Scattering of Dirac Fields in Black Hole Interiors

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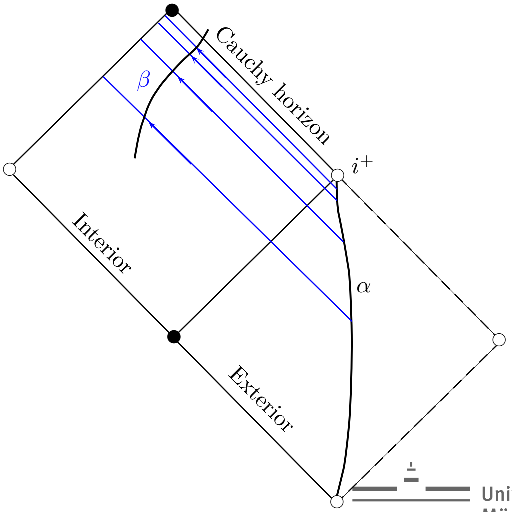
- Cauchy horizon:
  Data posed on a Cauchy hypersurface does not have unique solutions past CH.



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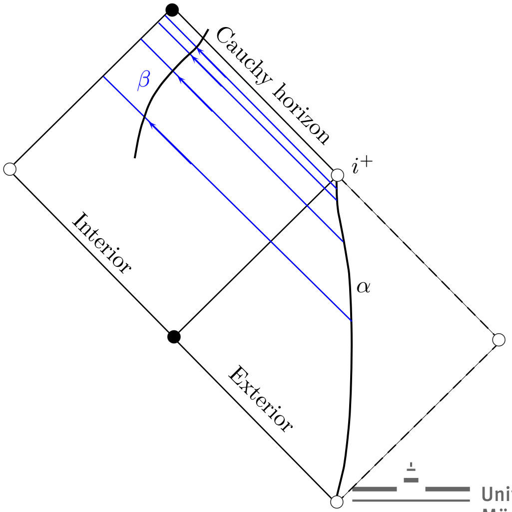
- Heuristic blue-shift:
  For a black hole with a Cauchy horizon, ingoing light rays build up infinitely, indicating blow-up.

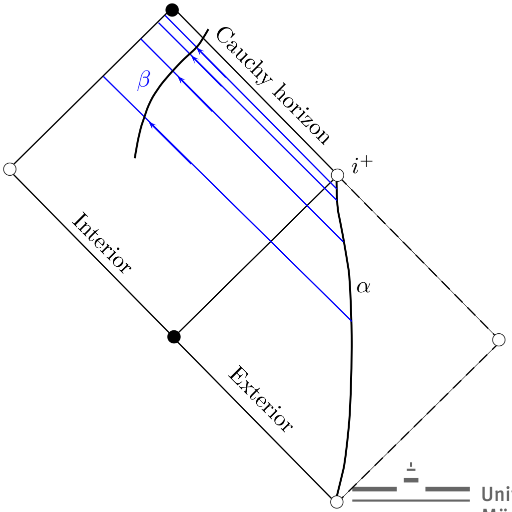


The Cauchy horizon and blue-shift

- Cauchy horizon:
  Data posed on a Cauchy hypersurface does not have unique solutions past CH.

- Heuristic blue-shift:
  For a black hole with a Cauchy horizon, ingoing light rays build up infinitely, indicating blow-up.

- Gravitational perturbations should behave similarly.



Previous scattering results for black hole interiors

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Penrose–Simpson '73, blue-shift instability of Maxwell, numerics.

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Kerr–Newman

Mokdad–P '23, scattering of Dirac, $\Lambda \in \mathbb{R}$.

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Black hole interiors

Illustrative example: Reissner-Nordström.
In Boyer-Lindquist coordinates
$(t, r, \theta, \varphi) \in \mathbb{R}_t \times (r_-, r_+)_r \times \mathcal{S}^2$,

$$g_{M,Q} := D(r)\, dt^2 - \frac{1}{D(r)} dr^2 - r^2 \mathring{g}\,,$$

with the horizon function

$$D(r) := \frac{(r - r_-)(r - r_+)}{r^2} < 0\,.$$

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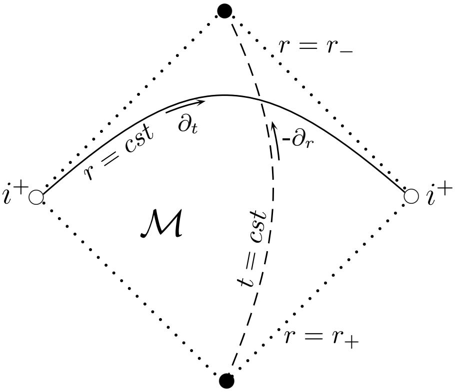
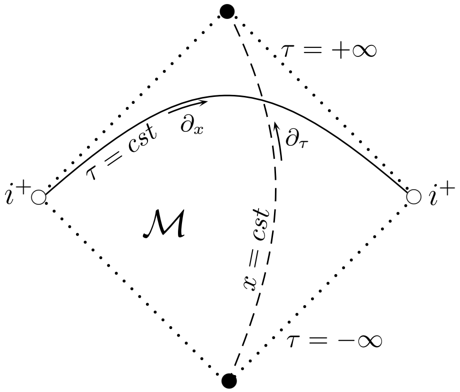


Illustrative example: Reissner-Nordström.
In Boyer-Lindquist coordinates
$(x, \tau, \theta, \varphi) \in \mathbb{R}_x \times (-\infty, +\infty)_\tau \times \mathcal{S}^2$,

$$g_{M,Q} = \widetilde{D}(r(\tau))d\tau^2 - \widetilde{D}(r(\tau))dx^2 - r(\tau)^2 \mathring{\not{g}}\,,$$

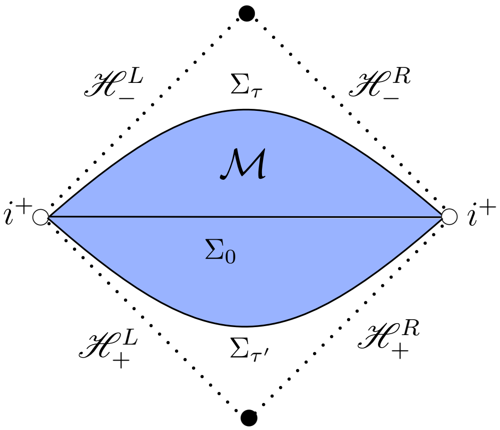
where

$$\widetilde{D}(r(\tau)) := -D(r(\tau)) > 0\,.$$



Take $(\mathcal{M} = \mathbb{R}_\tau \times \mathbb{R}_x \times \mathcal{S}^2, g = g_{M,Q,a,\Lambda})$.

$(\mathcal{M}, g)$ is globally hyperbolic:
$\implies$ each $\Sigma_\tau \cong \mathbb{R} \times \mathcal{S}^2 =: \Sigma$
$\implies$ $\mathcal{M} \cong \mathbb{R} \times \Sigma$.



We have the following asymptotics for functions of $r(\tau)$:

$$\begin{aligned} |r - r_\mp| &\sim e^{-2|\kappa_\mp||\tau|} \\ \widetilde{D} &\sim e^{-2|\kappa_\mp||\tau|} \end{aligned} \quad \text{as} \quad \tau \to \pm\infty \,.$$

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The Dirac equation

In a Newman-Penrose tetrad $\{l, n, m, \overline{m}\}$, four coupled complex scalar equations

$$n^a(\partial_a - iqA_a)\phi_0 - m^a(\partial_a - iqA_a)\phi_1 + (\mu_s - \gamma_s)\phi_0 + (\tau_s - \beta_s)\phi_1 = \frac{m}{\sqrt{2}}\chi^{0'}\,,$$

$$l^a(\partial_a - iqA_a)\phi_1 - \overline{m}^a(\partial_a - iqA_a)\phi_0 + (\alpha_s - \pi_s)\phi_0 + (\epsilon_s - \rho_s)\phi_1 = \frac{m}{\sqrt{2}}\chi^{1'}\,,$$

$$l^a(\partial_a - iqA_a)\chi^{0'} + m^a(\partial_a - iqA_a)\chi^{1'} + \overline{(\epsilon_s - \rho_s)}\chi^{0'} - \overline{(\alpha_s - \pi_s)}\chi^{1'} = -\frac{m}{\sqrt{2}}\phi_0\,,$$

$$n^a(\partial_a - iqA_a)\chi^{1'} + \overline{m}^a(\partial_a - iqA_a)\chi^{0'} - \overline{(\tau_s - \beta_s)}\chi^{0'} + \overline{(\mu_s - \gamma_s)}\chi^{1'} = -\frac{m}{\sqrt{2}}\phi_1\,.$$

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The system can be rewritten as a matrix equation for the *Dirac spinor*

$$\Psi = \begin{pmatrix} \phi_A \\ \chi^{A'} \end{pmatrix}, \quad A = 0, 1.$$

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Dirac $\rightarrow$ Schrödinger: the Hamiltonian

The Dirac equation in Schrödinger form for Ψ on $(\mathcal{M}, g_{M,Q,a,\Lambda})$ reads

$$i\partial_\tau \Psi = H(\tau)\Psi\,,$$

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where

$$H(\tau,\theta) := H_0(\tau) + \sqrt{\widetilde{D}(r(\tau))}\left(M_0(\tau,\theta) + H_1(\theta) + \sqrt{\Delta_\theta}\,\not{D}(\theta)\right)\,,$$

and (for $D = -i\partial$; unitary matrices)

$$H_0 := -\Gamma_z\left(D_x + \frac{a}{r^2+a^2}D_\varphi - \frac{qQr}{r^2+a^2}\right)\,,$$

$$M_0 := iM_- mr + M_+ ma\cos\theta\,,$$

$$H_1 := \Gamma_x \frac{i\Delta_\theta'}{4\sqrt{\Delta_\theta}} + \Gamma_y \frac{a\sin\theta}{\sqrt{\Delta_\theta}}\left(\lambda D_x + \frac{\Lambda a}{3}D_\varphi\right)\,,$$

$$\Delta_\theta := 1 + \frac{\Lambda a^2}{3}\,, \quad \not{D} := i\Gamma_x\left(\partial_\theta + \frac{\cot\theta}{2}\right) - i\Gamma_y\frac{\partial_\varphi}{\sin\theta}\,.$$

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A conserved quantity for the Dirac equation

The Dirac Hamiltonian $H(\tau)$ is symmetric on the space $L^2(\Sigma;\, \mathbb{C}^4)$.
Hence for two solutions $\Psi_1, \Psi_2 \in L^2(\Sigma;\, \mathbb{C}^4)$,

$$i\partial_\tau \langle \Psi_1, \Psi_2 \rangle_{L^2(\Sigma)} = \langle H\Psi_1, \Psi_2 \rangle_{L^2(\Sigma)} - \langle \Psi_1, H\Psi_2 \rangle_{L^2(\Sigma)} = 0\,.$$

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This means for data ψ of a solution Ψ

$$\forall \tau \in \mathbb{R}\,, \quad \|\Psi\|_{L^2(\Sigma)} = \|\psi\|_{L^2(\Sigma)}\;.$$

Define the Hilbert space:

$$\mathcal{H} := L^2(\Sigma;\, \mathbb{C}^4)\,.$$

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Functional spaces

Recall: $\mathcal{H} = L^2(\Sigma; \mathbb{C}^4)$.

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$H^1_{\mathrm{sp}}(\Sigma;\, \mathbb{C}^4)$ is the space of Dirac spinors ξ satisfying

$$\|\xi\|^2_{H^1_{\mathrm{sp}}(\Sigma)} := \|\xi\|^2_{\mathcal{H}} + \|\partial_x \xi\|^2_{\mathcal{H}} + \|\widetilde{\nabla}\xi\|^2_{\mathcal{H}} < \infty\,,$$

where $\widetilde{\nabla}$ is the spinorial connection on $\mathcal{S}^2$ with components

$$\widetilde{\nabla}_\theta \xi = \partial_\theta \xi \quad \text{and} \quad \widetilde{\nabla}_\varphi \xi = \partial_\varphi \xi + i\Gamma_z \frac{\cos\theta}{2}\xi\,, \quad \Gamma_z = \begin{pmatrix} \sigma_z & 0 \\ 0 & -\sigma_z \end{pmatrix}.$$

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One can verify that $\|\not{D}\xi\|^2_{\mathcal{H}} \approx \|\xi\|^2_{\mathcal{H}} + \|\widetilde{\nabla}\xi\|^2_{\mathcal{H}}$, and

$$\left\|\not{D}\xi\right\|_{\mathcal{H}} \lesssim \|\xi\|_{H^1_{\mathrm{sp}}(\Sigma)}\,, \quad \|H(\tau)\xi\|_{\mathcal{H}} \lesssim \|\xi\|_{H^1_{\mathrm{sp}}(\Sigma)}\,.$$

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Problem: $\nabla \sim \partial + \Gamma(\tau)$.

Time-dependent Hamiltonians and scattering

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Pass from a time-dependent IVP to one with "free" (time-independent) dynamics:

$$(\star) \quad \begin{cases} i\partial_\tau \Psi = H(\tau)\Psi \\ \Psi|_{\tau=0} = \psi \end{cases} \text{ on } \mathcal{H} \quad \longrightarrow \quad \begin{cases} i\partial_\tau \Psi^+ = H_0^+ \Psi^+ \\ \Psi^+|_{\tau=0} = \psi_0 \end{cases} \text{ on } \mathcal{H}\,.$$

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We would like:

$$\forall \psi \in \mathcal{H}\,, \quad \exists \psi_0 \in \mathcal{H} \,: \quad \left\| \mathcal{U}(\tau)\psi - e^{-iH_0^+ \tau}\psi_0 \right\|_{\mathcal{H}} \xrightarrow[\tau \to +\infty]{} 0\,,$$

where

$$\Psi(\tau) = \mathcal{U}(\tau)\psi \quad \text{and} \quad \Psi^+(\tau) = e^{-iH_0^+ \tau}\psi_0\,.$$

ψ_0 is called the *future scattering data* for $(\star)$. Equivalently,

$$\psi_0 = \lim_{\tau \to +\infty} e^{+iH_0^+ \tau}\,\mathcal{U}(\tau)\psi\,.$$

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Wave operators

This is equivalent to defining the *direct and inverse wave operators*

$$W^{\pm} := \underset{t\to\pm\infty}{s\text{–}\lim} W^{\pm}(t) \quad \text{and} \quad \Omega^{\pm} := \underset{t\to\pm\infty}{s\text{–}\lim} \Omega^{\pm}(t)\,,$$

where

$$W^{\pm}(t) := \mathcal{U}(t)^{-1} e^{-iH_0^{\pm} t} \quad \text{and} \quad \Omega^{\pm}(t) := e^{+iH_0^{\pm} t}\,\mathcal{U}(t)\,.$$

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Existence of $W^{\pm}$ and $\Omega^{\pm}$ $\implies$ existence, uniqueness, and asymptotic completeness of scattering data.

Existence is usually proven via *Cook's method*:

Check that $\left(\partial_t W^{\pm}(t)\psi,\, \partial_t \Omega^{\pm}(t)\psi\right) \in L^1(\mathbb{R};\, \mathcal{H} \times \mathcal{H})$ for each $\psi \in \mathcal{H}$.

Main result

Theorem (Mokdad-P '23)

Let $\psi \in \mathcal{H}$ and Ψ be a weak solution to the Dirac equation

$$\begin{cases} i\partial_\tau \Psi = H\Psi \,, \\ \Psi|_{\tau=0} = \psi \,. \end{cases}$$

with Hamiltonian $H = H_0 + \sqrt{\widetilde{D}}\left(\sqrt{\Delta_\theta}\not{D} + H_1 + M_0\right)$. Define $H_0^\pm := H_0|_{\tau\to\pm\infty}$. Then

$$\begin{aligned} W^\pm &:= \operatorname*{s\text{–}lim}_{\tau\to\pm\infty} \mathcal{U}(\tau)^{-1} e^{-i\tau H_0^\pm} \,, \\ \Omega^\pm &:= \operatorname*{s\text{–}lim}_{\tau\to\pm\infty} e^{+i\tau H_0^\pm} \mathcal{U}(\tau) \,. \end{aligned}$$

exist on $\mathcal{H}$.

Symmetry operators for special cases

Recall

$$H = \overbrace{\left(H_0 + imr\sqrt{\widetilde{D}}M_-\right)}^{\text{radial part}} + \sqrt{\widetilde{D}}\overbrace{\left(\sqrt{1 + \frac{\Lambda a \cos^2\theta}{3}}\not{D} + H_1 + M_+ ma\cos\theta\right)}^{\text{angular part}} .$$

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- Let $m = 0$. Then $H|_{m=0}$ commutes with $\left(\sqrt{\Delta_\theta}\not{D} + H_1\right)^2$.

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- Let $m = 0$. Then $H|_{m=0}$ commutes with $\left(\sqrt{\Delta_\theta}\not{D} + H_1\right)^2$.
- Let $a = 0$. Then $H|_{a=0}$ commutes with $\not{D}^2$ due to $[M_-, \Gamma_j] = 0$.

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- Let $m = 0$. Then $H|_{m=0}$ commutes with $\left(\sqrt{\Delta_\theta}\not{D} + H_1\right)^2$.
- Let $a = 0$. Then $H|_{a=0}$ commutes with $\not{D}^2$ due to $[M_-, \Gamma_j] = 0$.
- We define

$$Q := \left(\sqrt{\Delta_\theta}\not{D} + H_1\right)^2,$$

which is a **symmetry operator** when either $m = 0$ or $a = 0$.

Symmetry operators for special cases

For solutions Ψ with data ψ at $\tau = 0$, we have

$$(\star) \quad \begin{cases} i\partial_\tau \Psi = H\Psi \,, \\ \Psi|_{\tau=0} = \psi \,. \end{cases} \qquad \Longrightarrow \qquad \begin{cases} i\partial_\tau Q\Psi = HQ\Psi \,, \\ Q\Psi|_{\tau=0} = Q\psi \,. \end{cases}$$

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Hence: $\forall \tau \in \mathbb{R}\,, \quad \|Q\mathcal{U}(\tau)\psi\|_{\mathcal{H}} = \|Q\psi\|_{\mathcal{H}}$.

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Hence: $\forall \tau \in \mathbb{R} \,, \quad \|Q\,\mathcal{U}(\tau)\psi\|_{\mathcal{H}} = \|Q\psi\|_{\mathcal{H}}$.

Proof via Cook's method greatly simplified: $\forall \psi \in D := H^1_{\mathrm{sp}}(\Sigma;\, \mathbb{C}^4)$,

$$\left\|\partial_\tau \left(e^{+i\tau H_0^\pm}\,\mathcal{U}(\tau)\psi\right)\right\|_{\mathcal{H}}^2 = \|(H - H_0^\pm)\,\mathcal{U}(\tau)\psi\|_{\mathcal{H}}^2$$
$$\leq \|H_0 - H_0^\pm\|^2_{\mathcal{B}(D;\,\mathcal{H})}\,\|\psi\|_D^2 + \widetilde{D}\left\|Q^{1/2}\psi\right\|_{\mathcal{H}}^2 + m^2\widetilde{D}\,\|\psi\|_{\mathcal{H}}^2 \,.$$

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Failure to commute in the general case

Recall

$$H := \overbrace{\left(H_0 + {\color{red} imr\sqrt{\widetilde{D}}M_-}\right)}^{\text{radial part}} + \sqrt{\widetilde{D}}\overbrace{\left(\sqrt{\Delta_\theta}\not{D} + H_1 + {\color{blue} M_+ ma\cos\theta}\right)}^{\text{angular part}} .$$

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$$\text{Consider for } j \in \{0,1\}, \quad Q_j := \left(\sqrt{\Delta_\theta}\not{D} + H_1 + M_+ jma\cos\theta\right)^2.$$

Unfortunately,

- Q_1 commutes with the angular part, but does not commute with M_- due to terms like $\Gamma_x M_+$. $(\Gamma_x M_+ \Gamma_z = -\Gamma_z \Gamma_x M_+.)$
- Q_0 commutes with the radial part, but does not commute with $\cos\theta$ due to ∂_θ in $\not{D}$.

Therefore, given a solution Ψ to $(\star)$, $Q_j\Psi$ is not a solution.

Main difficulty: control of $\not{D}$

Cook's method will produce the term

$$\widetilde{D}\left\|\not{D}\,\mathcal{U}(\tau)\psi\right\|_{\mathcal{H}}^{2}.$$

No obvious way to commute $\not{D}$ through $\mathcal{U}(\tau)$. Even so,

Proposition (Mokdad-P '23)

Let $\psi \in D = H^1_{\mathrm{sp}}(\Sigma;\mathbb{C}^4)$. *Then* $\forall \tau \in \mathbb{R}$, *we have*

$$\left\|\not{D}\,\mathcal{U}(\tau)\psi\right\|_{\mathcal{H}} \lesssim \|\psi\|_{H^1_{\mathrm{sp}}(\Sigma)}.$$

The comparison operator B

We define $B := \not{D}^2 + {D_x}^2$ to have $\forall \psi \in D$

$$\left\| B^{1/2} \psi \right\|_{\mathcal{H}}^2 = \langle B\psi, \psi \rangle_{\mathcal{H}} = \| D_x \psi \|_{\mathcal{H}}^2 + \left\| \not{D} \psi \right\|_{\mathcal{H}}^2 \eqsim \| \psi \|_D^2 \, .$$

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This ensures:

- $\left\| \not{D} B^{-1/2} \right\|_{\mathcal{B}(\mathcal{H})} \leq 1,$
- $\forall \tau \in \mathbb{R}, \quad B^{1/2} \mathcal{U}(\tau) B^{-1/2} \in \mathcal{B}(\mathcal{H}).$

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- $\left\|\not{D}B^{-1/2}\right\|_{\mathcal{B}(\mathcal{H})} \leq 1$,
- $\forall\tau \in \mathbb{R}, \quad B^{1/2}\mathcal{U}(\tau)B^{-1/2} \in \mathcal{B}(\mathcal{H})$.

The last statement is an application of Grönwall's lemma for $k(\tau) = \left\|B^{1/2}\mathcal{U}(\tau)B^{-1/2}u\right\|_{\mathcal{H}}^2$ and requires showing that

$$B^{-1/2}[H, B]B^{-1/2} \in L^1(\mathbb{R}_\tau; \mathcal{B}(\mathcal{H}))\, .$$

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Proof of Proposition 4.1

Proposition (Mokdad-P '23)

Let $\psi \in D = H^1_{\mathrm{sp}}(\Sigma;\mathbb{C}^4)$*. Then* $\forall \tau \in \mathbb{R}$*, we have*

$$\left\| \not{D}\,\mathcal{U}(\tau)\psi \right\|_{\mathcal{H}} \lesssim \|\psi\|_{H^1_{\mathrm{sp}}(\Sigma)} \,.$$

Proof. Using the properties of B, we have for $\psi \in D$,

$$\begin{aligned} \left\| \not{D}\,\mathcal{U}(\tau)\psi \right\|_{\mathcal{H}} &= \left\| \not{D} B^{-1/2} B^{1/2} \mathcal{U}(\tau) B^{-1/2} B^{1/2} \psi \right\|_{\mathcal{H}} \\ &\leq \left\| \not{D} B^{-1/2} \right\|_{\mathcal{B}(\mathcal{H})} \left\| B^{1/2} \mathcal{U}(\tau) B^{-1/2} \right\|_{\mathcal{B}(\mathcal{H})} \left\| B^{1/2}\psi \right\|_{\mathcal{H}} \lesssim \|\psi\|_D \,. \end{aligned}$$

□

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Why B?

We recall the operators

$$B = \not{D}^2 + {D_x}^2\,,$$

$$Q_j = \left(\sqrt{\Delta_\theta}\not{D} + H_1 + M_+ jma\cos\theta\right)^2\,,\quad j \in \{0,1\}\,.$$

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B versus Q_j

- Downside: $[B, H]$ computation is a LARGE technical portion of the proof. Greatly shortened by using Q_j.
- Upside: B is simple and $B^{1/2}$ is equivalent to the H^1_{sp} −norm. Relating Q_j to the H^1_{sp} −norm seems unruly.

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Prospects

We have shown existence of the unitary *scattering operator* $S = \Omega^+ W^-$ on $\mathcal{H}$.

Using stationary scattering (reflection and transmission coefficients in the frequency domain), one hopes to prove blow-up using the poles of the holomorphic extension of the Fourier transform of S.[1]

[1] Upcoming work by N. Boussaid, T. Daudé, and M. Mokdad on the interior of RN(A)dS.

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Thank you!

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