Computing spectral curves for third order ODOs

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Functional Equations in Limoges

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I will present recent and ongoing joint work with M.A. Zurro

Algorithmic Differential Algebra and Integrability (ADAI)

MINISTERIO DE CIENCIA E INNOVACIÓN
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Contents

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The theory of commuting ODOs

The theory of commuting ODOs has broad connections with many branches of modern mathematics:

- Non-linear partial differential equations (find new exact solutions).
- Algebra (the Dixmier or Jacobian or Poisson conjectures, highly non-trivial and still open).
- Complex analysis. Deformation quantisation. ...

The theory of commuting ODOs

Non-linear differential equations (KdV, Boussinesq, KN...KP)

Korteweg-de Vries equation modeled the solitary waves (solitons) in shallow water.

$\Updownarrow$

COMMUTING ODOs $\rightleftarrows$ ALGEBRAIC CURVES

Commutative Ordinary Differential Operators.

By J. L. Burchnall and T. W. Chaundy.

(Communicated by A. L. Dixon, F.R.S.—Received December 22, 1926.—Revised February 1, 1928.)

Schur, Wallemberg, Baker, Krichever, Mumford ...

The theory of commuting ODOs

Non-linear differential equations (KdV, Boussinesq, KN...KP)

$\Updownarrow$

COMMUTING ODOs $\rightleftarrows$ ALGEBRAIC CURVES

DIRECT PROBLEM $\longrightarrow$

Implicitization

Inverse problem $\longleftarrow$

Parametrization

Beret's conjecture [Guo, Zheglov 2022].

Spectral problem

Schrödinger equation

$$\Psi_{xx} - u(x)\Psi = \lambda\Psi \tag{1}$$

with $u(x)$ satisfying a Korteweg de Vries (KdV) equation of the celebrated KdV hierarchy. For instance, the classical stationary KdV equation

$$u_{xxx} - 6uu_x = 0.$$

λ spectral parameter

(Drach's Ideology, 1919) Brehznev 2008, 2012, 2013.
Integrate (1) as an ODE to obtain a parametric solution $\Psi(x;\lambda)$

Spectral problem

(Σ, ∂) ordinary differential field
field of constants $C = \overline{C}$, characteristic 0.

Given

$$L \text{ in } \Sigma[\partial] \backslash C[\partial]$$

assuming

NON-TRIVIAL CENTRALIZER $\mathcal{Z}(L)$

Parametric solutions $\Psi(x; \lambda, \mu)$

$$L(\Psi) = \lambda \Psi, \;\; B(\Psi) = \mu \Psi$$

for $B \in \mathcal{Z}(L)$, $\partial(\lambda) = 0$, $\partial(\mu) = 0$.

Centralizers and spectral curves

Schur, Flanders, Krichever, Amitsur, Carlson, Ore....
[Goodearl, 1983]

$$\mathcal{Z}(L) = \{A \in \Sigma[\partial] \mid [L, A] = 0\}$$

- Trivial $\mathcal{Z}(L) = C[L]$
- Non-trivial $\mathcal{Z}(L)$ is a free $C[L]$-module, the cardinal of a basis divides $\mathrm{ord}(L)$.

$$\text{SPECTRAL CURVE } \Gamma := \mathrm{Spec}(\mathcal{Z}(L))$$

$\mathcal{Z}(L)$ maximal commutative domain in $\Sigma[\partial]$.

(with M.A. Zurro)

Computing defining ideals of space spectral curves for algebro-geometric third order ODOs. arXiv:2311.09988, 2023.

Contents

BC Ideal of a pair

Commuting P and Q in $\Sigma[\partial]$

$$e_{P,Q} : C[\lambda, \mu] \to \Sigma[\partial]$$

homomorphism of C-algebras defined by

$$g(P, Q) := e_{P,Q}(g) = e_{P,Q}(\sigma_{i,j}\lambda^i\mu^j) = \sigma_{i,j}P^iQ^j.$$

Define the Burchnall-Chaundy BC ideal of the pair P and Q as

$$\mathrm{BC}(P, Q) := \mathrm{Ker}(e_{P,Q}) = \{g \in C[\lambda, \mu] \mid g(P, Q) = 0\}.$$

Its elements are BC polynomials

Spectral curve of a pair

Commuting P and Q in $\Sigma[\partial]\backslash C[\partial]$

$\mathcal{Z}(P)$ finitely generated $C[P]$-module $\Rightarrow$ $\mathtt{BC}(P,Q)$ non zero ideal.

$\Sigma[\partial]$ domain $\Rightarrow$ $\mathtt{BC}(P,Q)$ prime ideal.

Spectral curve $\Gamma_{P,Q} := V(\mathtt{BC}(P,Q))$

Coordinate ring of $\Gamma_{P,Q}$

$$\frac{C[\lambda,\mu]}{\mathtt{BC}(P,Q)} \simeq C[P,Q].$$

Spectral curve of a pair

There exists an irreducible polynomial $f \in C[\lambda, \mu]$ such that

$$\mathtt{BC}(P, Q) = (f)$$

$$\Gamma_{P,Q} = \left\{ (\lambda_0, \mu_0) \in C^2 \mid f(\lambda_0, \mu_0) = 0 \right\}.$$

How do we compute f?

Computing BC ideals

Given (monic) $P,\ Q \in \Sigma[\partial]$, then $P - \lambda,\ Q - \mu$ in $\mathbb{D} = \Sigma[\lambda, \mu]$.

$\mathrm{ord}(P) = n$, $\mathrm{ord}(Q) = m$

$$h(\lambda, \mu) = \partial\mathrm{Res}(P - \lambda, Q - \mu) = \mu^n - \lambda^m + \dots$$

a non trivial polynomial in $\Sigma[\lambda, \mu]$

Generalize [Wilson, 1985], [Previato, 1991].

(RZ 2023) Arbitrary (Σ, ∂), $Const(\Sigma) = C = \overline{C}$.

If $[P, Q] = 0$ then $h(\lambda, \mu) \in \mathtt{BC}(P, Q)$.

1. Proof by Poisson's Formula $h(\lambda, \mu) \in C[\lambda, \mu]$.
2. Proof by elimination ideals $h(P, Q) = 0$.

Rosen-Morse potential $u_1 = \frac{-2}{\cosh^2(x)}$

$$L_1 = -\partial^2 + u_1,\ [L_1, A_3] = \mathrm{KdV}_0(\mathrm{u}_1) + \mathrm{KdV}_1(\mathrm{u}_1) = 0$$

$$\begin{aligned} f_1(\lambda,\mu) &= -\mu^2 - \lambda(\lambda-1)^2 = \\ &= \partial\mathrm{Res}(L_1-\lambda, A_3-\mu) = \end{aligned}$$

$$\begin{vmatrix} -1 & 0 & \frac{-2}{(\cosh(x))^2} - \lambda & 8\,\frac{\sinh(x)}{(\cosh(x))^3} & \frac{4}{(\cosh(x))^2} - 12\,\frac{(\sinh(x))^2}{(\cosh(x))^4} \\ 0 & -1 & 0 & \frac{-2}{(\cosh(x))^2} - \lambda & 4\,\frac{\sinh(x)}{(\cosh(x))^3} \\ 0 & 0 & -1 & 0 & \frac{-2}{(\cosh(x))^2} - \lambda \\ -1 & 0 & \frac{-3}{(\cosh(x))^2} + 1 & 9\,\frac{\sinh(x)}{(\cosh(x))^3} - \mu & \frac{3}{(\cosh(x))^2} - 9\,\frac{(\sinh(x))^2}{(\cosh(x))^4} \\ 0 & -1 & 0 & \frac{-3}{(\cosh(x))^2} + 1 & 3\,\frac{\sinh(x)}{(\cosh(x))^3} - \mu \end{vmatrix}$$

Elimination ideals

Left ideal

$$(P-\lambda, Q-\mu) = \{C(P-\lambda) + D(Q-\mu) \mid C, D \in \Sigma[\lambda,\mu][\partial]\}$$

Two sided ideals

$$\mathcal{E}(P-\lambda, Q-\mu) := (P-\lambda, Q-\mu) \cap \Sigma[\lambda,\mu].$$

and

$$\mathcal{E}_C(P-\lambda, Q-\mu) := (P-\lambda, Q-\mu) \cap C[\lambda,\mu].$$

By definition of the differential resultant

$$h(\lambda,\mu) = \partial\mathrm{Res}(P-\lambda, Q-\mu) \in \mathcal{E}_C(P-\lambda, Q-\mu).$$

Thus both elimination ideals are nonzero.

Elimination ideals

Commuting P and Q in $\Sigma[\partial]\backslash C[\partial]$, both of positive order,

$$f = \sqrt{h}, \text{ with } h = \partial\mathrm{Res}(P-\lambda, Q-\mu).$$

(RZ 2023)

1. The radical of the elimination ideal $\mathcal{E}_C(P-\lambda, Q-\mu)$ equals

$$\mathtt{BC}(P,Q) = (f).$$

2. The radical of the elimination ideal $\mathcal{E}(P-\lambda, Q-\mu)$ equals $[f]$.

Recall $f \in C[\lambda,\mu]$,

$$(f) = C[\lambda,\mu]f \text{ and } [f] = \Sigma[\lambda,\mu]f \text{ differential ideal.}$$

Contents

Spectral curve of L

Generalized Schur's Theorem [Goodearl, 1983]

$$\mathcal{Z}((L)) = \left\{ \sum_{j=-\infty}^{m} c_j Q^j \mid c_j \in C, m \in \mathbb{Z} \right\}$$

Commutative differential domain

$$\mathcal{Z}(L) = \mathcal{Z}((L)) \cap \Sigma[\partial]$$

$Spec(\mathcal{Z}(L))$ is an abstract algebraic curve Γ

Compute the defining ideal of Γ

Centralizer $\mathrm{ord}(L) = 3$

Given $L \in \Sigma[\partial] \backslash C[\partial]$, with $\mathcal{Z}(L) \neq C[L]$.

$\mathcal{Z}(L)$ is a free $C[L]$-module of rank 3.

$\{1, A_1, A_2\}$ basis of $\mathcal{Z}(L)$ as a $C[L]$-module. Each A_i is a monic operator in $\mathcal{Z}(L) \backslash C[L]$ of minimal order

$$o_i := \mathrm{ord}(A_i) \equiv i \ (mod\ 3).$$

$$\mathcal{Z}(L) = C[L] \oplus C[L]A_1 \oplus C[L]A_2 = C[L, A_1, A_2]$$

BC ideal $\mathrm{ord}(L) = 3$

$$e_L : C[\lambda, \mu_1, \mu_2] \to \Sigma[\partial]$$
$$e_{P,Q}(\lambda) = L, \;\; e_{P,Q}(\mu_1) = A_1, \;\; e_{P,Q}(\mu_2) = A_2.$$

Image of e_L,

$$\mathcal{Z}(L) = C[L, A_1, A_2]$$

Given $g \in C[\lambda, \mu_1, \mu_2]$ denote

$$g(L, A_1, A_2) := e_L(g).$$

$$\mathtt{BC}(L) := \mathrm{Ker}(e_L) = \{g \in C[\lambda, \mu_1, \mu_2] \mid g(L, A_1, A_2) = 0\}.$$

Spectral curve $\mathrm{ord}(L) = 3$

$$\mathrm{ord}(L) = 3 \text{ in } \Sigma[\partial], \;\; \mathcal{Z}(L) = C[L, A_1, A_2], \;\; \mathrm{ord}(A_i) \equiv_3 i$$

$$\begin{cases} f_i = \partial\mathrm{Res}(L - \lambda, A_i - \mu_i), \;\; i = 1, 2 \\ f_3^r = \partial\mathrm{Res}(A_1 - \mu_1, A_2 - \mu_2) \end{cases}$$

are irreducible in $C[\lambda, \mu_1, \mu_2]$ since

$$\mathtt{BC}(L, A_i) = (f_i) \text{ and } BC(A_1, A_2) = (f_3)$$

$$(0) \subset (f_i) \subset (f_1, f_2) \subseteq (f_1, f_2, f_3) \subseteq \mathtt{BC}(L), \;\; i = 1, 2.$$

$$\Gamma := V(\mathtt{BC}(L)) \;\subseteq\; \gamma := V(f_1, f_2, f_3) \;\subseteq\; \beta := V(f_1, f_2).$$

Space algebraic curve $\beta = V(f_1) \cap V(f_2)$ is the intersection of the irreducible surfaces defined by $f_1(\lambda, \mu_1) = 0$ and $f_2(\lambda, \mu_2) = 0$.

Spectral curve $\operatorname{ord}(L) = 3$

Theorem: (RZ 2023) $\mathtt{BC}(L)$ is a prime ideal

$$\mathtt{BC}(L) = (f_1, f_2, f_3)$$

Irreducible affine algebraic curve in C^3

$$\Gamma = V(\mathtt{BC}(L))$$

$$\mathcal{Z}(L) \simeq C[\Gamma] = \frac{C[\lambda, \mu_1, \mu_2]}{\mathtt{BC}(L)}$$

If $\mathrm{ord}(A_2) = 2$ then $A_1 = A_2^2$ implying that $f_3 = (\mu - \gamma^2)^2$.

$$\mathcal{Z}(L) = \mathcal{C}(A_2) = C[L, A_2] \simeq \frac{C[\lambda, \mu]}{(f_2)}$$

coordinate ring of a plane algebraic curve.

Planar spectral curve

[Dickson, Gesztesy, Unterkofler, 1999] $\Sigma = \mathbb{C}(x), \partial = d/dx$

$$L = \partial^3 - \frac{15}{x^2}\partial + \frac{15}{x^3} + h\ .$$

$$\mathcal{Z}(L) = C[L, A_1, A_2]\ ,\ \mathrm{ord}(A_1) = 4, \mathrm{ord}(A_2) = 8.$$

We compute the generators of the ideal $\mathtt{BC}(L) = (f_1, f_2, f_3)$ using differential resultants

$$f_1 = -\mu_1^3 + (\lambda - h)^4\ , f_2 = -\mu_2^3 + (\lambda - h)^8, f_3^4 = (\mu_2 - \mu_1^2)^4.$$

Since f_3 is the BC polynomial of A_1 and A_2 we have $A_2 = A_1^2$, implying that

$$\mathcal{Z}(L) = C[L, A_1] \simeq \frac{C[\lambda, \mu_1]}{(f_1)}$$

Non-planar spectral curves

(RZ 2022) $\Sigma = \mathbb{C}(x), \partial = d/dx$

$$L = \partial^3 - \frac{6}{x^2}\partial + \frac{12}{x^3} + h\ , \ \ h \in \mathbb{C}.$$

$\mathcal{Z}(L) = \mathbb{C}[L, A_1, A_2]$ with $\mathrm{ord}(A_1) = 4$, $\mathrm{ord}(A_2) = 5$.
Using differential resultants we compute

$$f_1 = -\mu^3 + (\lambda - h)^4,\ \ f_2 = -\gamma^3 + (\lambda - h)^5,\ \ f_3 = \gamma^4 - \mu^5.$$

$\mathtt{BC}(L) = (f_1, f_2, f_3)$ is a prime ideal.

First explicit example of a non-planar spectral curve.

The curve defined by $\mathtt{BC}(L)$ is a non-planar curve Γ parametrized by

$$\aleph(\tau) = (h - \tau^3, \tau^4, -\tau^5), \tau \in \mathbb{C}.$$

$\Sigma = \mathbb{C}(z = e^x), \partial = d/dx$

$$L = \partial^3 + \frac{24z}{(z+1)^2}\partial + \frac{-48z(z-1)}{(z+1)^3}, \quad \mathrm{ord}(A_1) = 4, \mathrm{ord}(A_2) = 5$$

Non-planar spectral curve Γ defined by the prime ideal

$$\mathtt{BC}(L) = (f_1, f_2, f_3)$$

$$\begin{aligned}
f_1 =& \partial\mathrm{Res}(L-\lambda, A_1-\mu_1) = 1 + \lambda^4 + \frac{44}{27}\lambda^2 - \mu_1^3 - 4\lambda^2\mu_1 + 3\mu_1^2 - 3\mu_1 \\
f_2 =& \partial\mathrm{Res}(L-\lambda, A_2-\mu_2) = \\
& \lambda^5 + 16(\mu_2-1)\lambda^2/3 + (4096\lambda)/729 - (\mu_2-1)^3 \\
f_3 =& \partial\mathrm{Res}(A_1-\mu_1, A_2-\mu_2) = \ldots
\end{aligned}$$

Contents

New coefficient field

$P, Q \in \Sigma[\partial]$

$$[P, Q] = 0 \Rightarrow \partial\mathrm{Res}(P - \lambda, Q - \mu) = f(\lambda, \mu)^r \in C[\lambda, \mu].$$

As differential operators in $\Sigma[\lambda, \mu][\partial]$,

$$\partial\mathrm{Res}(P - \lambda, Q - \mu) \neq 0 \Rightarrow \mathrm{gcrd}(P - \lambda, Q - \mu) = 1.$$

$$\Sigma(\Gamma_{P,Q}) = Fr\left(\frac{\Sigma[\lambda, \mu]}{[f]}\right)$$

As differential operators in $\Sigma(\Gamma_{P,Q})[\partial]$,

$$\partial\mathrm{Res}(P - \lambda, Q - \mu) = 0 \Rightarrow \mathrm{gcrd}(P - \lambda, Q - \mu) \neq 1.$$

New coefficient field

$[\mathrm{BC}(L)]$ is a prime differential ideal of $\Sigma[\lambda, \mu_1, \mu_2]$

Differential domain

$$\Sigma[\Gamma] = \frac{\Sigma[\lambda, \mu_1, \mu_2]}{[\mathrm{BC}(L)]}$$

Its fraction field

$$\Sigma(\Gamma)$$

is a differential field with the extended derivation.

Intrinsic right factor

$\mathrm{ord}(L) = 3$ in $\Sigma[\partial]$, $\mathcal{Z}(L) = C[L, A_1, A_2]$

Theorem: (RZ 2023) The greatest common right divisor in $\Sigma(\Gamma)[\partial]$

$$\partial + \phi = \mathrm{gcrd}(L - \lambda, A_1 - \mu_1, A_2 - \mu_2)$$

equals $\mathrm{gcrd}(L - \lambda, A_1 - \mu_1) = \mathrm{gcrd}(L - \lambda, A_2 - \mu_2)$
and divides $\mathrm{gcrd}(A_1 - \mu_1, A_2 - \mu_2))$.

Assume $L = \partial^3 + u_1\partial + u_0$

$$L - \lambda = \left(\partial^2 - \phi\partial + u_1 - 2\phi' + \phi^2\right) \cdot (\partial + \phi),$$

in $\Sigma(\Gamma)[\partial]$, under the condition

$$\phi^3 + u_1\phi - 3\,\phi\phi' - u_0 + \phi'' + \lambda = 0.$$

Non-planar spectral curve

$$\aleph(\tau) = (-\tau^3 + 1, \tau^4, -\tau^5),\ \tau \in \mathbb{C}.$$

The first differential subresultants of $L - \lambda$, $A_1 - \mu_1$ and $A_2 - \mu_2$ pairwise are equal to

$$\phi_{i,0} + \phi_{i,1}\partial,\ \ i = 1, 2, 3,\ \ j = 0, 1,$$

with

$$\begin{aligned}
\phi_{1,0} &= (1-\lambda)\mu_1 - \tfrac{4\mu_1}{x^3} + \tfrac{8(\lambda-1)}{x^4}, & \phi_{1,1} &= (\lambda-1)^2 - \tfrac{2\mu_1}{x^2} + 4\tfrac{(\lambda-1)}{x^3},\\
\phi_{2,0} &= (1-\lambda)^3 - \tfrac{4(1-\lambda)^2}{x^2} + \tfrac{8\mu_2}{x^4}, & \phi_{2,1} &= (\lambda-1)^3 - \tfrac{4(1-\lambda)^2}{x^2} + \tfrac{8\mu_2}{x^3},\\
\phi_{3,0} &= -\mu_2\left(\mu_1^2 + \tfrac{4\mu_2}{x^3} - \tfrac{8\mu_1}{x^4}\right), & \phi_{3,1} &= \mu_1^3 - \tfrac{2\mu_2^2}{x^2} + \tfrac{4\mu_2\mu_1}{x^3}.
\end{aligned}$$

We have $\mathrm{ord}(A_1) = 4$ and $\mathrm{ord}(A_2) = 5$ thus

$$\phi = \overline{\phi}_i = \frac{\phi_{0,i}}{\phi_{1,i}} + [\mathtt{BC}(L)], \quad i = 1, 2, 3.$$

$$\phi(\tau) := \phi_i(\aleph(\tau)) = \frac{-\tau^3 x^3 + 2\tau^2 x^2 - 4\tau x + 4}{(\tau^2 x^2 - 2\tau x + 2)x}.$$

Thus

$$L + \tau^3 - 1 = \left(\partial^2 + \phi(\tau)\partial + \phi(\tau)^2 + 2\phi(\tau)' - \frac{6}{x^2}\right) \cdot (\partial + \phi(\tau))$$

At every point $P_0 = \aleph(\tau_0)$ of the spectral curve Γ of L we recover a right factor $\partial + \phi(\tau_0)$, for $\tau_0 \neq 0$.

ADAI Goals

`Algorithmic Differential Algebra and Integrability (ADAI)`

Develop Picard-Vessiot (PV) theory for spectral problems.
Use effective differential algebra to develop symbolic algorithms:

- Parametric factorization of algebro-geometric ODOs.
- Existence and computation of spectral Picard-Vessiot fields. Differential field extension of $\Sigma(\Gamma)$, minimal containing all the solutions. In (MRZ 2021) for Schrödinger operators
- Compute integrable hierarchies and almost commuting basis.
- Compute new algebro-geometric ODOs, order ≥ 3.

- (MRZ 2020) J.J. Morales-Ruiz. S.L. Rueda, and M.A. Zurro. **Factorization of KdV Schrödinger operators using differential subresultants.** Adv. Appl. Math., 120:102065, 2020.
- (MRZ 2021) J.J. Morales-Ruiz. S.L. Rueda, and M.A. Zurro. **Spectral Picard-Vessiot fields for algebro-geometric Schrödinger operators .** Annales de l'Institut Fourier, Vol. 71, No. 3, pp. 1287-1324, 2021.
- (PRZ 2019) E. Previato, S.L. Rueda, and M.A. Zurro. **Commuting Ordinary Differential Operators and the Dixmier Test**. SIGMA Symmetry Integrability Geom. Methods Appl., 15(101):23 pp., 2019.
- (RZ 2021) S.L. Rueda and M.A. Zurro. **Factoring Third Order Ordinary Differential Operators over Spectral Curves.** See arXiv:2102.04733v1, 2021.