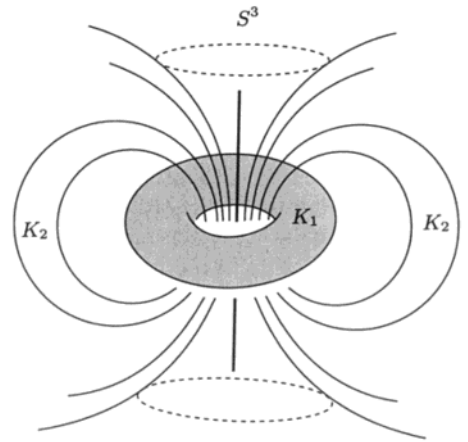
Institut de Mathématiques de Bourgogne
Mathematical Physics of Gravity and Symmetry
20/11/2024

Geometric actions and model spaces



Naber 2011

Glenn Barnich

Physique théorique et mathématique

Université libre de Bruxelles &
International Solvay Institutes

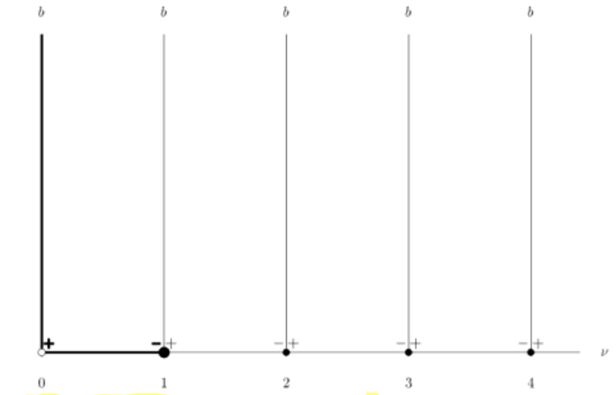


Figure 1: The coadjoint orbits of the Virasoro algebra.. The points of the figure are in one-

Balog et al. 97

Collaboration with T. Smoes to appear

Contents

3d gravity as group theory

Effective actions

Geometric actions and $\boxed{\text{model spaces}}$

Application to $SU(2)$

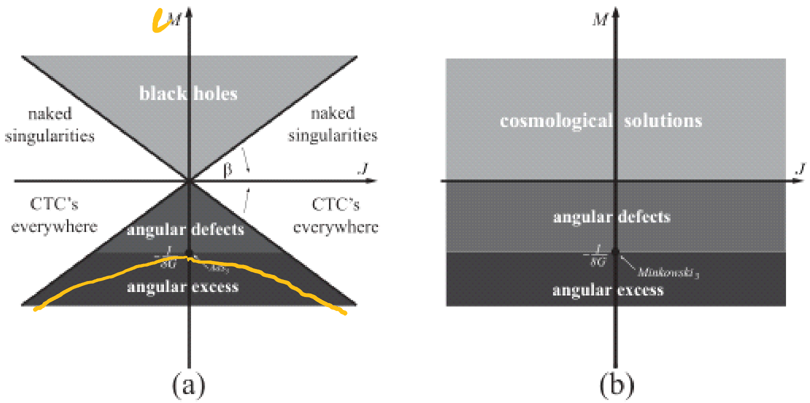
Why 3d gravity?

Toy model that separates 2 types of problems in 4d

gravitational waves
gravitons
outgoing radiation (news)

black holes, cosmologies
topological or boundary dof

$\boxed{\text{fast}}$



$\boxed{\text{slow}}$?

"zero mode solutions"

Gravity models: • AdS_3 gravity   Quantization?

general solution with Brown-Henneaux (FG) type boundary conditions

$$ds^2 = \frac{\ell^2}{r^2}\, dr^2 - \left(r\, dx^+ - \frac{8\pi G \ell}{r}\, b^-\, dx^-\right)\left(r\, dx^- - \frac{8\pi G \ell}{r}\, b^+\, dx^+\right)$$

$x^\pm = \frac{t}{\ell} \pm \varphi$, $\boxed{b^\pm(x^\pm + 2\pi) = b^\pm(x^\pm) \quad \text{arbitrary periodic functions}}$

conformal transformations $x^\pm \to f^\pm(x^\pm)$, $f(x^\pm + 2\pi) = f^\pm(x^\pm) + 2\pi$

residual diffeomorphisms

$\tilde{b}^\pm = Ad^*_{f^{-1}}\, b^\pm = (\partial_\pm f^\pm)^2\, b^\pm \circ f^\pm - c^\pm\, S_{x^\pm}[f^\pm]$

coadjoint representation of $\widehat{Diff(S^1)} \otimes \widehat{Diff(S^1)}$

$c^\pm = \frac{3\ell}{2G}$ $S_x[f] = \frac{1}{24\pi}\left[\partial_x^2 (\ln \partial_x f) - \frac{1}{2}(\partial_x \ln \partial_x f)^2\right]$

Schwarzian derivative

Asymptotically flat 3d metrics with Bondi-Sachs type boundary conditions

$$ds^2 = 2\left[8\pi G\, p\, du - dr + 8\pi G (j + u p')\, d\varphi \right] du + r^2 d\varphi^2$$

$p = p(\varphi)\,,\quad j = j(\varphi)$

finite BMS_3 transf.
$$\begin{cases} \tilde{p} = (f')^2\, p\circ f - c_2\, S_\varphi[f] \\ \tilde{j} = (f')^2 \left[j + \alpha p' + 2\alpha' p - \frac{c_2}{24\pi} \alpha''' \right]\circ f - c_1\, S_\varphi[f] \end{cases}$$

coadjoint representation of $\widehat{\mathrm{Diff}(S^1) \ltimes C^\infty(S^1)} = \widehat{BMS_3}$

zero mode solutions : $b_\pm(x^\pm),\ p(\varphi),\ j(\varphi)$ constants

Identification in non-radiative asymptotically flat spacetimes at $\mathscr{I}^+$

Back to S^2 & GR: BMS metric $\Longleftrightarrow$ $\boxed{\text{NP first order}}$ (similar analysis at $\mathscr{I}^-$)

Solution space, free data at $\mathscr{I}^+$: $\boxed{\Psi_2^0 + \bar{\Psi}_2^0 \,, \quad \Psi_1^0 \,, \quad \sigma^0}$ undetermined u-dependence

$\dot{\sigma}^0$ news

evolution equations. $\partial_u \Psi_2^0 = \eth \Psi_3^0 + \sigma^0 \Psi_4^0$, $\partial_u \Psi_1^0 = \eth \Psi_2^0 + 2\sigma^0 \Psi_3^0$

constraints $\Psi_2^0 - \bar{\Psi}_2^0 = \bar{\eth}^2 \sigma^0 - \eth^2 \bar{\sigma}^0 + \dot{\sigma}^0 \bar{\sigma}^0 - \sigma^0 \dot{\bar{\sigma}}^0$

$\Psi_3^0 = -\eth \dot{\bar{\sigma}}^0$, $\Psi_4^0 = -\ddot{\bar{\sigma}}^0$

additional data to construct solutions

$$\Psi_0 = \sum_{n \geq 0} \Psi_0^n (\zeta, \bar{\zeta}, u_0)\, r^{-5-n}$$

Transformation of (relevant) free data at $\mathscr{J}^+$ $\quad s = (Y, \bar{Y}, T)$, $\quad f = T + \frac{1}{2} u (\eth Y + \bar{\eth} \bar{Y})$

$$\delta_s \sigma^0 = [\, f \partial_u + Y \eth + \bar{Y} \bar{\eth} + \tfrac{3}{2} \eth Y - \tfrac{1}{2} \bar{\eth} \bar{Y} \,] \sigma^0 - \eth^2 f$$

$$\delta_s \dot{\sigma}^0 = [\, f \partial_u + Y \eth + \bar{Y} \bar{\eth} + 2 \eth Y \quad] \dot{\sigma}^0 - \tfrac{1}{2} \eth^2 (\underline{\eth Y} + \bar{\eth} \bar{Y})$$

EM tensor

Schwarzian derivative

$$\delta_s \psi_2^0 = [\quad ″ \quad ″ \quad ″ \quad + \tfrac{3}{2} \eth Y + \tfrac{3}{2} \bar{\eth} \bar{Y} \,] \psi_2^0 + 2 \eth f \, \psi_3^0$$

$$\delta_s \psi_1^0 = [\quad ″ \quad ″ \quad ″ \quad + 2 \eth Y + \bar{\eth} \bar{Y} \,] \psi_1^0 + 3 \eth f \, \psi_2^0$$

broken current algebra

$$J_s = \frac{i}{R^2} \int \left[(P_S \bar{P}_S)^{-1} J^u_s \, d\xi \wedge d\bar{\xi} + P_S^{-1} J^{\bar{\xi}}_s \, du \wedge d\xi - \bar{P}_S J^{\xi}_s \, du \wedge d\bar{\xi} \right]$$

$$\delta_{s_1} J_{s_2} + \boxed{\Theta_{s_2} (\delta_{s_1} X)} \approx - J_{[s_1, s_2]} + d L_{s_1, s_2}$$

non-conservation

$$d J_s + \Theta_s (\delta_{(0,0,1)} X) \approx 0$$

$s_1 : (Y_1, \bar{Y}_1, T_1) = (0, 0, 1)$ $\qquad \Theta_s (\delta X) \sim \dot{\sigma}^0 , \dot{\bar{\sigma}}^0$ vanishes in the absence of news time components

$$J^u_s = -\frac{1}{8\pi G}\Big\{\Big[\underbrace{\Psi^0_2+\bar\Psi^0_2}_{\text{BH}}+\underbrace{\sigma^0\dot{\bar\sigma}^0+\bar\sigma^0\dot\sigma^0}_{\text{gravitons}}\Big]f+\Big[\Psi^0_1+\sigma^0\eth\bar\sigma^0+\tfrac12\eth(\sigma^0\bar\sigma^0)\Big]Y+\Big[\bar\Psi^0_1+\bar\sigma^0\bar\eth\sigma^0+\tfrac12\bar\eth(\sigma^0\bar\sigma^0)\Big]\bar Y\Big\}$$

$$\Theta^u_s(\delta X) = \frac{1}{8\pi G}\Big[\dot{\bar\sigma}^0\delta\sigma^0+\dot\sigma^0\delta\bar\sigma^0\Big]f$$

charges $\quad Q_s = \int_{S^2,\,u=cte}\frac{i}{R^2}\frac{d\zeta d\bar\zeta}{P_s\bar P_s}J^u_s\,. \qquad \Theta_s[\delta X] = \int_{S^2,\,u=cte}\frac{i}{R^2}\frac{d\zeta d\bar\zeta}{P_s\bar P_s}\Theta^u_s[\delta X]$

algebra $\quad \delta_{s_1}Q_{s_2}+\Theta_{s_2}[\delta_{s_1}X] = -Q_{[s_1,s_2]}$

(non-)conservation of BMS_4 charges

G.B. & C.Troessaert JHEP (2011)

JHEP (2013)

$$\frac{d}{du}Q_s = \underbrace{-\int_{S^2,\,u=cte}\frac{i}{R^2\,8\pi G}\frac{d\zeta d\bar\zeta}{P_s\bar P_s}\Big[\dot{\bar\sigma}^0\delta_s\sigma^0+\dot\sigma^0\delta_s\bar\sigma^0\Big]}_{\text{fluxes}}$$

generalizes mass loss

$\boxed{\text{non-radiative spacetimes (no news)}}$ $\quad \sigma^0 = \sigma^0(\xi, \bar\xi, \cancel{u})$ $\quad (\Leftrightarrow \dot\sigma^0 = 0 = \psi_3^0 = \psi_4^0, \quad O_s[\delta X] = 0)$

compare "abstract" construction of bms_4^*

identification at $u = 0$ $\quad \mathcal{P} = -\frac{1}{2G}(\psi_2^0 + \bar\psi_2^0)$ $\quad \bar{\mathcal{J}} = -\frac{1}{2G}\big(\underbrace{\psi_1^0 + \sigma^0 \eth \bar\sigma^0 + \frac{1}{2}\eth(\sigma^0\bar\sigma^0)}_{\psi^0_{1\bar{\mathcal{J}}}}\big)$

super-momentum
= Bondi mass aspect

~~super~~-angular momentum
= Bondi angular momentum aspect

(pre) momentum map: $\mathcal{F}$. algebra of non-radiative free data

bms_4 representation δ_s, $\quad [\delta_{s_1}, \delta_{s_2}] = \delta_{[s_1, s_2]}$

$$\mu : \mathcal{F} \longrightarrow bms_4^*$$

$$\mu\left(-\tfrac{1}{2G}(\psi_2^0 + \bar\psi_2^0)\right) = \mathcal{P}, \quad \mu\left(-\tfrac{1}{2G}\psi^0_{1\bar{\mathcal{J}}}\right) = [\bar{\mathcal{J}}], \quad \mu \circ \delta_s = \mathrm{ad}_s^* \circ \mu$$

transformation laws at $u=0$ $\quad \delta_s(\psi_2^0+\bar\psi_2^0) = \left(Y\eth + \bar Y\bar\eth + \frac{3}{2}\eth Y + \frac{3}{2}\bar\eth\bar Y\right)(\psi_2^0+\bar\psi_2^0)$

$$\delta_s\psi_1^0 = \left[Y\eth + \bar Y\bar\eth + 2\eth Y + \bar\eth\bar Y\right]\psi_1^0 + \frac{1}{2}T\eth\left(\psi_2^0+\bar\psi_2^0+\bar\eth^2\cancel{\sigma^0}-\eth^2\cancel{\bar\sigma^0}\right)$$
$$+\frac{3}{2}\eth T\left(\psi_2^0+\bar\psi_2^0+\bar\eth^2\cancel{\sigma^0}-\eth^2\cancel{\bar\sigma^0}\right)$$

$$\delta_s\psi_{1\bar T}^0 = \left[Y\eth+\bar Y\bar\eth+2\eth Y+\bar\eth\bar Y\right]\psi^0_{1\bar T} + \frac{1}{2}T\eth\left(\psi_2^0+\bar\psi_2^0\right)+\frac{3}{2}\eth T\left(\psi_2^0+\bar\psi_2^0\right)$$

$$+\frac{1}{2}\bar\eth\left(T\bar\eth\eth\sigma^0-\bar\eth T\eth\sigma^0+3\eth T\bar\eth\sigma^0-3\bar\eth\eth T\sigma^0-\frac{3}{2}T\sigma^0\right)\boxed{-\frac{1}{2}\eth^3\left(T\bar\sigma^0\right)}$$

trivial!

Remark: electric case $\bar\eth^2\sigma_e^0 = \eth^2\bar\sigma_e^0 \iff \sigma_e^0 = \eth^2\chi_e$

$$\delta_s\chi_e = \left[Y\eth+\bar Y\bar\eth-\frac{1}{2}\eth Y-\frac{1}{2}\bar\eth\bar Y\right]-T$$

Newman Penrose JMP 1966

Strominger et al. 2013–

Compère et al. 2016

simplified pre-momentum map $\mu' : \mathcal{F}_e \longrightarrow \mathfrak{bms}_4^*$ (not physically relevant!)

$$\mu'\left[-\frac{1}{2G}\left(\psi_2^0+\bar\psi_2^0\right)\right]=T,\quad \mu'\left[-\frac{1}{2G}\psi_1^0\right]=[Y],\quad \mu'\circ\delta_s = \mathrm{ad}_s^*\circ\mu'$$

Summary: (covariant) phase space of

3d AdS or flat gravity : $\bigoplus_{\pm}$ virasoro$^*_{\pm}$ or bms_3^*

$$\mathrm{Ad}^*_G : \mathfrak{g}^* \to \mathfrak{g}^*, \quad \langle \underset{\in \mathfrak{g}^*}{Y}, g^{-1} \underset{\in \mathfrak{g}}{Z} g \rangle = \langle \underbrace{g Y g^{-1}}_{= \mathrm{Ad}^*_{g^{-1}} \omega}, Z \rangle$$

$\mathfrak{g}^*$: degenerate KKS Poisson structure

Partition of $\mathfrak{g}^*$ into coadjoint orbits

$$\mathfrak{g}^* \simeq \bigcup_{\Xi} \mathrm{Ad}^*_G \, Y^{\Xi}$$

Y^{Ξ} orbit representatives

individual coadjoint orbits : symplectic spaces that can be quantized classification of coadjoint orbits for $\widehat{\mathrm{Diff}}(S_1)$

Lazutkin & Pankratova, Kirillov, Witten, Balog et al. ...

Coadjoint Orbits of the Virasoro Group

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Abstract. The coadjoint orbits of the Virasoro group, which have been investigated by Lazutkin and Pankratova and by Segal, should according to the Kirillov-Kostant theory be related to the unitary representations of the Virasoro group. In this paper, the classification of orbits is reconsidered, with

Summary

orbit representatives given by

zero mode solutions (constant $L_0, \bar{L}_0$)

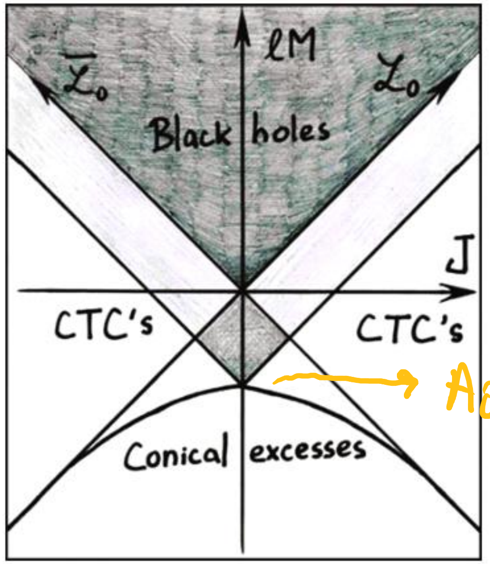
virasoro*

elliptic, hyperbolic, parabolic

special points BH's

+ exceptional orbits

(Oblak, PhD.)



AdS$_3$

quantization of ~~individual~~ some Virasoro orbits is understood

in terms of VIRREPS

Desideratum: quantization of collection of orbits

needed in order to account for BTZ black holes

of different M, J

Model space: classical G-invariant system whose quantization

gives each VIRREP of G with multiplicity one

From Geometric Quantization to Conformal Field Theory

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Abstract. Investigation of $2d$ conformal field theory in terms of geometric quantization is given. We quantize the so-called model space of the compact Lie group, Virasoro group and Kac-Moody group. In particular, we give a

Lagrangian version of the Drinfeld-Sokolov Hamiltonian reduction (see also [6]).

Here we will give a slightly different type of geometric construction in which all representations of the group are considered simultaneously and on the same footing. More precisely, using the path integral approach, we will quantize the so-called model space, i.e. such space that its quantization yields all representations of the group with multiplicity one. This space is larger than the coadjoint orbit (roughly speaking, it contains an extra variable which parametrizes the orbits and the conjugate moment). The corresponding Hilbert space splits into the direct sum

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Vol. 1, No. 2, 1981

0272-9903/81/01/0121-22$5.90

Models of Representations of Lie Groups*

I. N. Bernstein, I. M. Gelfand, and S. I. Gelfand

Virasoro Model Space

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Abstract. The representations of a compact Lie group G can be studied via the construction of an associated "model space." This space has the property that when geometrically quantized its Hilbert space contains every irreducible representation of G just once. We construct an analogous space for the group $\mathrm{Diff}\, S^1$. It is

We are dealing here with a model of the representations; namely one can introduce a scalar product on the space of analytic functions on the principal affine space so that in the decomposition of the resulting unitary representation of U into irreducible factors, all the irreducible representations of U occur with multiplicity one (see [1, 4]). Granted the naturalness

no explicit proposal / construction for generic G (?)

- Suppose G group of symmetries known but not necessarily fundamental theory

Effective actions

- construct model that can be quantized & has G as global symmetry group

Weinberg, Callen...

(Noether charges, current algebras)

$$S[g] = -\int d^4x \; \mathrm{Tr}\left[(\partial_\mu g\, g^{-1})(\partial^\mu g\, g^{-1}) \right]$$

(+ Poincaré invariance)

$$S[g] = \int dt \; \mathrm{Tr}\left[(\dot{g}\, g^{-1})(\dot{g}\, g^{-1}) \right]$$

(particle action)

· global right inv. $g \to g h_R$, $K = d(g \cancel{h_R}) \cancel{h_R}^{-1} g^{-1}$ RI Maurer-Cartan form

Exercise : express Lie group G & algebra $\mathfrak{g}$ theory in local coordinates

g^i " Euler angles " arbitrary e_α basis of $\mathfrak{g}$

generators of right / left translations = left / right invariant vector fields

$$g \overbrace{\frac{d}{dt} h_R(t)}^{X}\Big|_{t=0} \Big/ \overbrace{\frac{d}{dt} h_L(t)}^{X}\Big|_{t=0} g \qquad \vec{L}_\alpha = L_\alpha{}^i(g) \frac{\partial}{\partial g^i} \Big/ \vec{R}_\alpha = R_\alpha{}^i(g) \frac{\partial}{\partial g^i}$$

$$[\vec{L}_\alpha, \vec{L}_\beta] = f^\gamma_{\alpha\beta} \vec{L}_\gamma \,, \quad [\vec{R}_\alpha, \vec{R}_\beta] = -f^\gamma_{\alpha\beta} \vec{R}_\gamma \,, \quad [\vec{L}_\alpha, \vec{R}_\beta] = 0$$

" frames $e_\alpha{}^\mu \partial_\mu$, structure functions $\pm f^\gamma_{\alpha\beta}$ "

left / right invariant MC forms $\theta = g^{-1}dg \;/\; K = dg\,g^{-1}$

$\theta = e_\alpha L^\alpha{}_i\, dg^i \quad / \quad K = e_\alpha R^\alpha{}_i\, dg^i$

$L^\alpha{}_i L_\beta{}^i = \delta_\alpha{}^\beta = R^\alpha{}_i R_\beta{}^i$

$L^\alpha{}_i L_\alpha{}^j = \delta_i{}^j = R^\alpha{}_i R_\alpha{}^j$

$d\theta + \frac{1}{2}[\theta, \theta] = 0 \quad / \quad dK - \frac{1}{2}[K, K] = 0$

" coframes $e^\alpha{}_\mu dx^\mu$ "

Adjoint representation $\mathrm{Ad}_g e_\alpha = g\, e_\alpha\, g^{-1} = e_\beta R^\beta{}_i L_\alpha{}^i$

$$\boxed{S[g^i] = \int dt\, \frac{1}{2}\, g_{ij}\, \dot g^i \dot g^j}$$

$g_{\alpha\beta}$: Killing metric $g_{ij}(g) = g_{\alpha\beta} R^\alpha{}_i R^\beta{}_j$

geodesic flow on G

$\ddot g^i + \frac{1}{2}\Gamma^i{}_{jk}\, \dot g^j \dot g^k = 0$

Euler-Arnold equation

Global sym & Noether charges

$\delta_X g^i = L_\alpha{}^i X^\alpha = \xi^i_X$ Kof of g_{ij}

$Q_X = g_{ij}\, \xi^i_X\, \dot g^j$

Theorem (Arnold) geodesic flow on G $\Leftrightarrow$ $\boxed{\dot{\pi} = -ad^*_{g^{-1}\pi}\, \pi}$, $\pi \in \mathfrak{g}^*$

Proof = Hamiltonian analysis $\{g^i, p_j\} = \delta^i_j$, $\{g^i, g^j\} = 0 = \{p_i, p_j\}$

$$p_j = \frac{\partial L}{\partial \dot{g}^j} = g_{\alpha\beta} R^\alpha_i R^\beta_j \dot{g}^i$$

$$\Leftrightarrow \boxed{\underbrace{R_\beta{}^j p_j}_{= \pi_\beta}} = g_{\alpha\beta} R^\alpha{}_i \dot{g}^i \quad \Leftrightarrow \quad \dot{g}^i = R_\alpha{}^i g^{\alpha\beta} \pi_\beta$$

Non-Darboux coordinates $\boxed{\{\pi_\alpha, \pi_\beta\} = f^\gamma{}_{\alpha\beta}\, \pi_\gamma}$ KKS bracket

$$\{g^i, g^j\} = 0 \qquad \{g^i, \pi_\alpha\} = R_\alpha{}^i \qquad \pi_\alpha e^{*\alpha} \in \mathfrak{g}^*$$

$$S_H = \int dt \, [\, \pi_\alpha R^\alpha{}_i \, \dot{g}^i - H \,] \quad , \quad H = \frac{1}{2} \pi_\alpha g^{\alpha\beta} \pi_\beta$$

$$\boxed{\dot{\pi}_\alpha = \{ \pi_\alpha , H \} = f^\gamma{}_{\alpha\beta} \, \pi_\gamma \, g^{\beta\delta} \pi_\delta}$$

$$\dot{g}^i = \{ g^i , H \} = R_\alpha{}^i g^{\alpha\beta} \pi_\beta \iff$$ definition of momentum

Noether charges $\quad Q^{\pi}_X = \langle \pi , \mathrm{Ad}_g X \rangle$

Remarks: 1) More general Hamiltonians "inertia operators"

$$A : \mathfrak{g} \to \mathfrak{g}^* \qquad \text{invertible} \qquad H = \frac{1}{2} \pi_\alpha (A^{-1})^{\alpha\beta} \pi_\beta$$

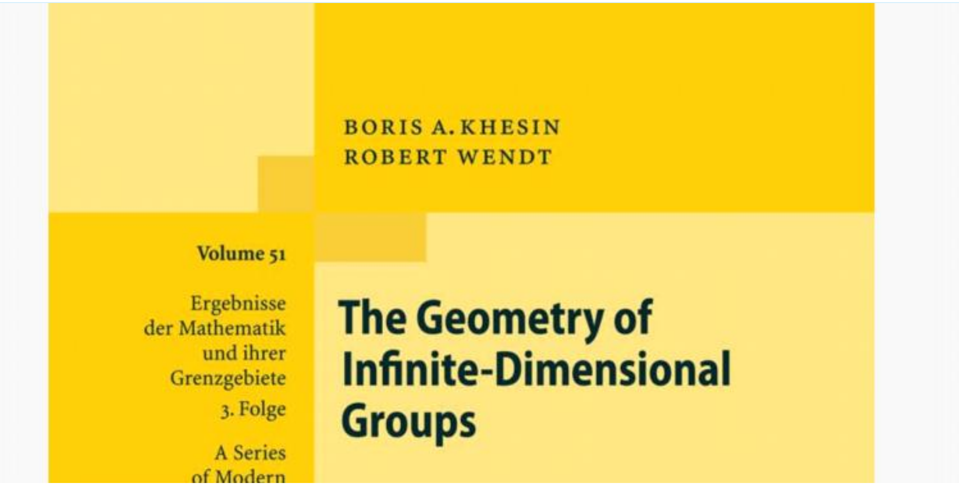
$$X^\beta \mapsto A_{\alpha\beta} X^\beta$$

$SO(3)$: Euler top

34 I. Preliminaries

Group	*Metric*	*Equation*
SO(3)	$\langle \omega, A\omega \rangle$	Euler top
$\mathrm{SO}(3) \ltimes \mathbb{R}^3$	quadratic forms	Kirchhoff equation for a body in a fluid
SO(n)	Manakov's metrics	n-dimensional top
Diff(S^1)	L^2	Hopf (or, inviscid Burgers) equation
Virasoro	L^2	KdV equation
Virasoro	H^1	Camassa–Holm equation
Virasoro	$\dot{H}^1$	Hunter–Saxton (or Dym) equation
SDiff(M)	L^2	Euler ideal fluid
SDiff(M)	H^1	averaged Euler flow
$\mathrm{SDiff}(M) \ltimes \mathrm{SVect}(M)$	$L^2 + L^2$	Magnetohydrodynamics
Maps(S^1, SO(3))	H^{-1}	Heisenberg magnetic chain

Table 4.1: Euler equations related to various Lie groups.



$\boxed{\text{2) Bi-Hamiltonian integrable systems}}$

$$\{\pi_\alpha, \pi_\beta\}_0 = f_{\alpha\beta}^{\gamma} Y_\gamma \qquad \text{2nd compatible Poisson bracket}$$

$$Y_\gamma(X) \in \mathfrak{g}^* \text{ fixed}$$

Geometric actions

$\boxed{\text{No Killing metric ?}}$

"Berry phase"

- use fixed coadjoint vector $Y_\alpha \in \mathfrak{g}^*$ to built first order action

$$S[g; Y, Z] = \int dt \left[\langle Y, \frac{dg}{dt} g^{-1} \rangle - \langle Y, \mathrm{Ad}_g Z \rangle \right]$$

$$= \int dt \left[Y_\alpha R^\alpha{}_i \dot{g}^i - Y_\alpha R^\alpha{}_i L_\beta{}^i Z^\beta \right]$$

$$\begin{cases} \delta_X g = gX \\ \delta_X g^i = L_\alpha{}^i X^\alpha \end{cases} \Leftrightarrow \frac{dX}{dt} = [X, Z]$$

cf. boosts

- first order

$$\begin{cases} a = \langle Y, dg\, g^{-1} \rangle & \text{potential 1-form} \\ \sigma = da = \langle Y, \frac{1}{2}[dg\, g^{-1}, dg\, g^{-1}] \rangle & \text{presymplectic 2-form} \end{cases}$$

gauge invariance H_Y little group of Y

$\delta_{\epsilon(t)} g = \epsilon(t)\, g$ little algebra $\mathcal{H}_Y$, $ad^*_{\epsilon(t)} Y = 0$

are these all gauge transformations ?

How many models $S[g; Y, Z]$ to study ?

$$\begin{cases} Y' = Ad^*_{h^{-1}} Y \\ Z' = Ad_k Z \end{cases} \qquad S[g; Y', Z'] = S[g'; Y, Z]$$

$g' = h\, g\, k$ field redefinition $\Rightarrow$ QM equivalent

only 1 representative needed per partition of $\mathfrak{g}^*, \mathfrak{g}$ into $\boxed{\text{coadjoint orbits} \\ \text{conjugacy classes}}$

Summary:

geometric action associated with single coadjoint orbit

$H_Y \backslash G$

Constrained Hamiltonian analysis (purely algebraic)

$\{g^i, p_j\} = \delta^i_j$, $\quad p_i = \frac{\partial \mathcal{L}}{\partial \dot{g}^i} = Y_\alpha R^\alpha{}_i$ primary constraint

$p_i \leftrightarrow \pi_\alpha = R_\alpha{}^i p_i$ $\quad \boxed{\phi^Y_\alpha = \pi_\alpha - Y_\alpha \approx 0}$

$S_{H_Y \backslash G} = \int dt \left[\pi_\alpha R^\alpha{}_i \dot{g}^i - H_z - w^\alpha \phi^Y_\alpha \right]$, $\quad H_z = \langle \pi, \mathrm{Ad}_g z \rangle = \pi_\alpha R^\alpha{}_i L_\beta{}^i z^\beta$

linear in π

Dynamics: $\quad \dot{g}^i = \{g^i, H_z + w^\alpha \phi^Y_\alpha\} = L_\alpha{}^i Y^\alpha + R_\alpha{}^i w^\alpha$ $\quad$ no 2nd order Lagrangian

$\dot{\pi}_\alpha = \{\pi_\alpha, H_z + w^\alpha \phi^Y_\alpha\} = \pi_\gamma f^\gamma{}_{\alpha\beta} w^\beta$

Noether charges: $\quad Q^\pi_X = \langle \pi, \mathrm{Ad}_g X \rangle$, $\quad \{Q^\pi_{X_1}, Q^\pi_{X_2}\} = -Q^\pi_{[X_1, X_2]}$

NB: associated to T^*G

Secondary constraints ?

$$\dot{\phi}^{\gamma}_{\alpha} \approx 0 \Leftrightarrow \boxed{\underbrace{\psi_{\gamma} f^{\gamma}_{\alpha\beta}}_{C_{\alpha\beta}} u^{\beta} = 0} \quad (*)$$ No, only restrictions on Lagrange multipliers

complete set of null eigenvectors $e_a{}^{\alpha}$ of $C_{\alpha\beta}$: $u^{\alpha} = e_a{}^{\alpha} u^a$ basis of $\mathcal{H}_{\psi} \in \mathfrak{g}$

arbitrary

adapted basis: $e_a{}^{\alpha}$, $e_A{}^{\alpha}$, $e^a{}_{\alpha}$, $e^A{}_{\alpha}$

$$e_a{}^{\alpha} e^b{}_{\alpha} = \delta_a^b, \quad e_A{}^{\alpha} e^b{}_{\alpha} = 0, \quad e_A{}^{\alpha} e^B{}_{\alpha} = \delta_A^B, \quad e_a{}^{\alpha} e^a{}_{\beta} + e_A{}^{\alpha} e^A{}_{\beta} = \delta^{\alpha}_{\beta}$$ $\left\{\begin{array}{l}\text{orthonormality}\\ \text{completeness}\end{array}\right.$

$$f^c{}_{ab} = 0, \quad C_{ab} = 0 = C_{aB}, \quad C_{AB} = \psi_c f^c{}_{AB} + \psi_C f^C{}_{AB}$$ invertible

subalgebra

$$(C^{-1})^{AB} C_{BC} = \delta^A_C \qquad (*) \Leftrightarrow u^A = 0, \; u^a \text{ arbitrary}$$

$\phi^{Y}{}_{a} \approx 0$ first class $\qquad \phi^{Y}{}_{A} \not\approx 0$ second class

solve 2nd class constraints & work with Dirac brackets

$$S\,[g^i, \pi_a, \pi_A, w^b, w^B; Y, Z]$$

(π_A, w^B) : auxiliary fields $\to$ solve in the action $\qquad \pi_A = Y_A, \quad w^B = 0$

$$\boxed{S^R_{H_Y\backslash G}[g^i, \pi_a, w^b; Y, Z] = \int dt\,\left[a^R_i\, \dot{g}^i - H^R_Y - w^a\, \phi^Y{}_a\right]}$$

$$a^R = (\pi_a R^a{}_i + Y_A R^A{}_i)\, dg^i, \qquad \sigma^R = d a^R$$

Dirac brackets :

$$\begin{array}{c|cc} & g^i & \pi_b \\ g^i & \left(\begin{matrix} C_{AB} R^A{}_i R^B{}_j & -R^b{}_i \\ R^a{}_j & 0 \end{matrix}\right. \\ \pi_a & & \end{array}\right) \quad \begin{array}{c|cc} & g^k & \pi_c \\ g^i & \left(\begin{matrix} R_C{}^j (C^{-1})^{CD} R_D{}^k & R_c{}^j \\ -R_b{}^k & 0 \end{matrix}\right. \\ \pi_b & & \end{array}\right) = \begin{pmatrix} \delta_i^k & 0 \\ 0 & \delta_c^a \end{pmatrix}$$

$$\nabla^R = \tfrac{1}{2}\nabla^R{}_{ij}\, dg^i dg^j + \nabla^R{}_i{}^b\, dg^i d\pi^b$$

$$\{g^j, g^k\}^* = R_C{}^j (C^{-1})^{CD} R_D{}^k, \quad \{g^j, \pi_c\} = R_c{}^j$$

$$\{\pi_b, \pi_c\}^* = f^d{}_{bc}\, \pi_d = 0$$

- $\widehat{\mathrm{Diff}(S^1)}$ typical little groups $U(1)$, $SL(2,\mathbb{R})$ $\Rightarrow$ at most 3 π_a's

g^i : ∞ dimensional, at most 3 gauge invariances

Unconstrained model : $\boxed{\text{drop all constraints} \quad \phi^{Y}_{\alpha} \approx 0}$

$$S_{T^*G}[g^i, \pi_\alpha; Y] = \int dt \left[\pi_\alpha R^\alpha{}_i \dot{g}^i - H_Z\right]$$

$$\dot{g}^i = \{g^i, H_Z\} = L_\alpha{}^i Z^\alpha \iff \frac{dg}{dt} g^{-1} = \mathrm{Ad}_g Z$$

$$\dot{\pi}_\alpha = \{\pi_\alpha, H_Z\} = 0$$

conserved charges : $\boxed{Q^\pi_X, \quad \pi_\alpha}$ $\boxed{\text{too large !}}$

level sets $\pi_\alpha = Y_\alpha$ Hamiltonian reduction $\longrightarrow$ do previous analysis

Proposal: Model space from S_{T^*G}:

$$\boxed{\text{impose only second class constraints } \phi_A^{Y\Xi} = 0}$$

(drop first class ones $\phi_a^{Y\Xi} = 0$)

$$S_{\Pi G_\Xi} = \int dt \left[\pi_\alpha R^\alpha{}_i \dot{g}^i - H_Z - u^A \phi_A^{Y\Xi}\right]$$

$\Leftrightarrow$ Model space from $S_{HY\Xi\backslash G}$:

$$\boxed{\text{promote } Y_a \in \mathcal{H}_{Y\Xi}^{\#} \text{ to new dynamical variables } \pi_a}$$

$$S^R_{\Pi G_\Xi} = \int dt \left[\pi_a R^a{}_i (\dot{g}^i - L_\alpha{}^i Z^\alpha) + Y_A^\Xi R^A{}_i (\dot{g}^i - L_\alpha{}^i Z^\alpha)\right]$$

Testing proposal on $SU(2)$ (everything known)

UIRREPS : $\mathcal{H}^j \, , \, j \in \mathbb{N}/2$

$\hat{J}_\pm \, , \, \hat{J}_3 \qquad |j, m\rangle \, , \quad m = -j, \ldots, +j$

Model space : $\bigoplus_{j \in \mathbb{N}/2} \mathcal{H}^j$

- (co)adjoint representation of $SU(2)$

$\Longleftrightarrow$ vector representation of $SO(3)$

$$R^\alpha{}_\beta = \frac{1}{2} \mathrm{Tr} \left(\sigma^\alpha g \sigma_\beta g^\dagger \right) \in SO(3)$$

- fixed vector $y_\alpha \in su^{(*)}(2) \simeq \mathbb{R}^3$

left invariant by $R(\hat{y}, \psi) \in SO(3)$

little group $H_{\vec{y}} \in SU(2) \ni e^{-\psi \frac{\hat{y}^\alpha}{2} i\sigma_\alpha} \quad 0 \leq \psi < 4\pi \quad \simeq U(1)$

coadjoint orbits : spheres S^2_y of radius $y \qquad \mathfrak{J} = y$

orbit representatives: $y^{\mathfrak{J}}_\alpha = y \, \delta^3_\alpha \, , \quad \begin{pmatrix} e^{-i\frac{\psi}{2}} & 0 \\ 0 & e^{i\frac{\psi}{2}} \end{pmatrix} \in H_y$

foliation $\quad su^{(*)}(2) = \bigcup_y \mathcal{R}_{SO(3)} \vec{y}^{\,y}$

Hopf fibration $\quad SU(2) \simeq U(1) \backslash S^2$

Adapted Euler angles $\quad g = e^{-\frac{\psi}{2} i\sigma_3} e^{-\frac{\theta}{2} i\sigma_2} e^{-\frac{\phi}{2} i\sigma_3}$

Borel gauge $\psi = 0$ (careful: gimbal lock $\Leftrightarrow$ Gribov obstruction!)

reduced phase space $\quad S_{H_Y}(SU(2)) = \int dt \left[Y \cos\theta \frac{d\phi}{dt} - \cos\theta\, Z \right]$

$$\alpha = Y \cos\theta \, d\phi \qquad \sigma = -Y \sin\theta \, d\theta \wedge d\phi$$

quantization of single orbit : integrality condition

$$\int_{S^2} \sigma = 2\pi\hbar n, \quad n \in \mathbb{Z} \implies Y = \hbar j, \quad j \in \frac{\mathbb{N}^*}{2}$$

construction of $|j, m\rangle$ through G-invariant polarization of $\psi(\theta, \phi)$

Starting from $T^*SU(2)$: phase space $\pi_\pm, \pi_3, \psi, \theta, \phi$

impose $\pi_+ = 0$ & compute Dirac brackets $\Rightarrow$ model space

+ $\pi_3 = 4$, $\psi = 0$ $\Rightarrow$ single orbit   Drawback: not G covariant

better: conversion to first class systems

impose only $\pi_+ = 0$ for model space
$\pi_3 = 4$ for single orbit

Dirac : impose first class constraints on states after quantization

$$\hat{G}_\alpha |\psi\rangle = 0$$

quantization of $T^*SU(2)$: basis for $L^2(SU(2))$

Wigner functions $D^j_{m'm}(\psi\theta\phi) = e^{-im'\psi}\, \overbrace{d^j_{m'm}(\theta)}^{\text{Jacobi polynomials}}\, e^{-im\phi}$

$D^{*j}_{m'm}$ carry representations of $\hat\pi_\pm = -i\hbar \vec{R}_\pm$, $\hat\pi_3 = -i\hbar \vec{R}_3$ & $\hat Q_\pm = -i\hbar \vec{L}_\pm$, $\hat Q_3 = -i\hbar \vec{L}_3$

SW spherical harmonics ${}_sY_{jm}(\theta,\phi) = (-)^{m-s}\sqrt{\frac{2j+1}{4\pi}}\, D^{j*}_{sm}\big|_{\psi=0}$ Borel gauge

$\hat\pi_\pm \leftrightarrow \eth, \bar\eth$

Spin-s Spherical Harmonics and $\eth$

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Geometry of the Hopf Bundle and spin-weighted Harmonics

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Numerical evolutions of fields on the 2-sphere using a spectral method based on spin-weighted spherical harmonics

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How should spin-weighted spherical functions be defined?

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Quantum model space from Dirac quantization of $T^*SU(2)$

$$\hat{\pi}_+ \left(a_j^{m'm} D^{+j}_{m'm} \right) = 0 \iff ð \left(a_j^{sm} \, {}_sY_{jm} \right) = 0$$

In the study of spin-weighted spherical harmonics it is useful to contemplate the following array:

$j=0$							1					
$\frac{1}{2}$						2		2				
1		$\overleftarrow{ð'}$			3		3		3		$\overrightarrow{ð}$	
$\frac{3}{2}$				4		4		4		4		
2			5		5		5		5		5	
$\frac{5}{2}$		6		6		6		6		6		6
$\vdots$	$\iddots$											$\ddots$
$s = \ldots$		$-\frac{5}{2}$	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$ $\ldots$

(4.15.60)

The numbers in this triangular array (which extends indefinitely downwards) represent the complex *dimensions* of the various spaces of spin-weighted spherical harmonics, as discussed in (4.15.43) *et seq.* Each of these spaces is characterized by its values of s and j, as shown. The dimension *zero* is assigned wherever a blank space appears in the array. The operator ð carries us a step of one s-unit to the right and ð′ one s-unit to the left. (From our earlier discussion, the j-value is not affected by ð or ð′.) Whenever such a step carries us off the array, the result of the operator ð or ð′ is zero. Note that the dimension remains constant whenever it does not drop to, or increase from, zero.

$$\iff \mathcal{H}_{MSU(2)} = \underset{j,m}{\mathrm{Span}} \left\{ {}_jY_{jm} \right\}$$

Single orbit: $\hat{\pi}_3 - q = 0$

$$\Rightarrow q = \hbar j, \quad j \text{ fixed}$$

Penrose & Rindler

$T^*SU(2)$ as constrained system

$SU(2) \ni g = \begin{pmatrix} \bar{z}_1 & \bar{z}_2 \\ -z_2 & z_1 \end{pmatrix}$ $\quad |z_1|^2 + |z_2|^2 = 1 = x^A x_A \leftarrow$ Euler angles

remove this constraint $\quad z_1 = x^0 + i x^3, \; z_2 = i(x^1 + i x^2) \quad x^A \in \mathbb{R}^4 - \{0\}$

$$= x^0 \sigma_0 + x^\beta(-i\sigma_\beta)$$

group law: nonzero (non-unimodular) quaternions $\mathbb{H}^*$

$T^*\mathbb{H}^* \to T^*SU(2)$ $\quad \begin{cases} \pi_0 = \frac{1}{2} x^A p_A \\ R = 1 \end{cases}$ second class constraints

$\to MSU(2) \quad \begin{cases} \pi_+ = 0 \\ \pi_- = 0 \end{cases}$

better constraints $R=1 \longleftrightarrow \pi_3 = R^2$

reduced phase space : $z_1, \bar{z}_1, z_2, \bar{z}_2$, no more π's

compute Dirac brackets : $\{z_1, \bar{z}_1\}^* = \frac{i}{2} = \{z_2, \bar{z}_2\}^*$, $\{z_1, z_2\}^* = 0 = \{z_1, \bar{z}_2\}^*$

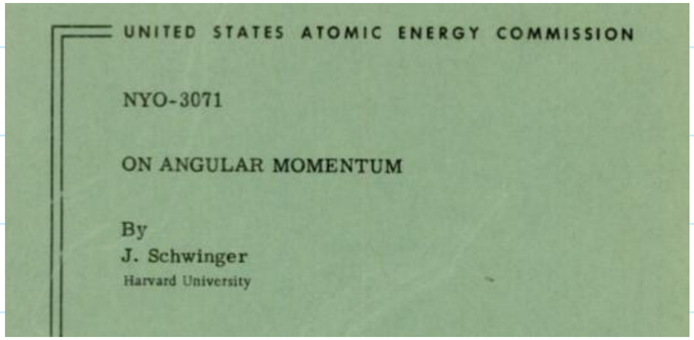
oscillators : $a_{1,2} = \sqrt{\frac{2}{\hbar}}\, \bar{z}_{1,2}$ $\quad a^*_{1,2} = \sqrt{\frac{2}{\hbar}}\, z_{1,2}$

reduced Noether charges
$$\begin{cases} q_+ = \hbar\, a^*_2 a_1 , \quad q_- = \hbar\, a^*_1 a_2 , \\ q_3 = \frac{\hbar}{2}(a^*_1 a_1 - a^*_2 a_2) \end{cases}$$

Model space in terms of two unconstrained oscillators

single orbit $R^2 = \frac{\hbar}{2}(a^*_1 a_1 + a^*_2 a_2) = 4$

quantum model space: $[\hat a_\xi , \hat a^+_{\xi'}] = \delta_{\xi\xi'}$ $\quad O_\alpha = \frac{\hat q_\alpha}{\hbar}$



Jordan-Schwinger map $\quad O_\alpha = \frac{1}{2} \hat a^+_\xi \, \tau_\alpha^{\xi\xi'} \, \hat a_{\xi'}$

orthonormal basis $\quad |n_1, n_2) = \frac{1}{\sqrt{n_1!}} \frac{1}{\sqrt{n_2!}} (\hat a^+_1)^{n_1} (\hat a^+_2)^{n_2} |0)$

$O^2 |n_1, n_2) = j(j+1) |n_1, n_2) \quad j = \frac{1}{2}(n_1 + n_2)$ $\quad$ (fixed j: single UIRREP)

$O_3 |n_1, n_2) = m |n_1, n_2) \quad m = \frac{1}{2}(n_1 - n_2) \quad |j+m, j-m) = |j, m\rangle$

better basis: coherent states holomorphic representation

$|a^\xi\rangle = e^{a^\xi \hat a^+_\xi} |0\rangle, \quad \psi(a^*_\xi) = \langle a^*_\xi | \psi\rangle, \quad \langle \phi | \psi \rangle = \int \prod_\xi \frac{da^*_\xi \, da^\xi}{2\pi i} e^{-a^*_\xi a^\xi} \phi^*(a^\xi) \psi(a^*_\xi)$

Conclusions · concrete proposal for model space for generic G

· works for $SO(2)$

· better understanding of SW spherical harmonics

basis for expansion of shear, news, Bondi mass & angular momentum aspects at $\mathcal{J}^+$

connection to expansions at i^0 ?

· application to Virasoro group & 3d gravity ?

J. Baez

Geometric Quantization (Part 5)

$S^{U(1)\backslash SU(2)}$

introduced in Part 4: quantization and projectivization. It's really the examples that bring the subject to life. They give new insights into hoary old topics in physics, and also raise some puzzles about the relation between classical and quantum mechanics.

I'll start with the classical spin-j particle and its quantization. I recently discovered through conversations on Twitter how few physicists have heard of the classical spin-j particle. They all know that the *quantum* spin-j particle has a Hilbert space $\mathbb{C}^{2j+1}$, an irreducible representation of $\mathrm{SU}(2)$. But the corresponding classical system whose quantization gives this Hilbert space seems remarkably little-known, especially given how simple it is. So, I'll describe it and its geometric quantization slowly and carefully, before feeding it into our functor