# Black hole perturbations in modified gravity as a first-order system

**Hugo Roussille**

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collab. with D. Langlois, K. Noui

2103.14750, 2312.11986

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# Introduction

- Modified gravity theories: predictions different from GR
- Important test: quasinormal modes of black holes
- Up to now, theoretical computations are rare
- Present a systematic algorithm to extract physical information
- Use it to perform numerical computation of modes

# Outline

1. Scalar-tensor theories and hairy black holes
    - Necessity for modified gravity
    - Black hole solutions
2. Quasinormal modes in GR
    - Perturbation setup
    - New challenges in modified gravity
3. Quasinormal modes in modified gravity
    - Asymptotics from the first order system
    - Application to new black hole solutions in modified gravity
    - Numerical results (BCL)

# Scalar-tensor theories and hairy black holes

# Motivations for modified gravity

## Heuristic approach

- Design new tests of GR beyond a null hypothesis check
- EFT of some high energy theory

| GR | Modified gravity | Quantum regime |
| --- | --- | --- |
| $g_{\mu\nu}$ | $g_{\mu\nu}+?$ | |

→ energy

## Issues of GR

- Big Bang singularity
- Black hole center singularity
- Cosmic expansion

$\Rightarrow$ Important to look for extensions of GR

$\Rightarrow$ Need to develop tests of these modified theories

# Building a modified gravity theory

General procedure to construct a modified gravity theory:

Break one or several hypotheses of **Lovelock's theorem** → Make sure the theory is not **pathological** (ghosts, instabilities...) → Take **experimental constraints** into account

## Different scales of tests

- Large scale: cosmology, growth rates of structures
- Weak fields: Solar System tests, orbits
- Strong fields, small scales: black holes (focus of this talk)

# Quasinormal modes and the ringdown

Ringdown of a merger: excited BH emits GW at precise frequencies, the **quasinormal modes**

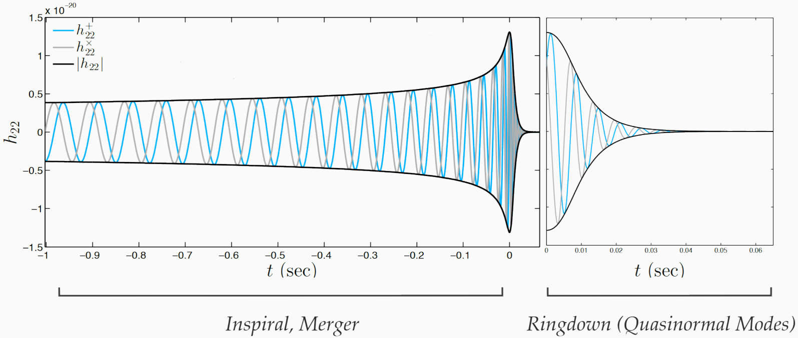
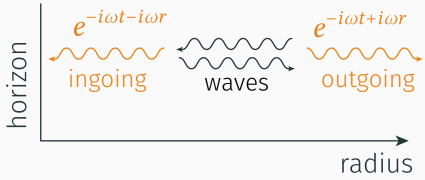



**Figure 1:** Ringdown phase of a binary black hole merger (L. London 2017)

# Computation and measurement

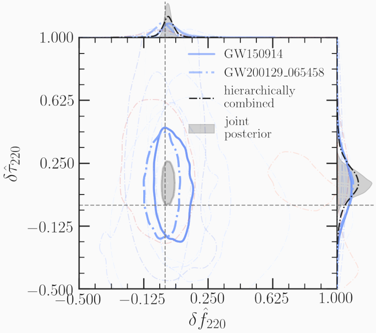
## Computation




- 2 boundary conditions: eigenvalue problem
- **Complex spectrum** due to energy loss
- Depend on background and theory

## Measurement

- Obtain from GW signal
- Compare to the theory [Abbott '21]:



# Horndeski theory of gravity

## Shift-symmetric Horndeski theory

$$S[g_{\mu\nu},\phi] = \int \mathrm{d}^4x \Big[F(X)R + P(X) + Q(X)\Box\phi + 2F'(X)\left(\phi_{\mu\nu}\phi^{\mu\nu} - (\Box\phi)^2\right),$$
$$+ G(X)G^{\mu\nu}\phi_{\mu\nu} + \frac{1}{3}G_X((\Box\phi)^3 - 3\Box\phi\phi_{\mu\nu}\phi^{\mu\nu} + 2\phi_{\mu\rho}\phi^{\rho\nu}\phi^{\mu}_{\nu})\Big]$$

$$\phi_\mu = \nabla_\mu\phi\,, \quad X = \phi_\mu\phi^\mu$$

- New scalar field: additional degree of freedom
- Easier to evade no-hair theorems [Hui, Nicolis '13]: new black hole solutions [Babichev, Charmousis+ '17; Van Aelst, Gourgoulhon+ '20; Achour, Liu+ '20]
- More involved dynamics in vacuum

# New black holes in Horndeski: BCL solution

Choice of Horndeski parameters [Babichev, Charmousis+ '17]:

$$F(X) = f_0 + f_1\sqrt{X} \quad P(X) = -p_1 X\,, \quad Q(X) = 0\,, \quad G(X) = 0$$

## Metric sector: RN with imaginary charge

$$\mathrm{d}s^2 = -A(r)\,\mathrm{d}t^2 + \frac{1}{A(r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$$

$$A(r) = 1 - \frac{r_m}{r} - \xi\frac{r_m^2}{r^2}\,, \quad \xi = \frac{f_1^2}{2f_0\,p_1\,r_m^2}$$

## Scalar sector

$$\phi = \psi(r)\,, \quad \psi'(r) = \pm\frac{f_1}{p_1\,r^2\sqrt{A(r)}}$$

$$X(r) = \frac{f_1^2}{p_1^2\,r^4}$$

# New black holes in Horndeski: 4dEGB solution

$D \to 4$ limit of higher-$D$ Gauss-Bonnet [Lu, Pang '20]:

$$F(X) = 1 - 2\alpha X \quad P(X) = 2\alpha X^2 \,, \quad Q(X) = -4\alpha X \,, \quad G(X) = -4\alpha \ln(X)$$

## Metric sector:

$$\mathrm{d}s^2 = -A(r)\,\mathrm{d}t^2 + \frac{1}{A(r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$$

$$A(r) = 1 - \frac{M(r)}{r} \,, \quad M(r) = \frac{2\mu}{1 + \sqrt{1 + 4\alpha\mu/r^3}}$$

## Scalar sector

$$\phi = \psi(r)$$

$$\psi'(r) = \frac{-1 + \sqrt{A}}{r\sqrt{A}}$$

# Quasinormal modes in GR

# Separating degrees of freedom

1. Start with the Einstein-Hilbert action $S[g_{\mu\nu}] = \int \mathrm{d}^4x \sqrt{-g}\, R$
2. Focus on **spherically symmetric BH**: Schwarzschild background
3. Perturb the metric: $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ and linearise Einstein's equations

   $\longrightarrow$ obtain **10 equations**
4. Decompose the components of $h_{\mu\nu}$ over spherical harmonics
5. Separate by parity: polar (even) and axial (odd) modes
6. Fix the gauge and Fourier transform [Regge, Wheeler '57; Zerilli '70]
7. Obtain the 2-dimensional systems

$$\frac{\mathrm{d}X_{\mathrm{odd}}}{\mathrm{d}r} = M_{\mathrm{odd}}(r)X_{\mathrm{odd}} \quad \text{and} \quad \frac{\mathrm{d}X_{\mathrm{even}}}{\mathrm{d}r} = M_{\mathrm{even}}(r)X_{\mathrm{even}}$$

# Schrödinger equation of propagation

$$\frac{\mathrm{d}X}{\mathrm{d}r} = MX\,, \quad M = \begin{pmatrix} * & * \\ * & * \end{pmatrix} \xrightarrow{\quad X = P(r)\tilde{X} \quad} \frac{\mathrm{d}\tilde{X}}{\mathrm{d}r_*} = \tilde{M}\tilde{X}\,, \quad \tilde{M} = \begin{pmatrix} 0 & 1 \\ V(r) - \omega^2 & 0 \end{pmatrix}$$

## Physical interpretation

$$\begin{cases} \tilde{X}_0' = \tilde{X}_1\,, \\ \tilde{X}_1' = (V(r) - \omega^2)\tilde{X}_0 \end{cases} \Rightarrow \quad \frac{\mathrm{d}^2\tilde{X}_0}{\mathrm{d}r_*^2} + \left(\omega^2 - V(r)\right)\tilde{X}_0 = 0\,, \quad \frac{\mathrm{d}r}{\mathrm{d}r_*} = A(r)$$

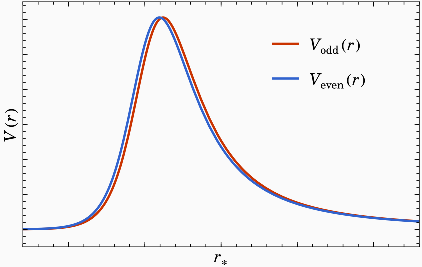
Schrödinger equation with potential $V$

$r_*$: “tortoise coordinate” such that $r = r_s$ is equivalent to $r_* = -\infty$

# Interpretation of the equation

**Horizon**

$$\tilde{X}_0 \sim e^{-i\omega(t \pm r_*)}$$




**Infinity**

$$\tilde{X}_0 \sim e^{-i\omega(t \pm r_*)}$$

- Recover the boundary conditions useful for QNM computation
- Numerical resolution of problem available [Leaver, Chandrasekhar '97]

# Summary: computation of modes in GR

Linearized Einstein's equations → Fix gauge → Separate by parity

**Odd parity:**

$$X' = \begin{pmatrix} * & * \\ * & * \end{pmatrix} X \longrightarrow$$

Wave equation:

$$Y'' + (\omega^2 - V_o)\,Y = 0$$

**Even parity:**

$$X' = \begin{pmatrix} * & * \\ * & * \end{pmatrix} X \longrightarrow$$

Wave equation:

$$Y'' + (\omega^2 - V_e)\,Y = 0$$

Fully decoupled system

→ Numerics

# What changes in scalar-tensor gravity

Linearized modified gravity equations → Fix gauge → Separate by parity

**Odd parity:**

$$X' = \begin{pmatrix} * & * \\ * & * \end{pmatrix} X$$

→ Wave equation:

$$Y'' + (\omega^2 - V_{\mathrm{o}})\, Y = 0$$

→ Numerics

**Even parity:**

$$X' = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} X$$

→ **In general:** no decoupling, usual numerics not possible

New degrees of freedom couple to gravitation

# Quasinormal modes in modified gravity

# First-order system and boundary conditions

## Main idea

Get boundary conditions and perform numerical computations **from the first-order system**

## Steps to perform

- Find asymptotic behaviour at the horizon and infinity
- Identify ingoing and outgoing modes
- Use a numerical method that does not require Schrödinger equations

# Example for axial Schwarzschild

Diagonalizing

$$M(r) = -i\begin{pmatrix} 0 & \omega^2 \\ 1 & 0 \end{pmatrix} + \frac{2}{r}\begin{pmatrix} 1 & 0 \\ -i\mu & 0 \end{pmatrix} + \mathcal{O}\left(\frac{1}{r^2}\right)$$

$$\tilde{M}(r) = \begin{pmatrix} -i\omega & 0 \\ 0 & +i\omega \end{pmatrix} + \frac{1}{r}\begin{pmatrix} 1 - i\mu\omega & 0 \\ 0 & 1 + i\mu\omega \end{pmatrix} + \mathcal{O}\left(\frac{1}{r^2}\right)$$

$$\frac{\mathrm{d}\tilde{X}}{\mathrm{d}r} = \tilde{M}\tilde{X} \implies \tilde{X} \sim \begin{pmatrix} e^{-i\omega r} & 0 \\ 0 & e^{+i\omega r} \end{pmatrix} \tilde{X}_c$$

One recovers ingoing and outgoing modes!

# Example for polar Schwarzschild

$$M(r) = \begin{pmatrix} 0 & 0 \\ i\omega^2/\lambda & 0 \end{pmatrix} r^2 + \mathcal{O}(r)$$

$$\frac{\mathrm{d}X}{\mathrm{d}r} = MX \implies X \sim \begin{pmatrix} 1 & 0 \\ \frac{i\omega^2}{\lambda}\frac{r^3}{3} & 1 \end{pmatrix} X_c$$

- No diagonalizing possible
- Asymptotic behaviour does not give ingoing and outgoing waves
- Reason of this problem: leading order is nilpotent

# Mathematical results

Solution from theory of meromorphic systems of linear ODEs [Wasow '65; Balser '99]

## Mathematical algorithm

- Main idea: apply a change of variables $X = P\tilde{X}$
- New system has

$$\tilde{M} = P^{-1}MP - P^{-1}\frac{\mathrm{d}P}{\mathrm{d}r}$$

- $\tilde{M}$ can be diagonal even if $M$ is not diagonalisable

$\Rightarrow$ important result: diagonalization is always possible order by order

# Principle of the algorithm

## Diagonalisable

$$P = I + \frac{1}{r}\Sigma_1 + \frac{1}{r^2}\Sigma_2 + \dots$$

Solve for $\Sigma_i$ order by order $\to$ final system diagonal

## Non-diagonalisable

$$M \sim \begin{pmatrix} \lambda & 0 & \dots \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{pmatrix} r^p$$

1. $P = \exp\left(\lambda r^{p+1}/(p+1)\right) I$
2. $P = \mathrm{diag}(1, r, r^2, \dots)$
3. Repeat

## Several eigenvalues

$$P = I + \frac{1}{r}\Sigma_1 + \frac{1}{r^2}\Sigma_2 + \dots$$

Solve for $\Sigma_i$ to decouple subsystems order by order

## General result

$$M = M_p r^p + M_{p-1} r^{p-1} + \dots \longrightarrow \tilde{M} = D_q r^q + D_{q-1} r^{q-1} + \dots$$

$$X \sim e^{D(r)} r^{D_{-1}} F(r) X_c$$

# Algorithm for polar Schwarzschild

$$M(r) = \begin{pmatrix} 0 & 0 \\ i\omega^2/\lambda & 0 \end{pmatrix} r^2 + \mathcal{O}(r)$$

Algorithm

$$P(r) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} r^2 + \begin{pmatrix} i & i \\ \frac{(2\lambda-3)\mu}{4\lambda} - \frac{i}{2\omega} & \frac{(2\lambda-3)\mu}{4\lambda} + \frac{i}{2\omega} \end{pmatrix} r + \mathcal{O}(1)$$

$$\tilde{M}(r) = \begin{pmatrix} -i\omega & 0 \\ 0 & +i\omega \end{pmatrix} + \frac{1}{r} \begin{pmatrix} 1 - i\mu\omega & 0 \\ 0 & 1 + i\mu\omega \end{pmatrix} + \mathcal{O}\left(\frac{1}{r^2}\right)$$

$$\frac{\mathrm{d}\tilde{X}}{\mathrm{d}r} = \tilde{M}\tilde{X} \implies \tilde{X} \sim \begin{pmatrix} e^{-i\omega r} & 0 \\ 0 & e^{+i\omega r} \end{pmatrix} \tilde{X}_c$$

One recovers ingoing and outgoing modes, and behaviour from axial modes

# Algorithm for BCL black hole: infinity

## BCL black hole

$$\mathrm{d}s^2 = -A(r)\,\mathrm{d}t^2 + A(r)^{-1}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\,, \quad \phi = \psi(r)$$

$$A(r) = 1 - \mu/r - \xi\mu^2/r^2\,, \quad \psi'(r) = \pm c\Big/ r^2\sqrt{A(r)}$$

$$M = \begin{pmatrix} \omega^2 & & \\ & & \\ & & \end{pmatrix} r^2 + \begin{pmatrix} 2\mu\omega^2 & -i\omega \\ & \\ & \end{pmatrix} r + \dots$$

algorithm

$$D = \begin{pmatrix} -i\omega & & & \\ & +i\omega & & \\ & & -\sqrt{2}\omega & \\ & & & +\sqrt{2}\omega \end{pmatrix} + \dots$$

$$\mathfrak{g}^{\infty}_{\pm}(t,r) = a_{\pm}\, e^{-i\omega t \pm i\omega r}$$

$$\mathfrak{s}^{\infty}_{\pm}(t,r) = b_{\pm}\, e^{-i\omega t \pm \sqrt{2}\omega r}$$

# Algorithm for BCL black hole: horizon

## BCL black hole

$$\mathrm{d}s^2 = -A(r)\,\mathrm{d}t^2 + A(r)^{-1}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\ , \quad \phi = \psi(r)$$

$$A(r) = 1 - \mu/r - \xi\mu^2/r^2\ , \quad \psi'(r) = \pm c\Big/ r^2\sqrt{A(r)}$$

algorithm

$$M = \begin{pmatrix} a_0 & & \\ & & \\ & & \end{pmatrix} \frac{1}{(r-r_+)^2} + M_{-1}\frac{1}{r-r_+} + \dots$$

$$D = \begin{pmatrix} -i\omega/c_0 & & & \\ & +i\omega/c_0 & & \\ & & 1/2 & 1 \\ & & & 1/2 \end{pmatrix} \frac{1}{r-r_+} + \dots$$

$$\mathfrak{g}_{\pm}^{r_+}(t,r) = a_{\pm}(r-r_+)^{\pm i\omega/c_0}\, e^{-i\omega t}$$

$$\mathfrak{s}_1^{r_+}(t,r) = b_1\sqrt{r-r_+}\, e^{-i\omega t}$$

$$\mathfrak{s}_2^{r_+}(t,r) = b_2\sqrt{r-r_+}\,\log(r-r_+)\, e^{-i\omega t}$$

# Algorithm for 4dEGB black hole: infinity

## 4dEGB black hole

$$A(r) = 1 - 2\mu \Big/ r\left(1+\sqrt{1+4\alpha\mu/r^3}\right)\ , \quad z = r/r_h\ , \quad \Omega = \omega r_h\,, \quad \beta = \alpha/r_h^2$$

$$M = \begin{pmatrix} \Omega^2/a_1 & & & \\ & & & \\ & & & \\ & & & \end{pmatrix} z^5 + \begin{pmatrix} \Omega^2/a_2 & & -i\Omega/a_1 & \\ & & & \\ & & & \\ & & & \end{pmatrix} z^4 + \ldots$$

algorithm ↓

$$D = \begin{pmatrix} -i\Omega & & & \\ & +i\Omega & & \\ & & & \\ & & & \end{pmatrix} + \begin{pmatrix} & & & \\ & & & \\ & & -i\sqrt{\lambda} & \\ & & & +i\sqrt{\lambda} \end{pmatrix} \frac{1}{z} + \ldots$$

$$\mathfrak{g}_\pm^\infty(t,r) \approx a_\pm\, e^{-i\Omega(t\pm z)}$$

$$\mathfrak{s}_\pm^\infty(t,r) \approx b_\pm\, e^{-i\omega t}\, z^{\pm i\sqrt{\lambda}}$$

# Algorithm for 4dEGB black hole: horizon

## 4dEGB black hole

$$A(r) = 1 - 2\mu \Big/ r\left(1 + \sqrt{1 + 4\alpha\mu/r^3}\right), \quad x = 1/\sqrt{r - r_h}, \quad \Omega = \omega r_h, \quad \beta = \alpha/r_h^2$$

algorithm

$$M = \begin{pmatrix} & & \\ & & \\ a_0 & & \end{pmatrix} + \ldots$$

$$D = \begin{pmatrix} -1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 2 \end{pmatrix} \frac{1}{x} + \ldots$$

$$\mathfrak{g}_{\pm}^{r_h}(t, r) = ?$$

$$\mathfrak{s}_{\pm}^{r_h}(t, r) = ?$$

No propagating modes

# "Recipe" for the computation of quasinormal modes

① Linearized Einstein's eqs → ② Gauge choice → ③ Back-ground → ④ First-order system → ⑤ Asymptoti-cal behaviour → ⑥ Numerical computa-tion

- Generic algorithm that should work for any modified gravity theory
- Go around the technical difficulties of finding a wave equation
- Caveat: we do not get the full decoupled equations for the modes $\Rightarrow$ no modified Regge-Wheeler equation
- Asymptotical behaviour is enough to obtain boundary conditions for numerical resolution

# Numerical method

Impose both boundary behaviours to the system $\mathrm{d}X/\mathrm{d}r = MX$ with the ansatz:

$$X = \underbrace{e^{+i\omega r} r^{i\mu\omega}}_{\text{Infinity}} \underbrace{\left(\frac{r-r_+}{r}\right)^{-i\omega/c_0}}_{\text{Horizon}} \sum_{n=0}^{\infty} \begin{pmatrix} a_n r^{m_1}(r-r_+)^{m_2} \\ b_n r^{m_3}(r-r_+)^{m_4} \\ c_n r^{m_5}(r-r_+)^{m_6} \\ d_n r^{m_7}(r-r_+)^{m_8} \end{pmatrix} \left(\frac{r-r_+}{r}\right)^{n}$$

- Singular behaviour is known from asymptotic diagonalisation
- Exponents $m_i$ known from transfer matrices $P_\infty$ and $P_h$ used for diagonalisation at infinity and horizon

# Condition for convergence

Convergence criterion: continued fraction method [Rosa, Dolan '12; Leaver, Chandrasekhar '97; Gautschi '67]

**Recurrence relation obtained after Gauss reduction**

$$\boldsymbol{X}_n = \begin{pmatrix} a_n & b_n & c_n & d_n \end{pmatrix}$$

$$\boldsymbol{\alpha}_0 \boldsymbol{X}_1 + \boldsymbol{\beta}_0 \boldsymbol{X}_0 = 0$$

$$\boldsymbol{\alpha}_n \boldsymbol{X}_{n+1} + \boldsymbol{\beta}_n \boldsymbol{X}_n + \boldsymbol{\gamma}_n \boldsymbol{X}_{n-1} = 0\,.$$

**Matrix-valued continued fraction method**

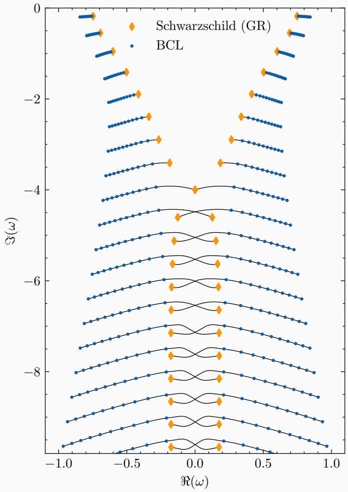
$$\boldsymbol{M}\boldsymbol{X}_0 = 0 \quad \Longrightarrow \quad \det(\boldsymbol{M}) = 0\,,$$

$$\boldsymbol{M} = \boldsymbol{\beta}_0 - \boldsymbol{\alpha}_0 \left[\boldsymbol{\beta}_1 - \boldsymbol{\alpha}_1 [\ldots]^{-1} \boldsymbol{\gamma}_2\right]^{-1} \boldsymbol{\gamma}_1$$

# BCL axial modes as proof of concept

Proof of concept for a $2 \times 2$ system

- Modification of the algebraically special mode
- Higher overtones are more sensitive to modifications
- Asymptotics: straight non-vertical line



# Conclusion

- Additional degrees of freedom make the computation of QNMs tricky
- First-order system analysis: extract boundary behaviour
- Use it to understand the physics of propagation and to get boundary conditions
- Method used in several setups to compute QNMs [Roussille, Langlois+ '24; Roussille, Larrouturou '24]

Thank you for your attention!