Towards an effective integro-differential elimination theory

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Introduction

Algebraic analysis is a mathematical theory which studies linear systems of PDEs using module theory, homological algebra...

It was developed by Malgrange, Bernstein, Kashiwara... in the 70's.

It nowadays plays a fundamental role in modern mathematics (algebraic geometry, representation theory, singularity theory...).

Question: What does algebraic analysis yield if we consider *rings of integro-differential operators* instead of rings of differential operators?

$$\dot{y}(t) + t^2\, y(t) + t \int_0^t y(\tau)\, d\tau - t \int_0^t \tau\, y(\tau)\, d\tau + (t-1)\, y(0) = 0$$

Integro-differential operators

$\mathbb{k}$ = a field of characteristic 0.

Let us consider the following $\mathbb{k}$-endomorphisms of $\mathbb{k}[t]$:

$$t : \begin{array}{ccc} \mathbb{k}[t] & \longrightarrow & \mathbb{k}[t] \\ p & \longmapsto & t\,p, \end{array} \quad \partial : \begin{array}{ccc} \mathbb{k}[t] & \longrightarrow & \mathbb{k}[t] \\ p & \longmapsto & p', \end{array} \quad I : \begin{array}{ccc} \mathbb{k}[t] & \longrightarrow & \mathbb{k}[t] \\ p & \longmapsto & \int_{t_0}^{t} p(\tau)d\tau. \end{array}$$

The *fundamental theorem of calculus* can be written as

$$\partial \circ I = 1,$$

where 1 denotes the identity endomorphism.

We can also see that:

$$\forall\, p \in \mathbb{k}[t], \quad (1 - I \circ \partial)(p) = p - \int_{t_0}^{t} \dot{p}(\tau)\, d\tau = p(t_0).$$

Fix $t_0 \in \mathbb{k}$ and consider the following endomorphism of $\mathbb{k}[t]$:

$$e = 1 - I \circ \partial : \begin{array}{ccc} \mathbb{k}[t] & \longrightarrow & \mathbb{k}[t] \\ p & \longmapsto & p(t_0). \end{array}$$

Definitions of $A_1(\mathbb{k})$ and $\mathbb{I}_1(\mathbb{k})$

Definition

$A_1(\mathbb{k})$ is the sub-$\mathbb{k}$-algebra of $\mathrm{end}_{\mathbb{k}}(\mathbb{k}[\mathrm{t}])$ generated by t and ∂.

Definition

$\mathbb{I}_1(\mathbb{k})$ is the sub-$\mathbb{k}$-algebra of $\mathrm{end}_{\mathbb{k}}(\mathbb{k}[\mathrm{t}])$ generated by t, ∂, I and e.

Identities of $\mathbb{I}_1(\mathbb{k})$: ($\circ$ is ommited)

$$\partial\, I = 1 \quad \text{: 1st fundamental thm}$$

$$I\,\partial = 1 - e \quad \text{: 2nd fundamental thm}$$

$$\partial\, p = p\,\partial + \dot{p} \quad \text{: Leibniz rule}$$

$$I\, p\,\partial = -I\,\partial\, p + p - e(p)\, e \quad \text{: integration by parts}$$

$$I\, p\, I = I(p)\, I - I\, I(p) \quad \text{: double integration}$$

$$e^2 = e,\ \partial\, e = 0,\ e\, p = e(p)\, e = p(t_0)\, e \quad \text{: relations with the evaluation}$$

Normal forms

A consequence of these identities is that every operator of $\mathbb{I}_1(\Bbbk)$ can be written in a canonical way (*normal form*).

Any operator of $\mathbb{I}_1(\Bbbk)$ can uniquely be written as

$$d = \underbrace{\sum_{i=0}^{m} a_i(t)\,\partial^i}_{\in A_1(\Bbbk)} + \sum_{j=0}^{p} b_j(t)\, I\, c_j(t) + \underbrace{\sum_{k=0}^{q} f_k(t)\, e\, \partial^k}_{\in \langle e \rangle},$$

where $a_i, b_j, c_j, f_k \in \Bbbk[t]$, $m, p, q \in \mathbb{N}$ and $\langle e \rangle$ is the only two-sided ideal of $\mathbb{I}_1(\Bbbk)$ generated by e, i.e., $\langle e \rangle = \mathbb{I}_1(\Bbbk)\, e\, \mathbb{I}_1(\Bbbk)$.

$\mathbb{I}_1(\mathbb{k})$ is not a noetherian ring

Theorem

$\mathbb{I}_1(\mathbb{k})$ is neither a left nor right noetherian ring.

For $N \in \mathbb{N}$, let us introduce

$$T_N = \sum_{k=0}^{N} \frac{t^k}{k!} \, e \, \partial^k \quad (\textit{Taylor operators for } t_0 = 0)$$

For instance, $T_0 = e$, $T_1 = e + t\, e\, \partial$. Notice that

$$e\,(e + t\,e\,\partial) = e^2 + e\,t\,e\,\partial = e^2 + 0 = e \;\Rightarrow\; \mathbb{I}_1\, T_0 \subset \mathbb{I}_1\, T_1.$$

More generally:

$$T_N = T_N\, T_{N+1} \;\Rightarrow\; \mathbb{I}_1\, T_N \subseteq \mathbb{I}_1\, T_{N+1}, \quad \mathbb{I}_1\, T_N \neq \mathbb{I}_1\, T_{N+1}.$$

Coherence definition

Finitely presented module

Let $\mathcal{R}$ be a ring and $\mathcal{M}$ a left $\mathcal{R}$-module finitely generated by $g_1, \ldots, g_p$. Then, we have the following surjective homomorphism:

$$\begin{array}{rcl} \pi : \mathcal{R}^{1\times p} & \longrightarrow & \mathcal{M} \\ e_i = (0 \ \ldots \ 1 \ \ldots 0) & \longmapsto & g_i, \quad i = 1, \ldots, p. \end{array}$$

$\mathcal{M}$ is said to be *left finitely presented* if the left $\mathcal{R}$-module

$$\ker \pi = \left\{ (\lambda_1, \ldots, \lambda_p) \in \mathcal{R}^{1\times p} \mid \pi(\lambda) = \sum_{i=1}^{p} \lambda_i \, g_i = 0 \right\}$$

is finitely generated. This is equivalent to the existence of a matrix $S \in \mathcal{R}^{q\times p}$ such that $\ker \pi = \mathrm{im}_{\mathcal{R}}(\cdot S)$, i.e., we have the following exact sequence:

$$\mathcal{R}^{1\times q} \xrightarrow{\cdot S} \mathcal{R}^{1\times p} \xrightarrow{\pi} \mathcal{M} \longrightarrow 0.$$

Coherent module

A left $\mathcal{R}$-module $\mathcal{M}$ is *coherent* if all of its finitely generated left $\mathcal{R}$-modules are left finitely presented.

$\mathbb{I}_1(\mathbb{k})$ is coherent

Coherence characterization

Let $\mathcal{R}$ be a ring. The following assertions are equivalent:

1. $\mathcal{R}$ is a left coherent ring.
2. i) For all $a \in \mathcal{R}$, $\mathrm{ann}_{\mathcal{R}}(.a) = \{r \in \mathcal{R} \mid r\, a = 0\}$ is a finitely generated left ideal.

   ii) For all pairs of ideals $\mathcal{I}$ and $\mathcal{J}$ finitely generated, the left ideal $\mathcal{I} \cap \mathcal{J}$ is finitely generated.

Theorem (*Bavula 2013*)

$\mathbb{I}_1(\mathbb{k})$ is a coherent ring, i.e., left coherent and right coherent.

END GOAL: Give an effective proof of this theorem.

$\Rightarrow$ *Effective development of an integro-differential elimination theory*

$$\forall\, R \in {\mathbb{I}_1}^{q \times p} \quad \exists\, Q \in {\mathbb{I}_1}^{r \times q} : \ \ker_{\mathbb{I}_1}(.R) = \mathrm{im}_{\mathbb{I}_1}(.Q)$$

$$R\,\eta = \zeta \;\Rightarrow\; Q\,\zeta = Q\,R\,\eta = 0 \quad (\textit{elimination of } \eta)$$

Situation

- The first point of the characterization of the coherence property is effective (Quadrat-Regensburger 20, Cluzeau, P., Quadrat 23).
- The extension of the matrix case of the first point makes effective the second point of the characterization, i.e., $\mathcal{I} \cap \mathcal{J}$ finitely generated where $\mathcal{I}$ and $\mathcal{J}$ are finitely generated, in the case where $\mathcal{I}$ and $\mathcal{J}$ are both included in $\langle e \rangle$.
- For the intersection $\mathcal{I} \cap \mathcal{J}$, where $\mathcal{I}$ or $\mathcal{J}$ is included in $\langle e \rangle$, we use the concept of **semisimple modules** (namely, direct sums of *simple* modules, e.g., $\mathbb{k}[t]^m$).

  A submodule of a semisimple module is semisimple

  Indeed, if $\mathcal{I} \subset \langle e \rangle$, then $\mathcal{I}$ semisimple and $\mathcal{I} \cap \mathcal{J} \subset \mathcal{I}$ is also semisimple.

Link intersection and annihilator

$\mathcal{I} = \langle u_1, \ldots, u_n \rangle$ and $\mathcal{J} = \langle v_1, \ldots, v_m \rangle$.

$$x \in \mathcal{I} \cap \mathcal{J} \Longleftrightarrow x = \sum_{i=1}^{n} a_i\, u_i = \sum_{j=1}^{m} b_j\, v_j \Longleftrightarrow \sum_{i=1}^{n} a_i\, u_i - \sum_{j=1}^{m} b_j\, v_j = 0$$

$$\Longleftrightarrow \begin{pmatrix} a_1 & \ldots & a_n \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} - \begin{pmatrix} b_1 & \ldots & b_m \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} = 0$$

$$\Longleftrightarrow \begin{pmatrix} a_1 & \ldots & a_n & -b_1 & \cdots - b_m \end{pmatrix} \underbrace{\begin{pmatrix} u_1 \\ \vdots \\ u_n \\ v_1 \\ \vdots \\ v_m \end{pmatrix}}_{R} = 0$$

$$\Longleftrightarrow \begin{pmatrix} a_1 & \ldots & a_n & -b_1 & \cdots - b_m \end{pmatrix} \in \ker_{\mathbb{I}_1}(\cdot R)$$

$\mathcal{I} \cap \mathcal{J}$ where $\mathcal{I}, \mathcal{J} \subset \langle e \rangle$

Theorem

Let $R \in \langle e \rangle^{q \times p}$ *and* $R = \sum_{k=0}^{n} R_k(t)\, e\, \partial^k$, *where* $R_k \in \mathbb{k}[t]^{q \times p}$. *Let* $m = \max_{k \in [\![0,n]\!]} \deg(R_k)$, *and*

$$C = \begin{pmatrix} R_0 & \dots & R_n \\ \vdots & & \vdots \\ R_0^{(m+1)} & \dots & R_n^{(m+1)} \end{pmatrix} \in \mathbb{k}^{q(m+2) \times p(n+1)}, \quad J_{m+1} = \begin{pmatrix} I_q \\ I_q\, \partial \\ \vdots \\ I_q\, \partial^{m+1} \end{pmatrix}.$$

Let $D \in \mathbb{k}[t]^{r \times q(m+2)}$ *be a full row rank matrices satisfying*

$$\ker_{\mathbb{k}[t]}(.C) = \mathrm{im}_{\mathbb{k}[t]}(.D)$$

and let us define $\begin{pmatrix} u_1 & \dots & u_r \end{pmatrix}^T = D\, J_{m+1} \in \mathbb{I}_1{}^{r \times q}$, *where* $u_1, \dots, u_r$ *belong to* $\mathbb{I}_1{}^{1 \times q}$. *Then, we have:*

$$\ker_{\mathbb{I}_1}(.R) = \mathrm{im}_{\mathbb{I}_1}(.D\, J_{m+1}) = \sum_{i=1}^{r} \mathbb{I}_1\, u_i.$$

Computation of $\mathcal{I} \cap \mathcal{J}$ where $\mathcal{I}, \mathcal{J} \subset \langle e \rangle$

Algorithm 1 Compute generators of $\mathcal{I} \cap \mathcal{J}$ where $\mathcal{I}, \mathcal{J} \subseteq \langle e \rangle$

Require: $p_1, \ldots, p_{n_1}$ generators of $\mathcal{I}$, $q_1, \ldots, q_{n_2}$ generators of $\mathcal{J}$

- Set $R = (p_1 \; \ldots \; p_{n_1} \; q_1 \; \ldots \; q_{n_2})^T$.
- Compute the matrix C corresponding to R.
- Compute D such that $\ker_{\mathbb{k}[t]}(.C) = \mathrm{im}_{\mathbb{k}[t]}(.D)$.
- Compute $u = (u_1, \; \ldots, \; u_r)^T = D\, J_{m+1}$, where $u_i = (u_{i,1} \;\; u_{i,2})$.

return $\{u_{1,1}\, p, \; \ldots, \; u_{n_1,1}\, p\}$

Semisimple structure

Consider $\mathcal{I} \subset \langle e \rangle$ generated by $a_1, \ldots a_q \in \langle e \rangle$.

GOAL : Find generators of $\mathcal{I}$ of the form *pure evaluation*, namely, evalutions of the form $e\, p(\partial)$ where $p \in \mathbb{k}[\partial]$.

Semisimple strucuture

Notations and definitions

- $A = \begin{pmatrix} a_1 & \dots & a_q \end{pmatrix} = \sum_{k=0}^{n} A_k(t)\, e\, \partial^k$ where $A_k \in \mathbb{k}[t]^{q\times 1}$
- $m = \max_{k\in[\![0,n]\!]} \deg(A_k)$
- $C = \begin{pmatrix} A_0 & \dots & A_n \\ \vdots & & \vdots \\ A_0^{(m+1)} & \dots & A_n^{(m+1)} \end{pmatrix} = \begin{pmatrix} A_0 & \dots & A_n \\ \vdots & & \vdots \\ A_0^{(m)} & \dots & A_n^{(m)} \\ 0 & \dots & 0 \end{pmatrix} = \begin{pmatrix} C' \\ 0 \end{pmatrix}$
- $D = \begin{pmatrix} D_0 & \dots & D_{m+1} \end{pmatrix}$, where $D_i \in \mathbb{k}[t]^{r\times q}$, is such as
$$\ker_{\mathbb{k}[t]}(.C) = \mathrm{im}_{\mathbb{k}[t]}(.D).$$
Note that $D = \begin{pmatrix} D' & 0 \\ 0 & I_q \end{pmatrix}$, where $D' \in \mathbb{k}[t]^{(r-q)\times q(m+1)}$
- $B = D\, J_{m+1} = \begin{pmatrix} \sum_{k=0}^{m} D'_i\, \partial^i \\ \partial^{m+1}\, I_q \end{pmatrix} = \begin{pmatrix} B' \\ \partial^{m+1}\, I_q \end{pmatrix}$
- $\mathcal{M} = \mathrm{coker}_{\mathbb{I}_1}(.B) = {\mathbb{I}_1}^{1\times q} \big/ ({\mathbb{I}_1}^{1\times r}\, B)$

Then, $\ker_{\mathbb{I}_1}(.A) = \mathrm{im}_{\mathbb{I}_1}(.B)$ and $\mathcal{M} \cong \mathrm{im}_{\mathbb{I}_1}(.A) = \sum_{i=1}^{q} \mathbb{I}_1\, a_i$.

Semisimple structure

$$\begin{array}{ccccccc}
{\mathbb{I}_1}^{1\times r} & \xrightarrow{.B} & {\mathbb{I}_1}^{1\times q} & \xrightarrow{.A} & \mathcal{I} = \mathrm{im}_{\mathbb{I}_1}(.A) & \longrightarrow & 0 \\
\| & & \| & & \uparrow \cong \psi: \ \psi(\pi(\lambda))=\lambda\, A & & \\
{\mathbb{I}_1}^{1\times r} & \xrightarrow{.B} & {\mathbb{I}_1}^{1\times q} & \xrightarrow{\pi} & \mathcal{M} = \mathrm{coker}_{\mathbb{I}_1}(.B) & \longrightarrow & 0
\end{array}$$

$\{y_i = \pi(e_i)\}_{1\leq i\leq q}$ generates $\mathcal{M}$.

$y = (y_1 \ \ldots \ y_q)^T$ satisfy the left $\mathbb{I}_1$-linear relations $B\, y = 0$.

In particular, we have:

$$\partial^{m+1} y = 0 \ \Leftrightarrow \ I^{m+1}\, \partial^{m+1} y = 0 \ \Leftrightarrow \ y = T_m\, y.$$

Moreover, we have:

$$y = T_m\, y = \sum_{k=0}^{m} \frac{t^k}{k!} \underbrace{e\, \partial^k\, y}_{z_k} = \sum_{k=0}^{m} \frac{t^k}{k!}\, z_k.$$

Then, the z_k's generate $\mathcal{M}$ and $\mathbb{I}_1\, z_k = \mathbb{I}_1\, e\, \partial^k\, y = \mathbb{k}[t]\, z_k$ yields

$$\mathcal{M} = \sum_{k=0}^{m} \mathbb{I}_1\, z_k = \sum_{k=0}^{m} \mathbb{k}[t]\, z_k.$$

Semisimple Structure

What are the relations between the z_k's?

$z = (z_0^T \ \dots \ z_m^T)^T$

- $\boxed{e\, z = z}$
- Since $0 = B'\, y = B' \sum_{k=0}^m \frac{t^k}{k!}\, z_k = \sum_{k=0}^m B' \left(\frac{t^k}{k!}\right) z_k$

  Set $P := \begin{pmatrix} B'(1) & B'(t) & \dots & B'\left(\frac{t^m}{m!}\right)\end{pmatrix} \in \mathbb{k}[t]^{(r-q)\times q(m+1)}$,

  $\boxed{P\, z = 0}$
- $$\mathcal{M} \cong \mathcal{M}' = \operatorname{coker}_{\mathbb{I}_1}\left(\cdot \begin{pmatrix} P \\ (1-e)\, I_{q(m+1)} \end{pmatrix}\right)$$
- $B' = \sum_{k=0}^m D_i'\, \partial^i \ \Rightarrow \ B'\left(\frac{t^k}{k!}\right) = D_0'\, \frac{t^k}{k!} + D_1'\, \frac{t^{(k-1)}}{(k-1)!} + \dots + D_k'$
- $$P = \underbrace{(D_0' \ \dots \ D_m')}_{D'} \underbrace{\begin{pmatrix} I_q & t\, I_q & \dots & \frac{t^m}{m!}\, I_q \\ 0 & I_q & \dots & \frac{t^{m-1}}{(m-1)!}\, I_q \\ \vdots & & \ddots & \vdots \\ 0 & \dots & & I_q \end{pmatrix}}_{U \text{ invertible}} = D'\, U.$$

Semisimple Structure

- We have

$$\ker_{\mathbb{k}[t]}(P.) = \ker_{\mathbb{k}[t]}(D'\,U.) = U^{-1}\,\ker_{\mathbb{k}[t]}(D'.) \\ = U^{-1}\,\mathrm{im}_{\mathbb{k}[t]}(C'.) = \mathrm{im}_{\mathbb{k}[t]}(U^{-1}\,C'.).$$

- We have

$$U^{-1}\,C' = \begin{pmatrix} I_q & -t\,I_q & \dots & \frac{(-t)^m}{m!}\,I_q \\ 0 & I_q & \dots & \frac{(-t)^{m-1}}{(m-1)!}\,I_q \\ \vdots & & I_q & \vdots \\ 0 & \dots & \dots & I_q \end{pmatrix} \begin{pmatrix} A_0 & \dots & A_n \\ \vdots & & \vdots \\ A_0^{(m)} & \dots & A_n^{(m)} \end{pmatrix} = C'(0)$$

- $\ker_{\mathbb{k}[t]}(P.) = \mathrm{im}_{\mathbb{k}[t]}(C'(0).) = \mathrm{im}_{\mathbb{k}[t]}(Q'.)$, where $Q' \in \mathbb{k}^{q(m+1)\times s}$ is a full column rank matrix and $s = \mathrm{rank}_{\mathbb{k}}(C'(0))$.
- D has a right inverse $\Rightarrow$ D' has a right inverse $\Rightarrow$ P has a right inverse $\Rightarrow$ $\mathrm{coker}_{\mathbb{k}[t]}(.P)$ is a free $\mathbb{k}[t]$-module of rank s.
- P has a right inverse $\Rightarrow$ Q' has a left inverse $T \in \mathbb{k}^{s\times q(m+1)}$.
- Set $w = T\,z = T\,e\,J_m\,y$. The entries w_i of the vector w are *pure evaluations* and $\mathcal{I} = \sum_{i=1}^{s} \mathbb{k}[t]\,\psi(w_i) = \mathbb{k}[t]^{1\times s}\,(T\,e\,J_m\,A)$.

Semisimple structure

Theorem

$\mathcal{I} = \mathbb{I}_1\, a_1 + \ldots \mathbb{I}_1\, a_q \subset \langle e \rangle$ and let $A = (a_1 \ \ldots \ a_q)^T$. Then, $\mathcal{I}$ is a semisimple $\mathbb{k}[t]$-module that can be generated by a finite set of pure evaluations.

- $C' = \begin{pmatrix} A_0 & \ldots & A_n \\ \vdots & & \vdots \\ A_0^{(m)} & \ldots & A_n^{(m)} \end{pmatrix}$
- $s = \mathrm{rank}_{\mathbb{k}}(C(0))$.
- $Q' \in \mathbb{k}^{(q(m+1)\times s}$ and $B \in {\mathbb{I}_1}^{r\times q}$ such as $\mathrm{im}_{\mathbb{k}}(Q'.) = \mathrm{im}_{\mathbb{k}}(C'(0).)$ and $\ker_{\mathbb{I}_1}(.A) = \mathrm{im}_{\mathbb{I}_1}(.B)$.
- $y = (\pi(e_1) \ \ldots \ \pi(e_q))^T$, where $\pi : {\mathbb{I}_1}^{1\times q} \longrightarrow \mathcal{M} = \mathrm{coker}_{\mathbb{I}_1}(.B)$
- $z = (e\, y \quad e\, \partial\, y \quad \ldots \quad e\, \partial^m\, y)^T$,
- $T \in \mathbb{k}^{s\times q(m+1)}$ a left inverse of Q',
- $w = T\, z \in \mathcal{M}^s$,

$$\mathcal{I} = \sum_{i=1}^{s} \mathbb{k}[t]\, \pi^{-1}(w_i)\, A = \mathbb{k}[t]^{1\times s}\, (T\, e\, J_m\, A).$$

Algorithm

Algorithm 2 Compute pure evaluation generators of a finitely generated evaluation ideal $\mathcal{I}$ as a $\mathbb{k}[t]$-module

Require: $a_1, \ldots, a_q$ generators of $\mathcal{I}$

- Set $A = (a_1 \;\ldots\; a_q)^T$ and compute the matrix C' .
- Compute a full column rank matrix Q' whose columns define a basis of $\mathrm{im}_{\mathbb{k}}(C'(0).)$.
- Compute a left inverse T of Q'.
- Compute $g = (g_1 \;\ldots\; g_s)^T = T\,(e \quad e\,\partial \;\ldots\; e\,\partial^m)^T\,A$

return $\{g_1, \ldots, g_s\}$.

Example

Consider $q = 2$; $m = 1$; $s = \mathrm{rank}_{\Bbbk}(C(0)) = 2$ and

$$A = \begin{pmatrix} t\,e + e\,\partial + (t+1)\,e\,\partial^2 \\ t\,e + t\,e\,\partial + 2\,t\,e\,\partial^2 \end{pmatrix} = \underbrace{\begin{pmatrix} t \\ t \end{pmatrix}}_{A_0} e + \underbrace{\begin{pmatrix} 1 \\ t \end{pmatrix}}_{A_1} e\,\partial + \underbrace{\begin{pmatrix} t+1 \\ 2\,t \end{pmatrix}}_{A_2} e\,\partial^2.$$

$$C' = \begin{pmatrix} t & 1 & t+1 \\ t & t & 2\,t \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} \text{ and } U = \begin{pmatrix} I_2 & t\,I_2 \\ 0 & I_2 \end{pmatrix}$$

$$Q = U^{-1}\,C' = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} Q_1 & Q_2 & Q_3 \end{pmatrix} \text{ and } Q' = \begin{pmatrix} Q_1 & Q_2 \end{pmatrix}$$

$$T = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad w = T\,z = \begin{pmatrix} e\,\partial \\ e \end{pmatrix} \quad g = w\,A = \begin{pmatrix} e + e\,\partial^2 \\ 0 \end{pmatrix}.$$

Then, we have

$$\mathcal{I} = \mathbb{I}_1\,e\,(1 + \partial^2) = \Bbbk[t]\,e\,(1 + \partial^2)$$

Perspectives

- We have also effectively proved that a finitely generated left ideal of $\langle e \rangle$ is principal.
- The semisimple structure of the ideals of $\langle e \rangle$ gives another effective proof of $\mathcal{I} \cap \mathcal{J}$ finitely generated, where $\mathcal{I}$ and $\mathcal{J}$ are two finitely generated left ideals in $\langle e \rangle$.
- The semisimple structure of the ideals of $\langle e \rangle$ gives a theoretical proof of $\mathcal{I} \cap \mathcal{J}$ finitely generated, where $\mathcal{I}$ and $\mathcal{J}$ are finitely generated ideals and one of them is in $\langle e \rangle$.

  We are now working on an algorithmic proof of this point.
- The last step for an algorithmic proof of the coherence of $\mathbb{I}_1$ is the case of $\mathcal{I} \cap \mathcal{J}$, where $\mathcal{I}$ and $\mathcal{J}$ are finitely generated ideals, $\mathcal{I} \notin \langle e \rangle$ and $\mathcal{J} \notin \langle e \rangle$.