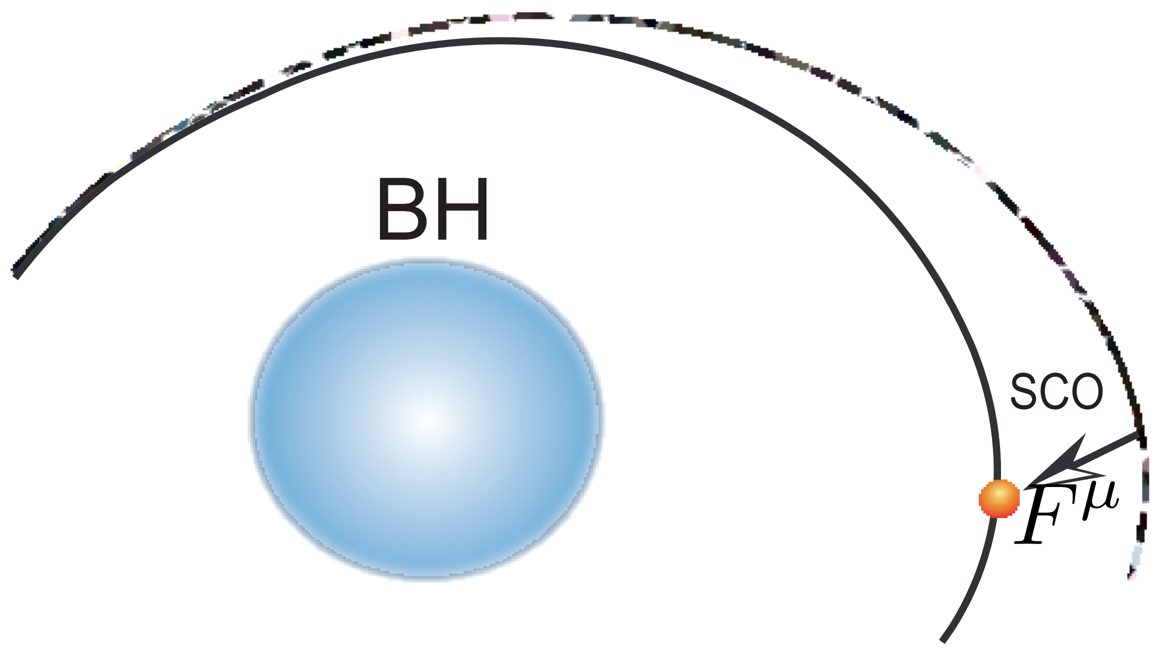

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- We can also look at this equation as describing geodesic motion in a perturbed (regularized) geometry.

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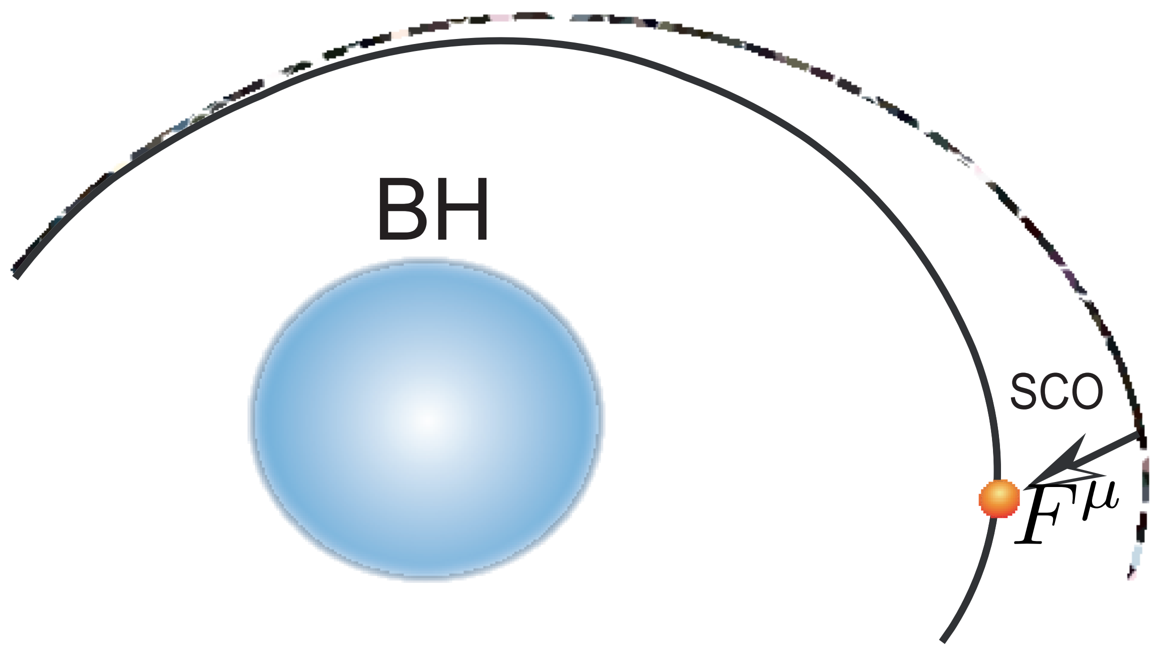

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• At the moment, it has been recently completed the computation of the 1st-order self-force for Kerr spacetimes. We need to implement it in a consistent evolution of the system and 2nd-order perturbations for gauge-invariant waveform generation.

A Hierarchical Approach to the Binary Black Hole Problem

Hierarchical Approach to the BBH Problem

Universality in Binary Black Hole Dynamics: An Integrability Conjecture

José Luis Jaramillo*[1], Badri Krishnan†[2], and Carlos F. Sopuerta‡[3]

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[2]Institute for Mathematics, Astronomy and Particle Physics Radboud University, Heyendaalseweg 135, 6525 AJ Nijmegen, The Netherland

[3]Institute of Space Sciences (ICE-CSIC and IEEC), Campus UAB, Carrer de Can Magrans s/n, 08193 Cerdanyola del Vallès, Spain

30 March 2023

arXiv:2305.08554

Abstract

The waveform of a binary black hole coalescence appears to be both simple and universal. In this essay we argue that the dynamics should admit a separation into 'fast and slow' degrees of freedom, such that the latter are described by an integrable system of equations, accounting for the simplicity and universality of the waveform.

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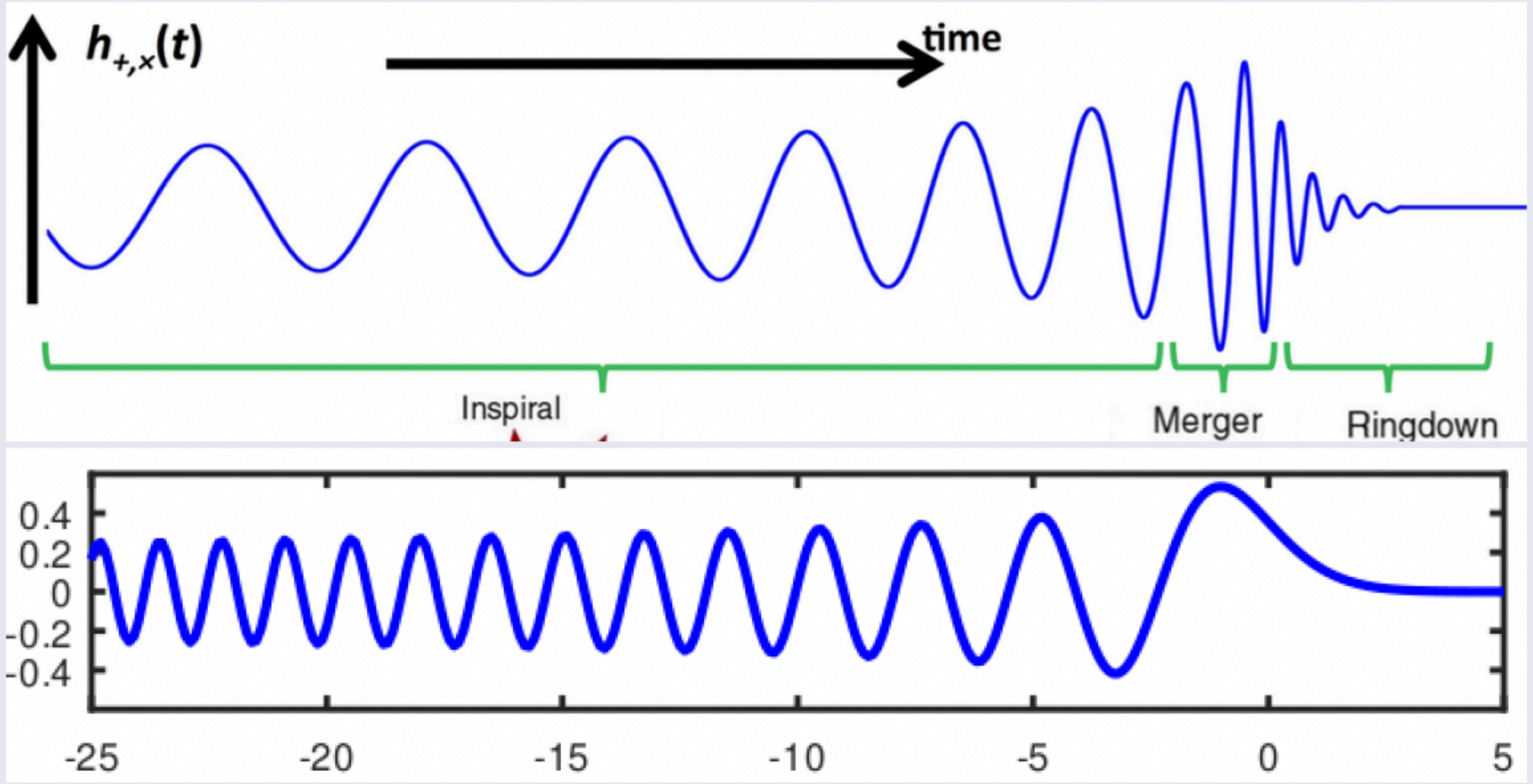
Hierarchical Approach to the BBH Problem

Asymptotic BBH Model	Mathematical/Physical Framework	Key Structures/Mechanisms
Fold-caustic model	Geometric Optics Catastrophe (singularity) Theory	Arnol'd-Thom's Theorem Classification of Stable Caustics
Airy function model	Fresnel's Diffraction, Semiclassical Theory	Universal Diffraction Patterns in Caustics
Painlevé-II model	Painlevé Transcendents and Integrability Self-force calculations and EMRBs	Painlevé property Non-linear Turning Points
KdV-like model	Inverse Scattering Transform and Integrability Dispersive Non-linear PDEs Critical Phenomena in Dispersive PDEs	Painlevé test, Lax pairs Darboux transformations, Soliton Scattering Universal Wave Patterns, Dubrovin's Conjecture
Propagation models on (anti)-Self-Dual backgrounds	Ward's Conjecture and Integrability Twistorial techniques	(anti-)Self-Dual DoF, Instantons, Tunneling Penrose Transform, 'Twistor' BBH data

Hierarchical Approach to the BBH Problem

Universality in (linear) dispersive wave equations: **“All caustics look the same”**

BBH merger waveform and Airy direct comparison: ignore ringdown and phase

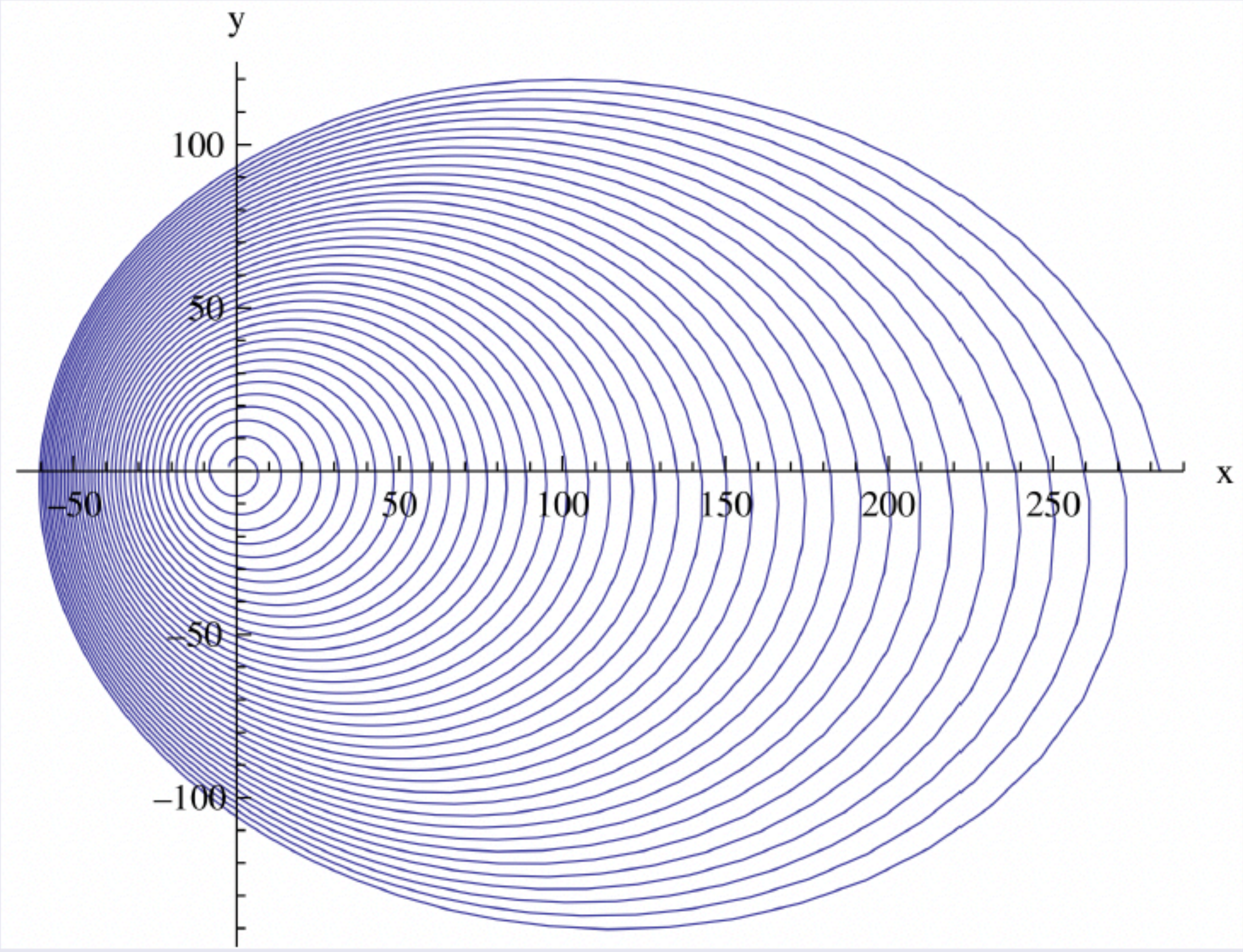


Problem: assess the BBH merger waveform as a modulated Airy function.

Damped orbital motion of a charged particle in a Coulomb potential [Rajeev 08]

Landau-Lifshitz equations (in the non-relativistic limit) for a charged particle in a Coulomb potential **solved exactly in terms of the Painlevé-II equation**:

$$\frac{d^2L}{dz^2} = -\frac{1}{2k^2}L^3 - \frac{\tau}{k}zL$$



$$\frac{d^2w}{dt^2} - tw - 2w^3 = 0$$

(Painlevé-II, "non-linear Airy")

EMRI transition to plunge: Painlevé-I dynamics [Ori & Thorne 00; Compère and Küchler 21]

Dynamics $\dfrac{d^2X}{dT^2} = -X^2 - T$ solved by $\dfrac{d^2w}{dt^2} = 6w^2 + t$ (Painlevé-I)

Hierarchical Approach to the BBH Problem

Ward's conjecture: integrability and self-duality

Conjecture: fundamental relation between integrable (or solvable) differential equation systems and reductions of (anti-)self-dual Yang-Mills equations.
In GR setting:

- Connection between self-dual Einstein vacuum equations and the self-dual Yang-Mills equations [Mason& Newman; Woodhouse; Dunajski...].
- Ansatz: **'Backgrounds' in the (anti-)self-dual sector of GR.** [Compare approach by Mason et al.]

Scattering on instantons

Gravitational (anti-)self-dual solutions, either complex or real in Euclidean GR:

- The notion of 'background' needs to be relaxed/generalised.
- Idea: *BBH merger waveforms addressed in terms of an appropriate notion of classical scattering on a (anti-)self-dual gravitational instanton.*

Remarks and Conclusions

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- Binary Black Holes are the main source of gravitational waves for current ground-based gravitational-wave detectors, They are also expected to be the main source for 3G and space-based detectors as they may provide revolutionary information about the nature of black holes and gravity.

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• In order to fulfill the exciting scientific program we need to have very precise models of their gravitational-wave emission (**waveform models**). In this sense, any analytical understanding of the dynamics of binary black holes would be crucial.

• Integrability has already played a major role in General Relativity and we expect it to be a maing guiding principle.

Many Thanks for your attention!