Memory effects in a gravitational plane wave

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Mathematical Physics of Gravity and Symmetry

Bourgogne University, Dijon

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Based on
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In collaboration with Jean-Philippe Uzan

Context and Motivations

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- Permanent shifts in relative observables associated to a couple of test particles after the passage of a gravitational wave:

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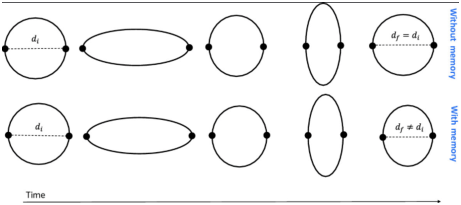

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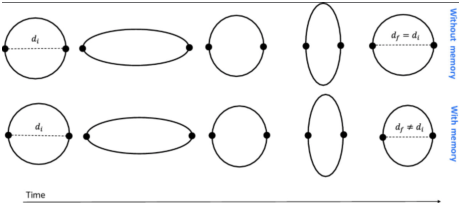

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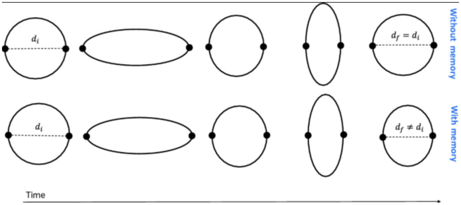

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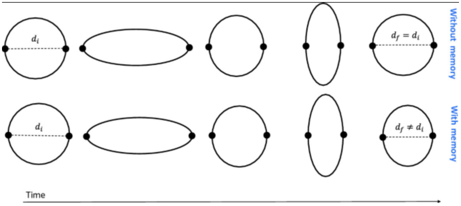

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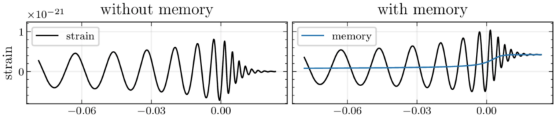

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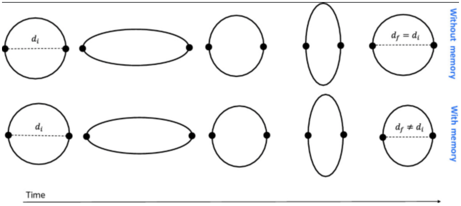

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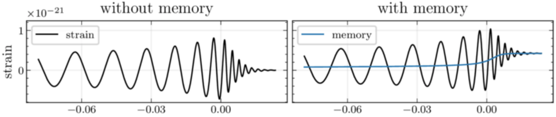

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History of memory effects in short:

- Linearized gravity: displacement memory [Zel'dovich and Polnarev '74]
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Relation between memory and the symmetries of spacetime

- Associated to the degeneracy of the vacuum in any gauge theory
- Gravitational memory effects $\leftrightarrow$ flux-balance laws $\leftrightarrow$ symmetries of open systems

$$\delta Q = \mathcal{F} \tag{1}$$

- Reveal the fine structure of the infrared regime of asymptotically flat gravity

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Memory effects tells us about the explicit and hidden symmetries of spacetime and vice-versa

- New memories recently identified: Spin memory, centered-of-masse memory, gyroscope [Pasterski, Strominger, Zhiboedov '16] [Nichols '18][Seraj, Oblak '23]
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How can we relate memory effects to explicit or hidden symmetries of spacetimes?

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- Need a careful classification of memories in pp-wave geometries

Step by step

- Identify all the symmetries of the geometries
- Solve the geodesic motion
- Construct the Fermi coordinates
- Solve the geodesic deviation equation
- Compare the asymptotic behavior $\rightarrow$ symmetry interpretation

Generalities on vacuum pp-wave

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- BJR coordinates are not global. Need to switch to Brinkman coordinates which play the role of null Fermi coordinates when studying memories ...

A new symmetry of vacuum gravitational plane wave

Raychaudhuri as a Schwarzian

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- First point: the Raychaudhuri equation admits a $\mathrm{SL}(2,\mathbb{R})$ symmetry

Conformal symmetry of the Raychaudhuri equation

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- Large freedom in the choice of wave-profile solving the Einstein equation
- Symmetry reduced version of a more general structure found later for any null hypersurfaces [Ciambelli, Leigh, Freidel '24]

Explicit and hidden symmetries of pp-waves
(for the geodesic motion)

Explicit and hidden symmetries

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- Consider first the conformal killing equations:

$$\mathcal{L}_\xi g_{\mu\nu} = \Omega g_{\mu\nu} \,,$$

which splits into

$$\begin{aligned}
\partial_v \xi^u &= 0 \,, \\
\partial_u \xi^v &= 0 \,, \\
\partial_i \xi^u + A_{ij} \partial_v \xi^j &= 0 \,, \\
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Conformal isometries of pp-waves

- At this level, the statements are valid also when $A_{12} \neq 0$
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- We are interested in the CKV which hold for any wave-profile $A_{ij}(u)$

Conformal isometries of pp-waves

- 5d isometry group : focus first on KV, i.e. on $\Omega = 0$

Explicit and hidden symmetries

Conformal isometries of pp-waves

- 5d isometry group : focus first on KV, i.e. on $\Omega = 0$
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$$\mathcal{N}^\alpha \partial_\alpha = \partial_v \ , \qquad P_+^\alpha \partial_\alpha = \partial_x \qquad P_-^\alpha \partial_\alpha = \partial_y$$

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$$H^{ij}(u_0, u) \equiv \int_{u_0}^{u} A^{ij}(w)\mathrm{d}w$$

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- In a more condensed form, the KV are given by

$$\mathcal{L}_\xi g_{\mu\nu} = 0 \qquad \xi^\alpha \partial_\alpha = \left\{h + b_i x^i\right\}\partial_v + \left[\chi^i + b_j H^{ij}(u_0, u)\right]\partial_i \,.$$

  with 5 parameters (h, χ^i, b_i)

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- Does the pp-wave admit more symmetries ? Hidden symmetries ?

Conformal isometries of pp-waves

- Consider the CVK: $\Omega \neq 0$
- If we look for a CKV for any A_{ij}, only one solution: HKV for $\Omega = 2$

$$Z^{\alpha}\partial_{\alpha} = 2v\partial_{v} + x^{i}\partial_{i}\,. \tag{13}$$

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Hidden symmetry of pp-waves

- Koutras theorem:
  if a spacetime admits both a gradient Killing vector and a HKV then it also admits a non-trivial rank-2 Killing tensor (KT) which generates an additional symmetry.
  With the gradient KV and the HKV given by

$$\xi_\alpha \mathrm{d}x^\alpha = (\partial_\alpha \Phi)\mathrm{d}x^\alpha \qquad Z_\alpha \mathrm{d}x^\alpha \tag{15}$$

the KT is explicitely given by

$$K_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \left[Z_{(\mu}\xi_{\nu)} - \Phi g_{\mu\nu}\right]\mathrm{d}x^\mu \mathrm{d}x^\nu. \tag{16}$$

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$$\boxed{K_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = 2v\mathrm{d}u^2 - u(2\mathrm{d}u\mathrm{d}v + A_{ij}\mathrm{d}x^i\mathrm{d}x^j) + A_{ij}x^j\mathrm{d}u\mathrm{d}x^i \,.} \tag{18}$$

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- All these symmetries hold for any wave-profile $A_{ij}(u)$!
- Can be use to integrate the geodesic motion ... and the geodesic deviation equation

Integrating the geodesic motion from the symmetries

Geodesic motion

- Lagrangian

$$L = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \dot{u}\dot{v} + \frac{1}{2} A_{ij} \dot{x}^i \dot{x}^j \tag{19}$$

gives the equations

$$\ddot{v} - \frac{1}{2} A'_{ij} \dot{x}^i \dot{x}^j = 0 \,, \qquad A_{ij} \ddot{x}^j + A'_{ij} \dot{x}^j \dot{u} = 0 \,, \qquad \ddot{u} = 0 \,, \tag{20}$$

with a prime refering to a derivative w.r.t. u.

- Phase space of geodesic motion

$$\{v, p_v\} = \{u, p_u\} = 1 \,, \qquad \{x^i, p_j\} = \delta^i{}_j \,. \tag{21}$$

with momenta

$$p_u = \frac{\delta \mathcal{L}}{\delta \dot{u}} = \dot{v} \,, \tag{22}$$

$$p_i = \frac{\delta \mathcal{L}}{\delta \dot{x}^i} = A_{ij} \dot{x}^j \,, \tag{23}$$

$$p_v = \frac{\delta \mathcal{L}}{\delta \dot{v}} = \dot{u} \,, \tag{24}$$

and hamltonian

$$H = p_u p_v + \frac{1}{2} A^{ij} p_i p_j = \frac{\epsilon}{2} \,. \quad \epsilon = \{0, -1\} \tag{25}$$

- What are the conserved charges ?

Algebraic integration of the geodesic flow

- Conserved charges

$$\xi^\alpha \partial_\alpha \to \mathcal{O} = \xi^\alpha p_\alpha \qquad K_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \to K = K^{\mu\nu} p_\mu p_\nu \tag{26}$$

- Translations: Since the Hamiltonian does not depend on neither v nor x^i, p_v and p_i are automatically conserved. We denote them as

$$\mathcal{N} = p_v\,, \qquad P_+ = p_x\,, \qquad P_- = p_y\,, \tag{27}$$

- Carrolian boost: The conserved charges generating the boosts are given by

$$\mathcal{B}_+ = H^{xx}(u) p_x + H^{xy}(u) p_y - p_v x(u)\,, \tag{28}$$

$$\mathcal{B}_- = H^{yy}(u) p_y + H^{yx}(u) p_x - p_v y(u)\,. \tag{29}$$

- Hidden Killing tensor charge: charge coming from the Killing tensor reads

$$\mathcal{K} = K^{\mu\nu} p_\mu p_\nu = p_v \mathcal{Z} - 2uH \qquad \text{with} \qquad \mathcal{Z} = 2 p_v v + p_i x^i\,, \tag{30}$$

- Charge algebra

$$\{P_\pm, \mathcal{B}_\pm\} = \mathcal{N}\,. \tag{31}$$

$$\{P_\pm, \mathcal{K}\} = \mathcal{N} P_\pm\,, \qquad \{B_\pm, \mathcal{K}\} = \mathcal{N} B_\pm\,, \qquad \{\mathcal{N}, \mathcal{K}\} = 2\mathcal{N}^2\,, \tag{32}$$

- Geodesic motion integrable since $(\mathcal{N}, \mathcal{B}_\pm, H)$ are in involution

Geodesic motion

Algebraic integration of the geodesic flow

- Trivial equation for $\dot{u}$:

$$u = p_v \tau = \mathcal{N}\tau. \tag{33}$$

- Transverse motion: Carrolian boosts allows to write the x and y trajectories as

$$x(u) = \frac{1}{\mathcal{N}}\left[H^{xx}(u_0,u)P_+ + H^{xy}(u_0,u)P_-\right] - \frac{\mathcal{B}_+}{\mathcal{N}}\,, \tag{34}$$

$$y(u) = \frac{1}{\mathcal{N}}\left[H^{yy}(u_0,u)P_- + H^{yx}(u_0,u)P_+\right] - \frac{\mathcal{B}_-}{\mathcal{N}}\,, \tag{35}$$

- Longitudinal motion: Combining the HKV and the KT charge give v-trajectory

$$v(u) = \frac{1}{2\mathcal{N}^2}\left[\epsilon u - H^{ij}(u_0,u)p_i p_j + p_i\mathcal{B}^i + \mathcal{K}\right]. \tag{36}$$

- Relations between initial conditions and conserved charges

$$v_0 \equiv v(u_0) = \frac{1}{2\mathcal{N}^2}\left[\epsilon u_0 - \mathcal{N} p_i x_0^i + \mathcal{K}\right]. \qquad x_0^i \equiv x^i(u_0) = -\frac{\mathcal{B}^i}{\mathcal{N}}\,. \tag{37}$$

- More important for the memories, the x-trajectory reduces to

$$\boxed{x^i(u) = \frac{1}{\mathcal{N}}\left[H^{ij}(u_0,u)p_j - \mathcal{B}^i\right]} \tag{38}$$

Properties of a timelike geodesic congruence

- With the exact solution to the geodesic, we have the 4-velocity

$$
\begin{aligned}
u^u &= \frac{\mathrm{d}u}{\mathrm{d}\tau} = \mathcal{N}\,, \\
u^i &= \frac{\mathrm{d}x^i}{\mathrm{d}\tau} = A^{ij} p_j\,, \\
u^v &= \frac{\mathrm{d}v}{\mathrm{d}\tau} = \frac{1}{2\mathcal{N}} \left(\epsilon - A^{ij} p_i p_j\right)\,.
\end{aligned} \tag{39}
$$

- We can compute the invariant quantities: expansion, shear and rotation

$$\Theta = \nabla_\mu u^\mu = \mathcal{N} \varrho \tag{40}$$

$$\sigma = \sigma_{\mu\nu}\sigma^{\mu\nu} = -\mathcal{N}^2 \left[\dot{A}^{ij}\dot{A}_{ij} + \frac{2}{3}\varrho(\varrho + \dot{A}_{ij}p^i p^j) - \frac{1}{9}\varrho^2 (p_i p^i)^2\right]\,, \tag{41}$$

$$\omega_{\alpha\beta} = \nabla_{[\alpha} u_{\beta]} = 0 \tag{42}$$

with $\varrho = A^{ij}\dot{A}_{ij}$

- $u^\alpha \partial_\alpha$ is hyper-surface orthogonal and the effect of the wave is to expand/contract and shear the congruence of geodesics

Constructing the Fermi coordinates

Fermi coordinates

- Geodesic deviation cannot be analyzed in an arbitrary coordinates system
- First task, construct the adapted Fermi normal coordinates (either null or timelike)
- For null geodesic, the procedure is closely related to the Penrose limit [Penrose '76]

Fermi coordinates

Adapted Fermi normal coordinates

- Pick up a null geodesic $\bar{\gamma}$ with tangent vector

$$\bar{u}^\mu \partial_\mu = \partial_u \qquad \bar{u}_\mu \mathrm{d}x^\mu = \mathrm{d}v\,. \tag{43}$$

- Introduce a set of adapted Fermi coordinates, X^I, with $I \in \{0, \ldots, 3\}$ related to the initial coordinates x^μ via

$$E^I{}_\mu \equiv \frac{\partial X^I}{\partial x^\mu}\,. \tag{44}$$

- Choose the "time-leg" such that the coordinate X^0 coincides with the affine parameter of the geodesic

$$\bar{E}^0{}_\mu \mathrm{d}x^\mu = \bar{u}_\mu \mathrm{d}x^\mu\,. \tag{45}$$

  Impose that i) the remaining legs be parallel transported along the null geodesic, and ii) the orthogonality relations

$$\bar{u}^\mu \nabla_\mu E^I{}_\nu = 0 \qquad g_{\mu\nu}|_{\bar{\gamma}} = \bar{E}^I{}_\mu \bar{E}^J{}_\nu \eta_{IJ} \tag{46}$$

- In our case, one obtains

$$\dot{E}^A{}_i = \frac{1}{2} A^{jk} \dot{A}_{ki} E^A_j\,, \qquad \dot{E}^i{}_A = -\frac{1}{2} A^{ik} \dot{A}_{kj} E^j{}_A \tag{47}$$

- Analyzing the effects in the Fermi coordinates requires to analytically solve this PT equation

Fermi coordinates

- Fermi coordinates : $X^0 = U$, $X^1 = V$ and $X^A = \{X, Y\}$ such that the transverse space w.r.t. the reference geodesic admits the coordinates $X^a \in \{V, X^A\}$.
- Then, the Taylor expansion of the BJR coordinates $x^\mu(U, V, X^A)$ up to second order reads

$$x^\mu(U, V, X^A) = x^\mu(U) + \bar{E}^\mu{}_a(U)X^a - \frac{1}{2}\bar{E}^\alpha{}_a(U)\bar{E}^\beta{}_b(U)\Gamma^\mu{}_{\alpha\beta}(U)X^aX^b\,. \tag{48}$$

- It follows that the Fermi and BJR coordinates are related by

$$u = U\,, \tag{49}$$

$$v = V + \frac{1}{4}\dot{A}_{ij}\bar{E}^i{}_A\bar{E}^j{}_B X^A X^B\,, \tag{50}$$

$$x^i = \bar{E}^i{}_A X^A\,. \tag{51}$$

- Ssetting $V = 0$ and $X^A = 0$, one recovers the position of the reference geodesic $\bar{\gamma}$.
- The vacuum GPW metric becomes in the Fermi coordinates

$$\boxed{\mathrm{d}s^2 = 2\mathrm{d}U\mathrm{d}V + \delta_{AB}\mathrm{d}X^A\mathrm{d}X^B + H_{AB}(U)X^AX^B\mathrm{d}U^2\,,} \tag{52}$$

with the wave-profile

$$H_{AB} = \frac{1}{2}E^i{}_A\,\partial_u\left(\dot{A}_{ij}E^j{}_B\right)\,. \tag{53}$$

- Known as the Brinkmann coordinates which are the one use to analyze the physical effects

Fermi coordinates

A quick look at Einstein equation in the Brinkmann form

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- Einstein equation translates into

$$H^A{}_A = 0 \qquad \to \qquad H_{AB} = \begin{pmatrix} H_+ & H_\times \\ H_\times & -H_+ \end{pmatrix} \tag{56}$$

- Any profile $(H_+, H_\times)$ is solution of Einstein equation
- Relation to initial description for the polarized wave-profile

$$H_+ = \frac{1}{2}\left[\frac{\ddot{A}_{11}}{A_{11}} - \frac{1}{2}\left(\frac{\dot{A}_{11}}{A_{11}}\right)^2\right] = -\frac{1}{2}\left[\frac{\ddot{A}_{22}}{A_{22}} - \frac{1}{2}\left(\frac{\dot{A}_{22}}{A_{22}}\right)^2\right]\,. \tag{57}$$

$$= \mathrm{Sch}[F] = -\mathrm{Sch}[G] \tag{58}$$

or with $A_{ii}(u) \equiv \mathcal{A}_i^2(u)$

$$H_+ = \frac{\ddot{\mathcal{A}}_1}{\mathcal{A}_1} = -\frac{\ddot{\mathcal{A}}_2}{\mathcal{A}_2} \tag{59}$$

- For a given example, either choose H_+ and determine (A_{11}, A_{22}) or the reverse: not so easy in practice

A quick look at Einstein equation in the Brinkmann form

- The vacuum GPW metric becomes in the Fermi coordinates

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What is the relationship of the Brinkman coordinates with the Penrose limit ?

A brief shortcut to the Penrose limit

- Concretely, pick up a null geodesic γ and construct null Fermi coordinates $X^A = (U, V, X^i)$ with $i \in (1, 2)$ adapted to the region around the geodesic

$$x^a = E^a_A X^A + E^a_\mu \bar{\Gamma}^\mu{}_{AB} X^A X^B + \mathcal{O}((X^A)^3) \tag{60}$$

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- Read the polarizations from the matrix $A_{ij}(U)$
- Full non-perturbative approach: we never ask that $A_{ij}(U)$ be "small"
- For pp-wave, the Penrose limit is exact $\to$ simply construct the Fermi or Brinkman coordinates

Geodesic deviation from the symmetries

Symmetries of the geodesic deviation equation

Geodesic deviation

- Consider two nearby curves $\bar{X}^\mu(\tau) := X^\mu(\tau, \sigma = 0)$ and $X^\mu(\tau, \sigma)$
- Their relative distance expands as folllows

$$\Delta X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma) - \bar{X}^\mu(\tau, 0) = \sigma N^\mu(\tau) \tag{66}$$

$$+ \sigma^2 \left(B^\mu - \bar{\Gamma}^\mu{}_{\alpha\beta} N^\alpha N^\beta \right)(\tau) + \mathcal{O}(\sigma^3) \tag{67}$$

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- First order deviation vector satisfies the following dynamics

$$\boxed{\frac{D^2 N^\mu}{\mathrm{d}\tau^2} = \bar{R}^\mu{}_{\alpha\beta\gamma} \bar{u}^\alpha \bar{u}^\beta N^\gamma} \tag{68}$$

- Second order deviation equation: Bazanski equation

$$\frac{D^2 b^\mu}{\mathrm{d}\tau^2} = \bar{R}^\mu{}_{\alpha\beta\gamma} \bar{u}^\alpha \bar{u}^\beta b^\gamma + \left[\bar{\nabla}_\nu \bar{R}_{\lambda\rho\sigma}{}^\mu - \bar{\nabla}_\lambda \bar{R}_{\nu\sigma\rho}{}^\mu \right] \bar{u}^\lambda \bar{u}^\sigma N^\rho N^\nu + 4 \bar{R}_{\lambda\rho\sigma}{}^\mu \bar{u}^\lambda N^\rho \bar{u}^\alpha \nabla_\alpha N^\sigma \tag{69}$$

- And so on ...

Parallel versus orthogonal deviation

- Split parallel and orthogonal contributions to the deviation: same for memories

$$N = N_\parallel + N_\perp \qquad N_\parallel = f(\tau)\bar{u} \qquad f(\tau) = C_1\tau + C_2\,. \tag{70}$$

- The remaining equation is hard to solve in general

What are the symmetry of this geodesic deviation equation ?

Symmetries of the geodesic deviation equation

Relation to hidden symmetries

- The solution space related to the explicit and hidden symmetries of the spacetime:
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$$L[N] = \frac{1}{2}\bar{g}_{\mu\nu}\bar{u}^{\alpha}\bar{u}^{\beta}\bar{\nabla}_{\alpha}N^{\mu}\bar{\nabla}_{\beta}N^{\nu} - \frac{1}{2}\bar{R}_{\mu\nu\alpha\beta}\bar{u}^{\mu}\bar{u}^{\alpha}N^{\nu}N^{\beta} \tag{71}$$

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- Admit a hidden symmetry given by

$$\delta N^{\mu}=\bar{K}^{\mu}{}_{\alpha_1....\alpha_p}\bar{u}^{\alpha_1}....\bar{u}^{\alpha_p} \qquad \rightarrow \qquad \delta L=\nabla_{\alpha}\left(N_{\mu}\bar{u}^{\alpha}\bar{u}^{\beta}\nabla_{\beta}\delta N^{\mu}\right) \tag{72}$$

where $\bar{K}^{\mu}{}_{\alpha_1....\alpha_p}$ is a rank-p (conformal) Killing tensor (similar to higher spin generators for the Laplacian) :

$$\nabla_{(\mu}K_{\nu\alpha_1....\alpha_p)}=\xi_{(\mu}h_{\nu\alpha_1....\alpha_p)} \tag{73}$$

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How can we use this symmetry to explicitly solve the GDE ?

Symmetries of the geodesic deviation equation

Integrability of the geodesic deviation equation (GDE)

- Consider the GDE associated to a timelike or null reference geodesic

$$\frac{D^2 N^\mu}{\mathrm{d}\tau^2} = \bar{R}^\mu{}_{\alpha\beta\gamma}\bar{u}^\alpha \bar{u}^\beta N^\gamma \qquad \text{with} \qquad \bar{u}^\alpha \bar{u}_\alpha = \epsilon \qquad \bar{u}^\alpha \nabla_\alpha \bar{u}^\mu = 0 \tag{74}$$

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- For radiative spacetime, relate memory effects to explicit and hidden symmetries of spacetime [Caviglia, Zordan, and Salmistraro '82][**BA** '24]

Geodesic deviation vector

- With the Fermi coordinates $X^0 = U$, $X^1 = V$ and $X^A = \{X, Y\}$ at hand, we can analyze the geodesic deviation
- The position and the velocity of the test particle are given in Brinkmann coordinates by

$$X^A = E^A{}_i \left(H^{ij} p_j - \mathcal{B}^i \right) , \tag{78}$$

$$\dot{X}^A = \dot{E}^A{}_i \left(H^{ij} p_j - \mathcal{B}^i \right) + E^A{}_i A^{ij} p_j \, . \tag{79}$$

- Parametrized by the two charges $(p_i, \mathcal{B}^i)$: 2d translations and carrolian boosts
- Consider the null reference geodesic with trajectory $\bar{X}^\mu$ and a second arbitrary test particle X^μ

$$\boxed{\zeta^A \equiv Y^A - \bar{X}^A \, ,} \tag{80}$$

encodes their relative distance

- We want to analyze

$$\zeta^A(u) = \zeta^i(u) E^A{}_i(u_0) \qquad \text{with} \qquad \boxed{\zeta^i(u) = \mathcal{A}_i(u) \left[H^{ii}(u_0, u) p_i - \mathcal{B}^i \right] \, .} \tag{81}$$

- Its dynamics satisfies the geodesic deviation equation

$$\ddot{\zeta}_A = R_{AUUB} \zeta^B \tag{82}$$

$$= R_{iuuj} E^i{}_A E^j{}_B \zeta^B \tag{83}$$

$$= \frac{1}{2} \left(\ddot{A}_{ij} - \frac{1}{2} A^{km} \dot{A}_{ki} \dot{A}_{mj} \right) E^i{}_A E^j{}_B \zeta^B \, . \tag{84}$$

Classification of memory effects

The three different types of memory effects

- Consider situations for which asymptotycally, i.e. for $u < u_0$ and $u > u_f$, one has

$$\zeta^i \neq 0 \qquad \text{and} \qquad \ddot{\zeta}^i = 0\,. \tag{85}$$

- **Velocity Memory (VM)**:
  When the relative velocity in the two asymptotic regions satisfies

$$\Delta\dot{\zeta} = \dot{\zeta}_f - \dot{\zeta}_0 \neq 0 \tag{86}$$

  $\rightarrow$ constant shift on the asymptotic value of the relative velocity.

- **Vanishing Velocity Memory (VM0)**: Subcase corresponding to

$$\Delta\dot{\zeta} = \dot{\zeta}_f - \dot{\zeta}_0 = 0 \tag{87}$$

  $\rightarrow$ no velocity memory but still interesting effects on the couple of test particles.

- **Displacement Memory (DM)**: subcase such that

$$\dot{\zeta}_f = 0 \qquad \text{and} \qquad \zeta_f \neq \zeta_0 \tag{88}$$

  $\rightarrow$ the relative velocity vanishes in the asymptotic future

Classification of memory effects

A brief look at the conclusions so far

- First work to analyze the memory effects in vacuum gravitational plane wave [Zhang, Duval, Gibbons, Horvathy '17 '18]
- VGPW only exhibits velocity memory effects / displacement memory effect can never occur

The Memory Effect for Plane Gravitational Waves

P.-M. Zhang[1*], C. Duval[2†], G. W. Gibbons[3,4,5‡], P. A. Horvathy[1,4§],

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(Dated: August 28, 2017)

Abstract

We give an account of the gravitational memory effect in the presence of the exact plane wave solution of Einstein's vacuum equations. This allows an elementary but exact description of the soft gravitons and how their presence may be detected by observing the motion of freely falling particles. The theorem of Bondi and Pirani on caustics (for which we present a new proof) implies that the asymptotic relative velocity is constant but not zero, in contradiction with the permanent displacement claimed by Zel'dovich and Polnarev. A non-vanishing asymptotic relative velocity might be used to detect gravitational waves through the "velocity memory effect", considered by Braginsky, Thorne, Grishchuk, and Polnarev.

Classification of memory effects

A brief look at the conclusions so far

- Very recently, two numerical examples where a displacement occurs have been presented in [Zhang, Horvathy '24]

Displacement within velocity effect

in

gravitational wave memory

P.-M. Zhang[1*], P. A. Horvathy[2,3†],

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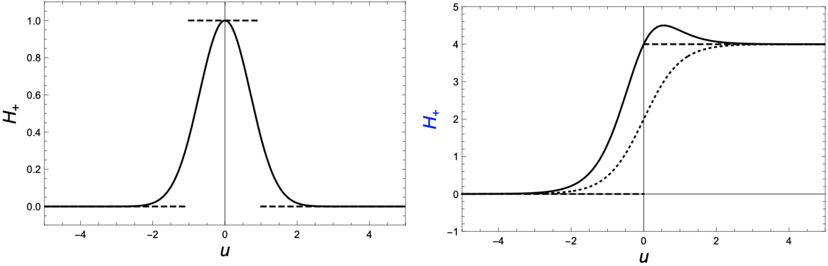
Abstract

Sandwich gravitational waves exhibit the *velocity memory effect* (VM) which however can become, for specific values of the wave parameters, pure *displacement* (DM) as suggested by Zel'dovich and Polnarev. Fixing such a "miraculous" value, the particle trajectory is an (approximate) standing wave characterized by a unique integer m, for which the particle does not absorb any energy from the passing wave. Our statements are illustrated by a simple Gaussian and by the Pöschl-Teller potential as profiles.

- No analytic conditions were provided leaving the question of the classification of the conditions to have a velocity versus a displacement memory open

Classification of memory effects

Pulse versus step profiles



- So far, only pulse profiles have been studied
- Full classification holds for both:
  very different conditions to realize memories depending on the nature of the wave
- Step profiles are more adapted to model realistic memory contributions [Favata '19, '20]

Memory effects for pulse profiles

- Focus on the polarized case: $\mathcal{A}_{12} = 0$ / Introduce $\mathcal{A}_i = \mathcal{A}_{ii}$
- Past and future asymptotic behavior of the pulse profile:

$$H_+(u) = \frac{\ddot{\mathcal{A}}_1}{\mathcal{A}_1} = -\frac{\ddot{\mathcal{A}}_2}{\mathcal{A}_2} = 0 \qquad \text{for} \qquad u < u_0 \qquad u > u_f \tag{89}$$

- To analyze the memory, we need the dynamics of the geodesic deviation vector

$$\dot{\zeta}^i(u) = \frac{\dot{\mathcal{A}}_i(u)}{\mathcal{A}_i(u)}\zeta^i(u) + \frac{p_i}{\mathcal{A}_i(u)}\,, \tag{90}$$

$$\ddot{\zeta}^i(u) = \frac{\ddot{\mathcal{A}}_i(u)}{\mathcal{A}_i(u)}\zeta^i(u). \tag{91}$$

- Asymptotic behavior of the relative acceleration

$$\ddot{\mathcal{A}}_i = 0 \qquad \rightarrow \qquad \ddot{\zeta}^i(u) = 0 \qquad \text{for} \qquad u < u_0 \qquad u > u_f \tag{92}$$

- Compute the asymptotic form of the relative distance and velocity $(\zeta_f, \dot{\zeta}_f, \zeta_0, \dot{\zeta}_0)$ in terms of the initial conditions $(p_i, \mathcal{B}_i)$ and the asymptotic wave form $(\mathcal{A}_0, \mathcal{A}_f)$.

- **Asymptotic past:** for $u < u_0$, one has

$$\mathcal{A}(u) = \dot{\mathcal{A}}_0(u-u_0) + \mathcal{A}_0 \,, \tag{93}$$

$$H(u,u_0) = \int_{u_0}^{u} \frac{1}{\mathcal{A}^2(u)} = \frac{u-u_0}{\mathcal{A}_0\mathcal{A}(u)} \qquad \text{such that} \qquad \mathcal{A}(u)H(u,u_0) = \frac{u-u_0}{\mathcal{A}_0} \,, \tag{94}$$

$$\zeta(u) = \dot{\zeta}_0(u-u_0) + \zeta_0 \tag{95}$$

- **Asymptotic future:** for $u > u_f$, one has

$$\mathcal{A}(u) = \dot{\mathcal{A}}_f(u-u_f) + \mathcal{A}_f \,, \tag{96}$$

$$H(u_0,u) = \int_{u_0}^{u_f} \frac{1}{\mathcal{A}^2(u)} + \int_{u_f}^{u} \frac{1}{\mathcal{A}^2(u)} = H_{0f} + \frac{u-u_f}{\mathcal{A}_f\mathcal{A}(u)} \,. \tag{97}$$

$$\zeta(u) = \dot{\zeta}_f(u-u_f) + \zeta_f \,. \tag{98}$$

- **Results:** relations between $(\zeta_f, \dot{\zeta}_f, \zeta_0, \dot{\zeta}_0)$ the different asymptotic quantities in terms of the initial conditions $(p_i, \mathcal{B}^i)$ and the asymptotic properties of the wave profile $(\mathcal{A}_0, \mathcal{A}_f, H_{0f})$

$$\boxed{\zeta_0 = -\mathcal{B}\mathcal{A}_0 \,, \qquad \dot{\zeta}_0 = \frac{p}{\mathcal{A}_0} - \mathcal{B}\dot{\mathcal{A}}_0} \tag{99}$$

$$\boxed{\zeta_f = \mathcal{A}_f(H_{0f}p - \mathcal{B}) \qquad \dot{\zeta}_f = \frac{p}{\mathcal{A}_f} + \dot{\mathcal{A}}_f(H_{0f}p - \mathcal{B})} \,, \tag{100}$$

Classification of memories for pulse profiles

- A **VM** occurs under the condition $\dot{\zeta}_f \neq \dot{\zeta}_0$.
  Since pulses profiles enjoy a constant asymptotic velocities

$$\ddot{A}_i = 0 \qquad \text{for} \qquad u < u_0 \qquad u > u_f \tag{101}$$

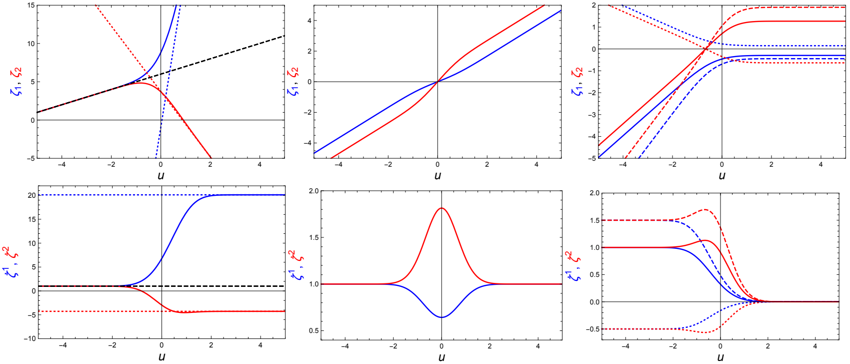
  they generically lead to a constant VM whatever the profile $H_+(u)$ and the constants of motions $(\mathcal{B}, p)$ or, similarly, whatever the initial conditions $(\zeta_0, \dot{\zeta}_0)$.
  This is a generic properties of any pulse.
- A **VM0** occurs in the special cases in which $\dot{\zeta}_f = \dot{\zeta}_0$. This translates into

$$\boxed{(\dot{\mathcal{A}}_f - \dot{\mathcal{A}}_0)\mathcal{B} = -p\left[\frac{\mathcal{A}_f - \mathcal{A}_0}{\mathcal{A}_f \mathcal{A}_0} - \dot{\mathcal{A}}_f H_{0f}\right] .} \tag{102}$$

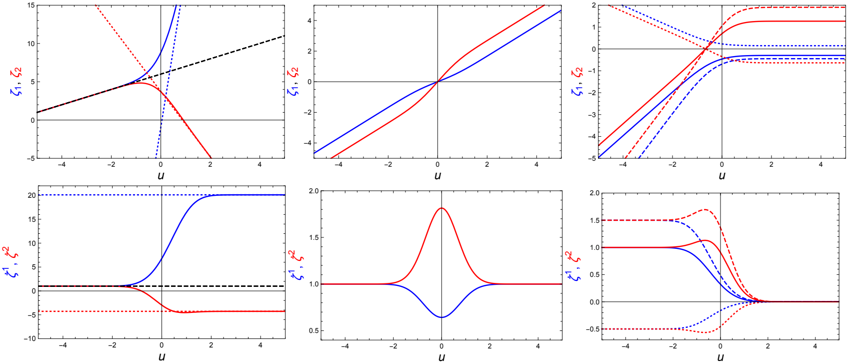
- A **DM** occurs when $\dot{\zeta}_f = 0$.
  This translates into a specific tuning between the two constants of motion,

$$\boxed{\dot{\mathcal{A}}_f \mathcal{B} = -p\left(\frac{1}{\mathcal{A}_f} + \dot{\mathcal{A}}_f H_{0f}\right) .} \tag{103}$$

- Lead to a finer classification depending on $\dot{\mathcal{A}}_f = \dot{\mathcal{A}}_0$ or $\dot{\mathcal{A}}_f \neq \dot{\mathcal{A}}_0$
- Let us see some examples.



- Profiles of the relative displacement (ζ_1, ζ_2) (upper line) and relative velocity $(\dot\zeta_1, \dot\zeta_2)$ (lower line) for $H_+ = e^{-u^2}$ with initial conditions that ensures $\dot{\mathcal{A}}_f - \dot{\mathcal{A}}_0 \neq 0$.



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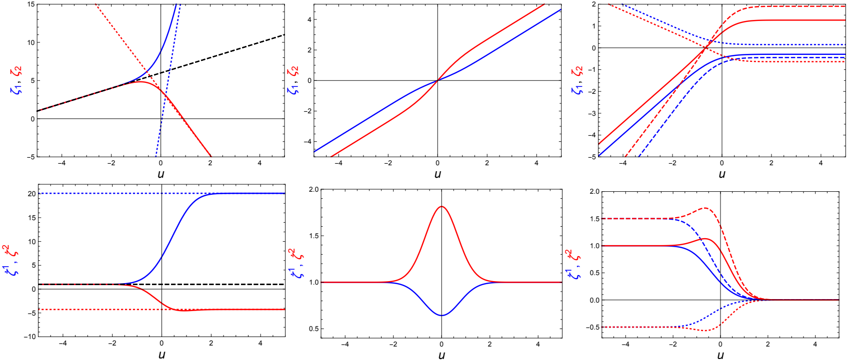
- Left panel: assume $p_i = (1, 1)$ and $\mathcal{B}_i = (-1, -1) \rightarrow$ clear non vanishing VM, so that $\dot{\zeta}_f \neq \dot{\zeta}_0$. Projected motion. Longitudinal position can be different so no colliding trajectories.



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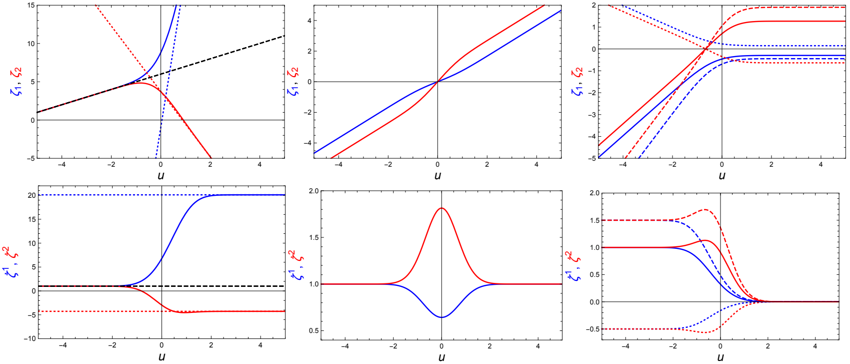
- Middle: $p_i \neq (0,0)$ and B_i is tuned to get a VM0 $\rightarrow$ a vanishing VM for which $\dot{\zeta}_f = \dot{\zeta}_0$.



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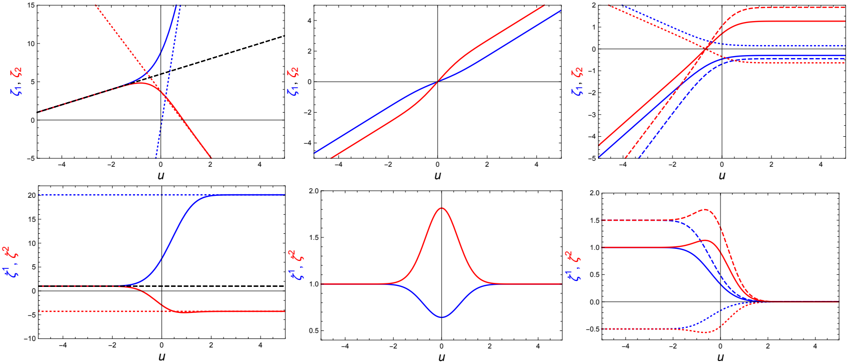
- Middle: $p_i \neq (0,0)$ and B_i is tuned to get a VM0 $\rightarrow$ a vanishing VM for which $\dot{\zeta}_f = \dot{\zeta}_0$.
- Right: $p_i = 1$ (Solid), $p_i = 1.5$ (Dahed) and $p_i = -0.5$ (Dotted) and B_i is tuned to $\dot{\zeta}_0 = p/\mathcal{A}_0 - \mathcal{B}\dot{\mathcal{A}}_0 \rightarrow$ pure constant DM



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New type of memory (middle column) simply switches the projected position of the two particles !

Memory effects for step profiles

Step profile

- $H_+ \to \lambda_{0,f} \neq 0$ respectively for $u < u_0$ and $u > u_f$ and $\lambda_{0,f} > 0$
- The solutions for the mode $\mathcal{A}_2$ (and hence ζ_2) are obtained from the expression of the mode $\mathcal{A}_1$ (and ζ_1) by the transformation $\lambda_{0,f} \to i\lambda_{0,f}$.

$$\mathcal{A}_1^- \sim \frac{1}{2}\left(\mathcal{A}_0 - \frac{\dot{\mathcal{A}}_0}{\lambda_0}\right) \mathrm{e}^{-\lambda_0(u-u_0)} \qquad \dot{\mathcal{A}}_1^- = -\lambda_0 \mathcal{A}_1^- \qquad \ddot{\mathcal{A}}_1^- = \lambda_0^2 \mathcal{A}_1^- \,. \tag{104}$$

$$\mathcal{A}_1^+ \sim \frac{1}{2}\left(\mathcal{A}_f + \frac{\dot{\mathcal{A}}_f}{\lambda_f}\right) \mathrm{e}^{+\lambda_f(u-u_f)} \qquad \dot{\mathcal{A}}_1^+ = \lambda_f \mathcal{A}_1^+ \qquad \ddot{\mathcal{A}}_1^+ = \lambda_f^2 \mathcal{A}_1^+ \,, \tag{105}$$

- Relative acceleration behaves as

$$\ddot{\zeta}_- \simeq -\frac{\lambda_0}{2}\left[\frac{p}{\mathcal{A}_0} + \lambda_0 \mathcal{B}\left(\mathcal{A}_0 - \frac{\dot{\mathcal{A}}_0}{\lambda_0}\right)\right] \mathrm{e}^{-\lambda_0(u-u_0)} \quad \text{when} \quad u \to -\infty \,. \tag{106}$$

$$\ddot{\zeta}_+ \simeq \frac{\lambda_f}{2}\left[\frac{p}{\mathcal{A}_f} + \lambda_f(H_{0f}p - \mathcal{B})\left(\mathcal{A}_f + \frac{\dot{\mathcal{A}}_f}{\lambda_f}\right)\right] \mathrm{e}^{\lambda_f(u-u_f)} \quad \text{when} \quad u \to +\infty \,. \tag{107}$$

- Choice of step profiles: $\lambda_0 = 0$ and $\lambda_f > 0$ free automatically gives $\ddot{\zeta}_- = 0$
- Requiring that $\ddot{\zeta} = 0$ when $u \gg u_f$ requires the condition

$$(\dot{\mathcal{A}}_f + \lambda_f \mathcal{A}_f)\mathcal{B} = p\left[H_0 f\left(\dot{\mathcal{A}}_f + \lambda_f \mathcal{A}_f\right) + \frac{1}{\mathcal{A}_f}\right] \tag{108}$$

Step profile Upon imposing the previous condition:

- **VM**: A wave with a step profile generically leads to a rather surprising velocity memory effect where

$$\dot{\zeta}_+ = \frac{1}{2}\left[\frac{p}{\mathcal{A}_f} + \lambda_f(pH_{0f} - \mathcal{B})\left(\mathcal{A}_f + \frac{\dot{\mathcal{A}}_f}{\lambda_f}\right)\right] e^{\lambda_f(u-u_f)} = 0 \tag{109}$$

$$\dot{\zeta}_- = \frac{1}{2}\left[\frac{p}{\mathcal{A}_0} + \mathcal{B}\left(\lambda_0\mathcal{A}_0 - \dot{\mathcal{A}}_0\right)\right] e^{\lambda_0(u-u_0)} = \frac{1}{2}\left[\frac{p}{\mathcal{A}_0} - \mathcal{B}\dot{\mathcal{A}}_0\right] \tag{110}$$

so that

$$\dot{\zeta}_- \neq 0 \qquad \text{while} \qquad \dot{\zeta}_+ = 0$$

The wave cancels the relative motion, i.e. the relative velocity, between the two particles.

- **VM0**: It occurs when one further has $\dot{\zeta}_- = 0$, which requires $\frac{p}{\mathcal{A}_0} = \mathcal{B}\dot{\mathcal{A}}_0$. This means that a VM0 requires that

$$\boxed{\frac{1}{\mathcal{A}_0\dot{\mathcal{A}}_0} = H_{0f} + \frac{1}{\mathcal{A}_f(\dot{\mathcal{A}}_f + \lambda_f\mathcal{A}_f)},} \tag{111}$$

which is a non-trivial property of the spacetime geometry.

- **DM**: A displacement memory effect requires

$$\dot{\zeta}_+ = \dot{\zeta}_- = 0 \qquad \text{while} \qquad \zeta_+ \neq \zeta_-$$

This implies that one needs $\dot{\mathcal{A}}_+ \neq \dot{\mathcal{A}}_-$ together with the constraint (111).

Conclusion and perspectives

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- Hidden symmetries (i.e. conformal Killing tensor) generate solutions of the GDE
  $\rightarrow$ link between hidden symmetries and memories for radiative spacetime [**BA** '24]
- Complete classification for the conditions relating both the wave-profile and initial conditions of relative motion to exhibit a velocity or a displacement memory effects in a vacuum gravitational plane wave [**BA**, Uzan '24]

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- Application to more complicated radiative systems: Robinson-Trautman geometries
- Revisit the memories in asymptotically flat spacetime : what are their hidden symmetries ? can we find asymptotic Killing tensors for asymptotically flat spacetime ? Work in progress

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Next goals

- Application to more complicated radiative systems: Robinson-Trautman geometries
- Revisit the memories in asymptotically flat spacetime : what are their hidden symmetries ? can we find asymptotic Killing tensors for asymptotically flat spacetime ? Work in progress
- Study the memories of extended quadrupolar bodies described by Dixon's theory

$$\frac{Dp^\mu}{\mathrm{d}\tau} = -\frac{1}{2} R^\mu{}_{\nu\alpha\beta} v^\nu S^{\alpha\beta} - \frac{1}{6} J^{\alpha\beta\gamma\delta} \nabla^\mu R_{\alpha\beta\gamma\delta} \tag{112}$$

$$\frac{DS^{[\mu\nu]}}{\mathrm{d}\tau} = 2p^{[\mu} v^{\nu]} + \frac{4}{3} R^{[\mu}{}_{\alpha\beta\gamma} J^{\nu]\alpha\beta\gamma} \tag{113}$$

$\rightarrow$ quasi-conserved charges and Killing-Yano symmetries [Compere, Druart '23]: are there new memories to identify ?

Thank you